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MINIMAL *CR* SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE WITH PARALLEL SECTION IN THE NORMAL BUNDLE

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ABSTRACT. In this paper we prove that if the minimum of the sectional curvatures of a compact n-dimensional minimal generic submanifold M of a complex projective space is $1/n$, then M is the geodesic minimal hypersphere.

Introduction

In [2] we proved that if the minimum of the sectional curvatures of a compact real minimal hypersurface of a complex m-dimensional projective space CP^m is $1/(2m-1)$, then M is the geodesic hypersphere. This result was generalized in [9] to the case of M is a generic submanifold with flat normal connection.

The purpose of the present paper is to study minimal CR submanifolds of CP^m with parallel normal section in the normal bundle and prove a generalization of theorems in [2] and [9] (see also [3]).

In §1 we state general formulas on CR submanifolds of a Kaehlerian manifold, especially those of a complex space form. §2 is devoted to the study CR submanifolds with nonvanishing parallel normal section of the normal bundle of M, and compute the restricted Laplacian for the second fundamental form in the direction of the parallel normal section.

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As applications of this, in §3, we prove an integral formula. In the last §4, we prove our main theorem by the integral formula given in §3.

1. Preliminaries

Let $\tilde{M}$ be a complex m-dimensional Kaehlerian manifold with almost complex structure J and with Kaehlerian metric g. Let M be a real n-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$. We denote by the same g the Riemannian metric tensor field induced on M from that of $\tilde{M}$. We denote by $\tilde{\nabla}$ the operator of covariant differentiation with respect to the metric g on $\tilde{M}$ and by ∇ the one in M. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y) \text{ and } \tilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. A and B appearing here are both called the second fundamental forms of M and are related by

$$g(B(X,Y),V) = g(A_V X, Y).$$

The second fundamental form A_V in the direction of the normal vector V can be considered as a symmetric (n,n)-matrix.

The covariant derivative $\nabla_X A$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form of M is said to be *parallel in the direction of* V. If the second fundamental form is parallel in any direction, it is said to be *parallel*.

The *mean curvature vector* ν of M is defined to be $\nu = (\mathrm{Tr}B)/n$, where $\mathrm{Tr}B$ denoting the trace of B. If $\nu = 0$, then M is said to be *minimal*. If the second fundamental form A vanishes identically, then M is said to be *totally geodesic*. A vector field V normal to M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M. A parallel normal vector field $V(\neq 0)$ is called an *isoperimetric section* if $\mathrm{Tr}A_V$ is constant, and is called a *minimal section* if $\mathrm{Tr}A_V$ is zero.

For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. Similarly, for any vector field V normal to M, we put

$$JV = tV + fV,$$

where tV is the tangential part and fV the normal part of JV. We then have [10]

$$g(PX,Y)+g(X,PY) = 0, \quad g(fV,U) + g(V,fU) = 0$$
$$g(FX,V) + g(X,tV) = 0.$$

Moreover, we see

$$P^2 = -I - tF, \quad FP + fF = 0, \quad Pt + tf = 0, \quad f^2 = -I - Ft.$$

We define the covariant derivatives of P, F, t and f by

$$(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y, \qquad (\nabla_X F)Y = D_X(FY) - F\nabla_X Y,$$
$$(\nabla_X t)V = \nabla_X(tV) - tD_X V, \qquad (\nabla_X f)V = D_X(fV) - fD_X V,$$

respectively. We then have [10]

$$(\nabla_X P)Y = A_{FY}X + tB(X,Y), \tag{1.1}$$

$$(\nabla_X F)Y = -B(X,PY) + fB(X,Y), \tag{1.2}$$

$$(\nabla_X t)V = A_{fV}X - PA_V X, \tag{1.3}$$

$$(\nabla_X f)V = -FA_V X - B(X, tV). \tag{1.4}$$

A submanifold M of a Kaehlerian manifold $\tilde{M}$ is called a *CR submanifold* of $\tilde{M}$ if there exists a differentiable distribution $H : x \to H_x \subset T_x(M)$ on M satisfying the following conditions:

(1) H is invariant with respect to J, namely, $JH_x \subset H_x$ for each point x in M, and

(2) the complementary orthogonal distribution $H^\perp : x \to H_x^\perp \subset T_x(M)$ is anti-invariant with respect to J, namely, $JH_x^\perp \subset T_x(M)^\perp$ for each point x in M.

