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THE MINIMUM THEOREM FOR THE RELATIVE ROOT NIELSEN NUMBER

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ABSTRACT. In [8], we introduce the relative root Nielsen number $N(f;X,A,c)$ for maps of pairs of spaces $f:(X,A)\to(Y,B)$. From it, we obtain some immediate consequences of the definition and illustrate it by some examples. We consider the question whether there exists a map $g:(X,A)\to(Y,B)$ homotopic to a given map $f:(X,A)\to(Y,B)$ which has precisely $N(f;X,A,c)$ roots, that is, the minimum theorem for $N(f;X,A,c)$.

1. Introduction

Topological coincidence theory about the study of the equation $f(x)=g(x)$ for maps $f,g:X\to Y$ specializes in two important ways. One such specialization arises when $X=Y$ and $g:X\to X$ is the identity map. So the equation becomes the fixed point equation $f(x)=x$. Then, when Brooks investigated coincidence theory in [2], he identified an interesting special case, that in which $g:X\to Y$ is the constant map $g(x)=a$ for some $a\in Y$. The coincidence equation becomes in this setting the root equation $f(x)=a$, so-called because it generalizes the equation $P(x)=0$ for a polynomial P, the solution to which are the roots of the polynomial.

A very clear presentation of many of the results in root theory can be found in Kiang's book [4] and we refer to [4] whenever possible. The minimum theorem 2.3 is a main goal of this paper. So we assume that the reader is familar with the theorems for maps of Lin Xiaosong [7]. I would like to thank X.Zhao for his helpful comments about this paper.

Let $f:X\to Y$ be a map (continuous function) and $c\in Y$. Two solutions (roots) x_0 and x_1 to $f(x)=c$ are equivalent iff there is a path p in X from x_0 to x_1 such that $[f\circ p]=[c]$. (Here $[f\circ p]$ denotes the

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fixed-end-point homotopy class containing $f \circ p$ and c is used both to denote the point $c \in Y$ as well as the constant path at $c \in Y$).

This equivalence is induced an equivalence relation; an equivalence class of roots is calleded *root class.* The set of roots of $f(x) = c$ is denoted by $\Gamma(f, c)$, the set of root classes by $\Gamma'(f, c)$.

Let $H : f_0 \simeq f_1 : X \to Y$ be a homotopy between f_0 and f_1, c a given point in Y, and x_i a root of the equation

$$f_i(x) = c, \quad i = 0, 1$$

and R_i a root class in $\Gamma'(f_i, c)$ containing x_i. If there exists in X a path p from x_0 to x_1 such that

$$[\triangle(H, p)] = [c],$$

then x_0 and x_1 are said to be in correspondence under H and denoted by $x_0 H x_1$, where $\triangle(H, p)$ is a diagonal path defined by $\triangle(H, p)(t) = H(p(t), t), 0 \leq t \leq 1$. This relati on $x_0 H x_1$ induces a correspondence from R_0 to R_1 under H, which is denoted by $R_0 H R_1$.

Let $f : X \to Y$ be a mapping under $H : f \simeq H(\cdot, 1) : X \to Y$ a homotopy. Let the root class in $\Gamma'(f, c)$ be denoted by R. If the root class R of the equation $f(x) = c$ corresponds to a root class $\in \Gamma'(H(\cdot, 1), c)$ under any such H, then R is called an *essential root class.* Denote the set of essential root classes of the equation $f(x) = c$ by $\Gamma^*(f, c)$. The number $\#\Gamma^*(f, c)$ of elements in $\Gamma^*(f, c)$ is called the *Nielsen number* of $f(x) = c$ and is denoted by $N(f, c)$.

$N(f, c)$ is clearly a lower bound for the number of solutions of $f(x) = c$. If g is homotopic to f, then $N(f, c) = N(g, c)$.

Let $f : (X, A) \to (Y, B)$ be a map of pair of compact connected ANR's. We shall write $\bar{f} : A \to B$ be a restriction of f to A, and $\Gamma(\bar{f}, c) = \Gamma(f, c) \cap A$ if $c \in B$. Throughout this paper c will be a point lies in $B(\subset Y)$.

DEFINITION 1.1. Let $f : (X, A) \to (Y, B)$ be a map of pair of spaces. A root class $R \in \Gamma'(f, c)$ of $f : X \to Y$ is a *common root class* of f and $\bar{f}$ if $R \cap \Gamma_e(\bar{f}, c) \neq \varnothing$ where $\Gamma_e(\bar{f}, c)$ is the essential root set of $\bar{f} : A \to B$. It is an *essential common root class* of f and $\bar{f}$ if it is an essential root class of $f : X \to Y$ and a common root class of f and $\bar{f}$.

DEFINITION 1.2. Let $(X,A),(Y,B)$ be pairs of compact connected ANR's. If $f:(X,A)\to(Y,B)$ is a map, the *relative root Nielsen number* $N(f;X,A,c)$ is defined as

$$N(f;X,A,c)=N(f,c)+N(\bar{f},c)-N(f,\bar{f},c),$$

where $N(f,\bar{f},c)$ is the number of essential common root classes of f and $\bar{f}$.

