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G(f)-SEQUENCES AND FIBRATIONS

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ABSTRACT. For a fibration (E, B, p) with fiber F and a fiber map f, we show that if the inclusion $i : F \longrightarrow E$ has a left homotopy inverse, then $G_n^f(E, F)$ is isomorphic to $G_n^f(F, F) \oplus \pi_n(B)$. In particular, by taking f as the identity map on E we have $G_n(E, F)$ is isomorphic to $G_n(F) \oplus \pi_n(B)$.

1. Introduction

D. H. Gottlieb [1, 2] introduced the subgroup $G_n(X)$ of $\pi_n(X)$. In [5, 13], the authors and J. Kim introduced subgroups $G_n(X, A)$ and $G_n^{Rel}(X, A)$ of $\pi_n(X)$ and $\pi_n(X, A)$ respectively and showed that they fit together into a G-sequence

$$\cdots \xrightarrow{j_\sharp} G_{n+1}^{Rel}(X, A) \xrightarrow{\partial} G_n(A) \xrightarrow{i_\sharp} G_n(X, A) \longrightarrow \cdots \xrightarrow{\partial} G_1(A) \xrightarrow{i_\sharp} G_1(X, A)$$

where $i_\sharp, j_\sharp$ and ∂ are restrictions of the usual homomorphisms of the homotopy sequence

$$\cdots \xrightarrow{j_\sharp} \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_\sharp} \pi_n(X) \longrightarrow \cdots \xrightarrow{\partial} \pi_1(A) \xrightarrow{i_\sharp} \pi_1(X).$$

Here we extend the concept of the above G-sequence into the $G(f)$-sequence for any self-map $f : (X, A) \to (X, A)$. In [7, 12], S. H. Lee and the second author showed that it is still exact in the extended concept when $i : A \to X$ has a left homotopy inverse or is null homotopic . In this paper, we will show that for a fibration (E, B, p) with fiber F and a fiber map f, if the inclusion $i : F \longrightarrow E$ has a left homotopy inverse, then $G_n^f(E, F)$ is isomorphic to $G_n^f(F, F) \oplus \pi_n(B)$. Especially if we take f to be the identity, then we have $G_n(E, F)$ is isomorphic to $G_n(F) \oplus \pi_n(B)$.

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2. Definitions

In this paper, all spaces are finite connected CW-complexes, all topological pairs are CW-pairs with base point and all subspaces mentioned contain the same base point as their total spaces. We denote by A^A the subspace of the function space X^A consisting of $f \in X^A$ such that $f(A) \subset A$. Let us take x_0 as the base point of X and its subspaces. Let I^n be the n-dimensional cube, let ∂I^n be its boundary and let J^{n-1} be the union of all $(n-1)$ faces of I^n except for the initial face. We use the same notation ω for the evaluation maps of X^X and X^A into X at the base point x_0 and use i as the inclusion map.

The Gottlieb groups $G_n(X)$ are defined by $G_n(X) = \{[h] \in \pi_n(X) | \exists$ map $H : X \times I^n \longrightarrow X$ such that $[H|_{x_0 \times I^n}] = [h]$ and $H|_{X \times u} = 1_X$ for $u \in \partial I^n\}$. In [2], Gottlieb showed that the Gottlieb groups (or evaluation subgroups of the homotopy groups) $G_n(X)$ is the image of $\omega_\sharp : \pi_n(X^X, 1_X) \to \pi_n(X, x_0)$. He used these groups to obtain some results about the identifications of topological spaces and to study a fixed point theory and a fibration theory. Since then, many authors [4, 8, 9, 11, 15] have studied and generalized $G_n(X)$.