We put $\dim H = h$, $\dim H^\perp = q$ and $codim M = 2m - n = p$. If $q = 0$, then a CR submanifold M is called an *invariant submanifold* of $\tilde{M}$, and if $h = 0$, then M is called an *anti-invariant submanifold* of $\tilde{M}$. If $p = q$, then a CR submanifold M is called a *generic submanifold* of $\tilde{M}$ (see [10]).

In the following, we suppose that M is a CR submanifold of a Kaehlerian manifold $\tilde{M}$. Then

$$FP = 0, fF = 0, tf = 0, Pt = 0, \tag{1.5}$$

$$P^3 + P = 0, f^3 + f = 0. \tag{1.6}$$

Equations in (1.6) show that P is an f-structure in M and f is an f-structure in the normal bundle of M (see [8]). From (1.1) we obtain

$$A_{FX}Y - A_{FY}X = 0 \quad \text{for } X, Y \in H^\perp \tag{1.7}$$

We have the following decomposition of the tangent space $T_x(M)$ at each point x of M:

$$T_x(M) = H_x(M) + H_x^\perp(M),$$

where $H_x(M) = JH_x(M)$ and $H_x^\perp(M)$ is the orthogonal complement of $H_x(M)$ in $T_x(M)$. Then $JH_x^\perp(M) = FH_x^\perp(M) \subset T_x(M)^\perp$. Similarly, we have

$$T_x(M)^\perp = FH_x^\perp(M) + N_x(M),$$

where $N_x(M)$ is the orthogonal complement of $FH_x^\perp$ in $T_x(M)^\perp$. Then $JN_x(M) = fN_x(M) = N_x(M)$.

We take an orthonormal basis $e_1, \cdots, e_{2m}$ of $\tilde{M}$ such that, restricted to $M, e_1, \cdots, e_n$ are tangent to M. Then $e_1, \cdots, e_n$ form an orthonormal basis of M. We can take $e_1, \cdots, e_n$ such that $e_1, \cdots, e_q$ form an orthonormal basis of $H_x^\perp(M)$ and $e_{q+1}, \cdots, e_n$ form an orthonormal basis of $H_x(M)$. Moreover, we can take $e_{n+1}, \cdots, e_{2m}$ of an orthonormal basis of $T_x(M)^\perp$ such that $e_{n+1}, \cdots, e_{n+q}$ form an orthonormal basis of $FH_x^\perp(M)$ and $e_{n+q}, \cdots, e_{2m}$ form an orthonormal basis of $N_x(M)$. In case of need, we can take $e_{n+1}, \cdots, e_{n+q}$ such that $e_{n+1} = Fe_1, \cdots, e_{n+q} = Fe_q$. Unless otherwise stated, we use the conventions that the ranges of indices are respectively:

$$\begin{aligned} &i, j, k = 1, \cdots, n; && x, y, z = 1, \cdots, q; \\ &a, b, c = n+1, \cdots, 2m; && \lambda, \mu, \nu = n+q+1, \cdots, 2m. \end{aligned}$$

We denote by $\tilde{M}^m(c)$ an m-dimensional complex space form of constant holomorphic sectional curvature c. Then equations of Gauss and Codazzi of M are given respectively by

$$\begin{aligned} R(X,Y)Z = &\tfrac{1}{4}c\{g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX \\ &- g(PX,Z)PY + 2g(X,PY)PZ\} \\ &+ A_{B(Y,Z)}X - A_{B(X,Z)}Y, \end{aligned} \tag{1.8}$$

where R being the Riemannian curvature tensor of M,

$$\begin{aligned} &g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z) \\ &\quad = g((\nabla_X B)(Y,Z), V) - g((\nabla_Y B)(X,Z), V) \\ &\quad = \tfrac{1}{4}c\{g(PY,Z)g(FX,V) - g(PX,Z)g(FY,V) \\ &\quad\quad + 2g(X,PY)g(FZ,V)\}, \end{aligned} \tag{1.9}$$

where ∇B is defined to be

$$(\nabla_X B)(Y,Z) = D_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

We define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V.$$

Then we have the equation of Ricci

$$\begin{aligned} & g(R^{\perp}(X,Y)V,U) + g([A_U, A_V]X,Y) \\ (1.10) \qquad & = \tfrac{1}{4}c\{g(FY,V)g(FX,U) - g(FX,V)g(FY,U) \\ & + 2g(X,PY)g(fV,U)\}. \end{aligned}$$