Hence $N(f;X,A,c)$ is a finite integer ≥ 0 and equals $N(f,c)$ if $X=A$ or $A=\varnothing$.

The number $N(f;X,A,c)$ defined above satisfies the usual basic two properties (lower bound property and homotopy invariance) and we show in some examples that it can be easy to find $N(f;X,A,c)$ in [8].

2. The Minimum Theorem

In this section, we consider the whether there exists a map $g:(X,A)\to(Y,B)$ homotopic to given map $f:(X,A)\to(Y,B)$ which has precisely $N(f;X,A,c)$ roots.

To prove the Minimum Theorem we shall need additional properties of (X,A) which is introduced in the next definitions.

DEFINITION 2.1. A subspace A of a space X can be *by-passed* if every path in X with end points in $X-A$ is homotopic to a path in $X-A$ keeping end point fixed.

The subspace A in Examples 1-4 of [8] can be by-passed.

DEFINITION 2.2. Suppose $P,Q\subset X$ are subpolyhedra of the unbounded n-manifold X and that $p+q=n$, where $p=\dim P, q=\dim Q$. We say P is *transverse* to Q in X if

(1) $P\cap Q$ consists of a finite set of points,
(2) for each $x\in P\cap Q$, there are neighborhoods U_1,U_2,U_3 of x in P,Q,X such that (U_1,U_2,U_3) is p.l. homeomorphic to a nbd of 0 in $(\mathbb{R}^p\times 0, 0\times\mathbb{R}^q,\mathbb{R}^p\times\mathbb{R}^q)$.

Suppose P and Q have opposite intersection number at x and y. With these assumptions, we have

WHITNEY'S LEMMA. *Let $x, y \in P \cap Q$ be transverse intersection points and α a path from x to y. If dim$P$, dim$Q \geq 3$, then there exists an isotopy of the embedding $P \hookrightarrow X$, keeping the outside of an arbitrary neighborhood of α fixed, moving P to P', such that $P' \cap Q = P \cap Q - \{x, y\}$.*

The Whitney's Lemma enables us to cancel double points. If each of P, Q and X are orientable, then we can attach a sign to an intersection point $x \in P \cap Q$, and the idea is to give conditions under which can cancel a pair of intersections of opposite sign; in other words find an ambiant isotopy of P which removes this pair from the set of intersections of P and Q.

Now we can prove the Minimum Theorem. (Compare with [3] Theorem 2.4).

THEOREM 2.3 (MINIMUM THEOREM). *Let X and Y are connected closed orientable n-manifolds and let $f : (X, A) \to (Y, B)$ be a map of pairs such that*

(1) *A and B are closed orientable submanifolds of X and Y resp.,*
(2) *whenever the component A_j of A is mapped to the component B_c of B,which contains c, then dimA_j = dim$B_c \geq 3$,*
(3) *A can be by-passed in X.*

Then there is a map $g : (X, A) \to (Y, B)$ which is homotopic to f with exact $N(f; X, A, c)$ roots

PROOF. The proof of Theorem can be done by following 3 steps. □

Step 1. With above assumption, we may assume that $\Gamma(\bar{f}, c)$ contains exactly $N(\bar{f}, c)$ points which lying in bdA and every map $f : X \to Y$ is ($\varepsilon-$)homotopic to a map $f' : X \to Y$ such that f' has finitely many roots.

PROOF. As Lin Xiaosong Theorem B, on each component A_j of A, use the Whitney's trick for the root class, we may eliminate a pair of points in any root class of $\Gamma(\bar{f}, c)$ with opposite local degrees. Repeat such a procedure finitely many times, we may asume that there are exactly $N(\bar{f}, c)$ point in $\Gamma(\bar{f}, c)$. And by HEP and transversality theory, the graph f is homotopic to a graph f' of a map f' such that the graph of f is transverse to $X \times \{c\}$ in $X \times Y$ Then f' has finitely many roots. □

Step 2. Let A can be by-passed in X. Then no two points in $\Gamma(f, c) - A$ are in the same class.

PROOF. From Step 1, we may assume that every root class $R - A$ in $\Gamma(f, c) - A$ contains finitely many, say $m(R)$ points and f has local degree sign in R at each point in $\Gamma(f, c) - A$. As Lin's proof of Theorem A and B, using the fact that A can be by-passed in X, Lin's construction can be carried out in $X - A$ so that $R - A$ contains only one point at last. □

Step 3. If two points $y \in \Gamma(f, c) - A$ and $\bar{x} \in \Gamma(\bar{f}, c)$ are Nielsen related, then there exists a homotopy $F : (X \times I, A \times I) \to (Y, B)$ constant on A and in a neighborhood of $\Gamma(f, c) - \{\bar{x}, y\}$ from $(f, \bar{f})$ to $(g, \bar{g})$ such that $\Gamma(g, c) = \Gamma(f, c) - \{y\}$.