In [13], the second author and J. Kim introduced $G_n^f(X, A)$ for any map $f : (A, x_0) \to (X, x_0)$. The subgroup is defined by $G_n^f(X, A) = \{[h] \in \pi_n(X) | \exists$ map $H : A \times I^n \longrightarrow X$ such that $[H|_{x_0 \times I^n}] = [h]$ and $H|_{A \times u} = f$ for $u \in \partial I^n\}$. These groups are called the *generalized evaluation subgroups* of the homotopy groups. We also show that the generalized evaluation subgroup $G_n^f(X, A)$ is the image of $\omega_\sharp : \pi_n(X^A, f) \to \pi_n(X, x_0)$. Especially, if $i : A \to X$ is the inclusion, we denote $G_n^i(X, A)$ by $G_n(X, A)$. $G_n(X, A)$ has always contained $G_n(X)$ and

$$G_n(X, A) = \begin{cases} G_n(X) & \text{for } A = X \\ \pi_n(X) & \text{for } A = \{x_0\}. \end{cases}$$

In [5], the authors introduced the subgroup $G_n^{Rel}(X, A)$ of the relative homotopy group $\pi_n(X, A)$ which is defined by $G_n^{Rel}(X, A) = \{[h] \in \pi_n(X, A) | \exists$ map $H : (X \times I^n, A \times \partial I^n) \longrightarrow (X, A)$ such that $[H|_{x_0 \times I^n}] = [h]$ and $H|_{X \times u} = 1_X$ for $u \in J^{n-1}\}$. Equivalently, $G_n^{Rel}(X, A)$ is the image $\omega_\sharp : \pi_n(X^A, A^A, i) \to \pi_n(X, A, x_0)$, where A^A is the subspace of X^A which consists of maps from A into itself. In [7], the relative homotopy Jiang group $G_n^{Rel}(f)$ is defined by $G_n^{Rel}(f) = \{[h] \in \pi_n(X, A) | \exists$

map $H : (X \times I^n, A \times \partial I^n) \longrightarrow (X, A)$ such that $[H|_{x_0 \times I^n}] = [h]$ and $H|_{X\times u} = f$ for $u \in J^{n-1}\}$ for $n \geq 2$ for a self-map f of a pair (X, A).

3. $G(f)$-sequences and fiber space

Let f be a self-map of (X, A). The inclusion map i and the evaluation map ω induce the following commutative diagram

$$\begin{array}{ccccccccccc}
\to & \pi_n(A^A, \bar{f}) & \xrightarrow{i_\sharp} & \pi_n(X^A, f_A) & \xrightarrow{j_\sharp} & \pi_n(X^A, A^A, \bar{f}) & \xrightarrow{\partial} & \pi_{n-1}(A^A, \bar{f}) & \to \\
& \downarrow \omega_\sharp & & \downarrow \omega_\sharp & & \downarrow \omega_\sharp & & \downarrow \omega_\sharp & \\
\to & G_n^f(A, A) & \xrightarrow{i_\sharp} & G_n^f(X, A) & \xrightarrow{j_\sharp} & G_n^{Rel}(f) & \xrightarrow{\partial} & G_{n-1}^f(A, A) & \to \\
& \downarrow \cap & & \downarrow \cap & & \downarrow \cap & & \downarrow \cap & \\
\to & \pi_n(A) & \xrightarrow{i_\sharp} & \pi_n(X) & \xrightarrow{j_\sharp} & \pi(X, A) & \xrightarrow{\partial} & \pi_{n-1}(A) & \to
\end{array}$$

where the top and bottom rows are exact and the middle sequence form a chain complex. We call the middle sequence the *$G(f)$-sequence of (X, A) for the self-map f*. Especially if we take f to be the identity, then the $G(f)$-sequence of (X, A) for the self-map f is just the G-sequence. This sequence is not necessarily exact(see Theorem 3.4 [5]) and there are theorems (see [7], [12]) describing under what conditions the $G(f)$-sequence of (X, A) becomes an exact sequence as follows.

THEOREM 3.1. *Let f be a self-map of (X, A). If the inclusion $i : A \to X$ has a left homotopy inverse or is homotopic to the constant map, then the $G(f)$-sequence of (X, A) is exact.*

Let $p : E \to B$ be a fibration with the fiber $F = p^{-1}(b_0)$, where b_0 is a base point of B. Then p induces a homomorphism $p_\sharp : \pi_n(E, F) \to \pi_n(B, b_0)$ which is one-to-one and onto.