From the equation of Gauss (1.8), the Ricci tensor S of M is given by

$$\begin{aligned} S(X,Y) &= \tfrac{1}{4}c\{(n-1)g(X,Y) + 3g(PX,PY)\} \\ &+ \sum \mathrm{Tr}A_a g(A_a X,Y) - \sum g(A_a^2 X,Y), \end{aligned} \qquad (1.11)$$

A_a being the second fundamental form in the direction of e_a.

2. Lemmas

First of all, we prepare some lemmas for later use. For any vector field X on a Riemannian manifold M we have generally (see [6])

$$\begin{aligned} \mathrm{div}(\nabla_X X) - \mathrm{div}((\mathrm{div}X)X) = S(X,X) + \tfrac{1}{2}|L(X)g|^2 \\ - |\nabla X|^2 - (\mathrm{div}X)^2, \end{aligned}$$

where $L(X)g$ denotes the Lie derivative of g with respect to the vector field X, and $|Y|$ denotes the length of a vector field Y on M with respect to g.

Let M be an n-dimensional minimal CR submanifold of a complex space form $\tilde{M}^m(c)$. Suppose that U is a parallel section of the normal bundle of M. Then (1.3) implies

$$\nabla_X(tU) = A_{fU}X - PA_U X$$

for any vector field X tangent to M. Thus we have

$$\mathrm{div}(tU) = \sum g(\nabla_i tU, e_i) = \mathrm{Tr}A_{fU} - \mathrm{Tr}PA_U = 0.$$

From this and (2.1) we obtain

$$\operatorname{div}(\nabla_{tU}tU) = S(tU,tU) + \tfrac{1}{2}|L(tU)g|^2 - |\nabla tU|^2. \tag{2.2}$$

On the other hand, by (1.11), we have

$$\begin{aligned} S(tU,tU) =&\tfrac{1}{4}c(n-1)g(tU,tU) \\ &+ \sum g(A_a tU, tU)\mathrm{Tr}A_a - \sum g(A_a tU, A_a tU). \end{aligned} \tag{2.3}$$

We also have

$$\begin{aligned} |\nabla tU|^2 =&\mathrm{Tr}A_{fU}^2 + \mathrm{Tr}A_U^2 \\ &- 2\mathrm{Tr}A_U A_{fU} P - \sum g(A_U te_a, A_U te_a). \end{aligned} \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2), we have

$$\begin{aligned} \operatorname{div}(\nabla_{tU}tU) =&\tfrac{1}{4}c(n-1)g(tU,tU) + \tfrac{1}{2}|L(tU)g|^2 \\ &+ \sum g(A_a tU, tU)\mathrm{Tr}A_a - \sum g(A_a tU, A_a tU) \\ &- \mathrm{Tr}A_{fU}^2 - \mathrm{Tr}A_U^2 \\ &+ 2\mathrm{Tr}A_U A_{fU} P + \sum g(A_U te_a, A_U te_a). \end{aligned} \tag{2.5}$$

Since we have

$$\begin{aligned} (L(tU)g)(X,Y) &= g(\nabla_X tU, Y) + g(\nabla_Y tU, X) \\ &= g((A_U P - P A_U)X, Y), \end{aligned}$$

we obtain

$$|L(tU)g|^2 = |[P, A_U]|^2 = 2\{\mathrm{Tr}(A_U P)^2 - \mathrm{Tr}A_U^2 P^2\}.$$

Therefore, we have

LEMMA 2.1. *Let M be an n-dimensional minimal CR submanifold of a complex space form $\tilde{M}^m(c)$. If U is a parallel section of the normal bundle of M, then*

$$\begin{aligned} \operatorname{div}(\nabla_{tU}tU) =&\tfrac{1}{4}c(n-1)g(tU,tU) + \tfrac{1}{2}|[P, A_U]|^2 \\ &- \sum g(A_a tU, A_a tU) - \mathrm{Tr}A_{fU}^2 - \mathrm{Tr}A_U^2 \\ &+ 2\mathrm{Tr}A_U A_{fU} P + \sum g(A_U te_a, A_U te_a). \end{aligned} \tag{2.6}$$