PROOF. Since $\bar{x}, y$ are Nielsen related and $A \subset X$ can be by-passed, there is a path ω establishing the Nielsen relation between y and $\bar{x}$ and satisfying $\omega[0, 1) \subset X - A$, $\omega(t) \notin \Gamma(f, c)$ for $0 < t < 1$. Let us fix a subset $U \subset X$ homeomorphic to $\mathbb{R}^{n-1} \times (-\infty, 1]$ such that under this homeomorphism $\omega(t) = (0, t) \in \mathbb{R}^{n-1} \times [0, 1]$

$$U \cap A \subset \mathbb{R}^{n-1} \times 1, \quad U \cap \Gamma(f, c) = \{\bar{x}, y\}.$$

On the other hand, the constant path c^* in Y is homotopic to $f\omega$. Choose an open neighborhood $V \subset Y$ of c. Since $f\omega$ and c^* are fixed end point homotopic, there exist homotopices $f|_t : \omega[0, 1] \to Y$ from the restriction of f to V constant on $\omega[0, \varepsilon] \cup \omega[1 - \varepsilon, 1]$ for some $\varepsilon > 0$. Moreover, by the assumption $\dim B_c \geq 3$, we may assume that $\Gamma(f|_t, c) = \{\bar{x}, y\}$ for each t.

Fix two closed balls K_0 and K_1 in $U \subset X$ centered at $\bar{x}$ and y resp. such that

$$K_0 \cap \omega[0, 1] \subset \omega[0, \varepsilon]$$

$$K_1 \cap \omega[0, 1] \subset \omega[1 - \varepsilon, 1]$$

Define $F' : X \times 0 \cup (K_0 \cup \omega[0,1] \cup K_1) \times I \to Y$ by

$$F'(x,t) = \begin{cases} f|_t(x), & \text{if } x \in \omega[\varepsilon, 1-\varepsilon] \\ f(x) & \text{otherwise.} \end{cases}$$

Fix a retraction $r : X \times [0,1] \to (X \times 0) \cup (K_0 \cup \omega[0,1] \cup K_1) \times [0,1]$ such that $r(x,t) = (x,0)$ for x lying outside a neighborhood of $\omega[0,1], r(A \times [0,1]) \subset A$ and $r^{-1}(\{\bar{x}, y\} \times [0,1]) = \{\bar{x}, y\} \times [0,1]$. We put $F_1(x) = F'r(x,1)$. Now F_1 sent $\omega[0,1]$ into V hence $F_1(U_1) \subset V$ for some neighborhood U_1 of $\omega[0,1]$ in U. In U_1 we fix another Euclidean neighborhood U_2 such that $\omega[0,1) \subset U_2 \subset U_1 - A$ and any point $x \in (cl\, U_2 - \bar{x})$ is uniquely labelled as $x = t\bar{x} + (1-t)x_1$ where $x_1 \in ((bd\, U_2) - \bar{x})$. Finally we put

$$F(x,t) = \begin{cases} tF_1(\bar{x}) + (1-t)F_1(x_1) & \text{for } x = t\bar{x} + (1-t)x_1 \in cl\, U_2 \\ F_1(x) & \text{otherwise.} \end{cases}$$

Now, put $g(x) = F(x,t)$, then g is homotopic to F_1 rel. $(X - U_2)$ hence

$$\Gamma(g,c) = \Gamma(F_1,c) - \{y\}.$$

□

REMARK. We have thought the number $N(f; X, A, c)$ of a map $f : (X,A) \to (Y,B)$ and assume that $c \in B$. But if $c \notin B$, then as we have shown in Example 3 of [8], above Minimum Theorem does not holds. That is, even though $N(f; X, A, c) = 0$, there exist a map $g(\simeq f) : (X,A) \to (Y,B)$ such that it must have a root.

So we leave it an open problem to find a sharp lower bound if $c \notin B$.

References

[1] R. Brooks, *Certain subgroups of the fundamental group and the numberof roots of $f(x) = a$*, Amer. J. Math. **95** (1973), 720–728.

[2] ———, *Coincidences, roots and fixed points*, Doctoral Dissertation, Univ. of Calfornia, Los Angeles (1967).

[3] J. Jezierski, *A relative coincidence Nielsen numbers*, preprint.

[4] T. Kiang, *The theory of fixed point classes*, Springer-Verlag, Heidelberg (1989).

[5] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, Spr- inger-Verlag, Berlin, Heidelberg, New York (1972).
[6] H. Schirmer, *A relative Nielsen number*, Pacific J. Math. **122** (1986), 459–473.
[7] L. Xiaosong, *On the root classes of mapping*, Acta math. Sinica, New Series **2** (1986), 199–206.
[8] K. Y. Yang, *A relative root Nielsen number*, Comm. Korean Math. Soc. **11** (1996), 245-252.

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