LEMMA 3.2. *Let $p : E \to B$ be a fibration with the fiber F and $f : E \to E$ be a fiber map . The restriction $p_\sharp|_{G_n^{Rel}(f)} : G_n^{Rel}(f) \to \pi_n(B)$ is one to one and onto.*

PROOF. It is sufficient to show that $p_\sharp|_{G_n^{Rel}(f)}$ is onto. Let $[h] \in \pi_n(B)$. Then $h : (I^n, \partial I^n) \to (B, b_0)$ is a pair map. Define

$$H : E \times I^{n-1} \times 0 \cup F \times I^{n-1} \times I \to B$$

by

$$H|_{E\times I^{n-1}\times 0} = pf|_E \text{ and } H|_{F\times I^{n-1}\times I} = hk|_{I^n},$$

where $k : (I^n, \partial I^n) \to (I^n, \partial I^n)$ is a homeomorphism such that $kk = 1$ and $k(J^{n-1}) = I^{n-1} \times 0$ from [3, 11.6]. The map H is well-defined and continuous. Since $(E \times I^{n-1}, F \times I^{n-1})$ is a CW-pair, there is an extension $\bar{H} : E \times I^n \to B$ of H by the absolute homotopy extension property. Consider the following diagram

$$\begin{array}{ccc} E \times I^{n-1} \times 0 & \xrightarrow{f\pi_1} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ E \times I^{n-1} \times I & \xrightarrow{\bar{H}} & B \end{array}$$

where $\pi_1 : E \times I^{n-1} \times 0 \to E$ is the projection, there is a map $\theta : E \times I^n \to E$ such that $\theta|_{E\times I^{n-1}\times 0} = f\pi_1$ and $p\theta = \bar{H}$ since $p : E \to B$ is a fibration. Define $\bar{\theta} : E \times I^n \to E$ by $\bar{\theta} = \theta(1 \times k)$. Then

$$\bar{\theta}|_{E\times J^{n-1}} = \theta(1 \times k)|_{E\times J^{n-1}} = \theta|_{E\times I^{n-1}\times 0} = f \text{ and } \bar{\theta}(F \times \partial I^n) \subset F.$$

Let $\alpha = \bar{\theta}|_{e_0\times I^n} : (I^n, \partial I^n, J^{n-1}) \to (E, F, e_0)$, where $e_0 \in F$ is the base point of E. Then $[\alpha] = [\bar{\theta}|_{e_0\times I^n}] \in G_n^{Rel}(f)$ because $\bar{\theta}$ is an affiliated map of α. Since

$$\begin{aligned} p_\sharp[\alpha] = [p\alpha] &= [p\bar{\theta}|_{e_0\times I^n}] = [p\theta(1 \times k)|_{e_0\times I^n}] \\ &= [\bar{H}(1 \times k)|_{e_0\times I^n}] = [hkk] = [h], \end{aligned}$$

so we show $[h] \in p_\sharp(G_n^{Rel}(f))$. □

Corollary 3.3. *For any fibration $F \xrightarrow{i} E \xrightarrow{p} B$ and a fiber map $f : E \to E$, we have $d(\pi_{n+1}(B)) \subseteq G_n^f(F, F)$ for all n, where d arises the homotopy sequence of the fibration.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} G_{n+1}^{Rel}(f) & \xrightarrow{\partial} & G_n^f(F, F) \\ {\scriptstyle p_\sharp}\downarrow & & \downarrow{\scriptstyle i} \\ \pi_{n+1}(B) & \xrightarrow{d} & \pi_n(F) \end{array}$$

where ∂ is the boundary homomorphism and i is the inclusion. Since $p_\sharp$ is a bijective map, we have

$$d(\pi_{n+1}(B)) = i\partial p_\sharp^{-1}(\pi_{n+1}(B)) = i\partial(G_{n+1}^{Rel}(f)) \subseteq i(G_n^f(F,F)) = G_n^f(F,F).$$

□

The pair (X,A) is a *relative G-pair* if $\pi_n(X,A) = G_n^{Rel}(X,A)$ for all n [14].