LEMMA 2.2. *Let M be an n-dimensional minimal CR submanifold of a complex space form $\tilde{M}^m(c)$. If U is a parallel section of the normal bundle of M, then*

$$(2.7)\quad \begin{aligned}(\nabla^2 A)_U X = &\sum (R(e_i, X)A)_U e_i \\ &+ \tfrac{1}{4}c\{-A_{FX}tU - tB(tU, X) + 3PA_U PX \\ &- g(X, tU)\sum A_a te_a - 2\sum g(A_a te_a, X)tU \\ &+ PA_{fU}X - 2A_{fU}PX\}.\end{aligned}$$

PROOF. From the assumption we see

$$\sum (\nabla_i A)_U e_i = 0.$$

Thus, from (1.9), we have

$$\begin{aligned}(\nabla^2 A)_U X &= \sum (\nabla_i \nabla_i A)_U X \\ &= \sum (R(e_i, X)A)_U e_i + \tfrac{1}{4}c\sum\{g((\nabla_i F)e_i, U)PX \\ &\quad + g(Fe_i, U)(\nabla_i P)X - g((\nabla_i F)X, U)Pe_i \\ &\quad - g(FX, U)(\nabla_i P)e_i + 2g((\nabla_i P)e_i, X)tU \\ &\quad + 2g(Pe_i, X)(\nabla_i t)U\}.\end{aligned}$$

Using (1.1), (1.2) and (1.3), we have our equation. □

3. Integral formulas

Let M be an n-dimensional minimal CR submanifold of a complex projective space CP^m with constant holomorphic sectional curvature 4. We suppose that there is a parallel unit normal section μ in the normal bundle of M. Then we have

(3.1)

$$\begin{aligned}g(\nabla^2 A_\mu, A_\mu) = &\sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) \\ &+ 3\{\mathrm{Tr}(A_\mu P)^2 - \sum g(A_a te_a, A_\mu t\mu) - \mathrm{Tr} PA_\mu A_{f\mu}\}.\end{aligned}$$

Since μ is parallel, (1.10) implies

$$\sum\{g(A_a t\mu, A_\mu te_a) - g(A_a te_a, A_\mu t\mu)\} = q-1,$$

$$\sum g([A_{f\mu}, A_\mu]e_i, Pe_i) = 2\mathrm{Tr}A_\mu A_{f\mu}P = -2hg(f\mu, f\mu).$$

Therefore, we have

$$\begin{aligned}&\mathrm{Tr}(A_\mu P)^2 - \sum g(A_a te_a, A_\mu t\mu) - \mathrm{Tr}PA_\mu A_{f\mu}\\ &= \tfrac{1}{2}\|[P, A_\mu]\|^2 - \mathrm{Tr}A_\mu^2 + \sum g(A_\mu te_a, A_\mu te_a) - \sum g(A_a t\mu, A_\mu te_a)\\ &\quad + (q-1) - \mathrm{Tr}PA_\mu A_{f\mu}.\end{aligned}$$

Substituting this equation into (3.1), we have

$$\begin{aligned}g(\nabla^2 A_\mu, A_\mu) =& \sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) + 3\{\tfrac{1}{2}\|[P, A_\mu]\|^2\\ &- \mathrm{Tr}A_\mu^2 + \sum g(A_\mu te_a, A_\mu te_a)\\ &- \sum g(A_a t\mu, A_\mu te_a) + (q-1) - \mathrm{Tr}PA_\mu A_{f\mu}\}.\end{aligned} \tag{3.2}$$

By Lemma 2.1, (3.2) becomes

$$\begin{aligned}&- g(\nabla^2 A_\mu, A_\mu) - 2(n-q) + (q-1) + \tfrac{1}{2}\|[P, A_\mu]\|^2\\ &- 7\mathrm{Tr}PA_\mu A_{f\mu} + 2\mathrm{Tr}A_{f\mu}^2\\ &= \mathrm{Tr}A_\mu^2 - \sum g((R(e_i, e_j)A)_\mu e_i, A_\mu e_j) - 2\mathrm{div}(\nabla_{t\mu} t\mu)\\ &+ \sum\{3g(A_a t\mu, A_\mu te_a) - 2g(A_a t\mu, A_a t\mu) - g(A_\mu te_a, A_\mu te_a)\}.\end{aligned} \tag{3.3}$$