COROLLARY 3.4. *If $p : E \to B$ is a fibration with fiber F, then the CW-pair (E,F) is a relative G-pair .*

PROOF. It follows immediately from Lemma 3.2. □

By Corollary 3.3, we have $d(\pi_n(B)) \subset G_n^f(F,F)$ and hence the following sequence form a chain complex.

$$\cdots \to G_n^f(F,F) \xrightarrow{i_\sharp} G_n^f(E,F) \xrightarrow{p_\sharp} \pi_n(B) \xrightarrow{d} \cdots \to G_1^f(F,F) \to G_1^f(E,F) \to \pi_1(B).$$

This sequence is called the $G(f)$- *sequence of the fibration* for the fiber map f. When does the $G(f)$- sequence of a fibration to be exact ? The following lemma gives us an answer for this question.

LEMMA 3.5. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and f be a fiber map. If the inclusion map i is null-homotopic or has a left homotopy inverse, then the $G(f)$-sequence of the fibration is exact.*

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccccc}
\cdots \to G_n^f(F,F) & \xrightarrow{i_\sharp} & G_n^f(E,F) & \xrightarrow{j_\sharp} & G_n^{Rel}(f) & \xrightarrow{\partial} & G_{n-1}^f(F,F) \to \cdots \\
\| & & \| & & \cong \downarrow p_\sharp & & \| \\
\cdots \to G_n^f(F,F) & \xrightarrow{i_\sharp} & G_n^f(E,F) & \xrightarrow{\bar{p}_\sharp} & \pi_n(B) & \xrightarrow{d} & G_{n-1}^f(F,F) \to \cdots
\end{array}$$

where $\bar{p}_\sharp = p_\sharp \circ j_\sharp$ and $d = \partial \circ p_\sharp^{-1}$. Since the $G(f)$-sequence of the pair (E,F) is exact by Theorem 3.1 and $p_\sharp$ is an isomorphism by Lemma 3.2, the $G(f)$-sequence of the fibration is also exact. □

THEOREM 3.6. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and f be a fiber map. If the inclusion map i has a left homotopy inverse. Then we have $G_n^f(E,F) \cong \pi_n(B) \oplus G_n^f(F,F)$.*

PROOF. Let $r : E \to F$ be a left homotopy inverse of i. By the above lemma, we have the following exact $G(f)$-sequence of fibration

$$\cdots \to G_n^f(F,F) \xrightarrow{i_\sharp} G_n^f(E,F) \xrightarrow{p_\sharp} \pi_n(B) \xrightarrow{d} G_{n-1}^f(F,F) \to \cdots .$$

Since $i_\sharp$ is a monomorphism, we have $d(\pi_n(B)) = 0$ and $p_\sharp$ is an epimorphism. Now we will show the homomorphism $r_\sharp : \pi_n(E) \to \pi_n(F)$ induces a homomorphism $r_\sharp : G_n^f(E,F) \to G_n^f(F,F)$. Let $[\alpha]$ be an element of $G_n^f(E,F)$. Then there exists a homotopy $H : F \times I^n \longrightarrow E$ such that

$$[H|_{x_0 \times I^n}] = [\alpha] \text{ and } H|_{F\times u} = f_F \text{ for } u \in \partial I^n.$$

If we define $\bar{H} = rH : F \times I^n \longrightarrow F$, then we have

$$[\bar{H}|_{x_0 \times I^n}] = [r\alpha] = r_\sharp[\alpha] \text{ and } \bar{H}|_{F\times u} = rf_F = \bar{f} \text{ for } u \in \partial I^n.$$

Therefore $r_\sharp([\alpha])$ belongs to $G_n^f(F,F)$ and hence $r_\sharp$ induces a homomorphism $r_\sharp : G_n^f(E,F) \to G_n^f(F,F)$. If we define a homomorphism $\phi : G_n^f(E,F) \to \pi_n(B) \oplus G_n^f(F,F)$ by $\phi(\alpha) = (p_\sharp(\alpha), r_\sharp(\alpha))$, it is easy to show that ϕ is an isomorphism. □

COROLLARY 3.6. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. If the inclusion map i has a left homotopy inverse, then we have $G_n(E,F) \cong \pi_n(B) \oplus G_n(F)$.*

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