THEOREM 3.1. *Let M be a compact n-dimensional minimal CR submanifold of CP^m with parallel unit normal section μ in the normal bundle of M. If $f\mu = 0$, then*

$$\begin{aligned}0 \le& \int_M \{|\nabla A_\mu|^2 - 2(n-q) + (q-1)\\ &+ \tfrac{1}{2}\|[P, A_\mu]\|^2 + 2g(A_\lambda t\mu, A_\lambda t\mu)\} \star 1\\ =& \int_M \{\mathrm{Tr}A_\mu^2 - \sum g(R(e_i, e_j)A_\mu)e_i, A_\mu e_j)\} \star 1.\end{aligned} \tag{3.4}$$

PROOF. We have

$$\frac{1}{2}\triangle \mathrm{Tr} A_\mu^2 = g(\nabla^2 A_\mu, A_\mu) + |\nabla A_\mu|^2.$$

Thus we have

$$-\int_M g(\nabla^2 A, A) * 1 = \int_M |\nabla A|^2 * 1.$$

Let us put $T(X,Y) = (\nabla_X A_\mu)Y - g(PX,Y)t\mu - g(Y,t\mu)PX$. Then we have $|T|^2 = |\nabla A_\mu|^2 - 2(n-q) \geq 0$ by (1.9). Thus we have

$$|\nabla A_\mu|^2 - 2(n-q) \geq 0,$$

and the equality holds if and only if

$$(\nabla_X A_\mu)Y = g(PX,Y)t\mu + g(Y,t\mu)PX.$$

Here, we can take $e_{n+1}, \cdots, e_{n+q}$ such that $e_{n+1} = Fe_1, \cdots, e_{n+q} = Fe_q$. Then, we obtain

$$\sum\{3g(A_a t\mu, A_\mu te_a) - 2g(A_a t\mu, A_a t\mu) - g(A_\mu te_a, A_\mu te_a)\} = -2\sum g(A_\lambda t\mu, A_\lambda t\mu)$$

by (1.7), where $\lambda = n+q+1, \cdots, 2m$. Therefore, we have (3.4). □

4. Main theorems

THEOREM 4.1. *Let M be a compact n-dimensional minimal CR submanifold of CP^m with parallel unit normal section μ in the normal bundle such that $f\mu = 0$. If the minimum of the sectional curvatrues of M is $1/n$, then $q = 1, |\nabla A_\mu|^2 = 2(n-1)$ and $PA_\mu = A_\mu P$.*

PROOF. We choose an orthonormal frame $\{e_i\}$ of M such that $A_\mu e_i = \lambda_i e_i$ $(i = 1, \cdots, n)$. We denote by K_{ij} the sectional curvature of M spanned by e_j and e_i. Then we have

$$\begin{aligned}&\sum g((R(e_i, e_j)A_\mu)e_i, A_\mu e_j)\\ &= \sum\{g(R(e_i, e_j)A_\mu e_i, A_\mu e_j) - g(A_\mu R(e_i, e_j)e_i, A_\mu e_j)\}\\ &= \tfrac{1}{2}\sum(\lambda_i - \lambda_j)^2 K_{ij}\\ &\geq (1/2n)\sum(\lambda_i - \lambda_j)^2 = \mathrm{Tr}A_\mu^2.\end{aligned}$$

Consequently, we see

$$\mathrm{Tr}A_\mu^2 - \sum g((R(e_i, e_j)A_\mu)e_i, A_\mu e_j) \leq 0.$$

From this and Theorem 3.1 we have our assertion.

EXAMPLE. We consider the standard fibration

$$S^1 \longrightarrow S^{2n+1} \longrightarrow CP^n,$$

where S^k denotes the Euclidean sphere of curvature 1. In S^{2n+1} we have the family of generalized clifford surfaces whose spheres lie in complex subspaces (cf. [4]):

$$M_{2p+1,2q+1} = S^{2p+1}(((2p+1)/2n)^{\frac{1}{2}}) \times S^{2q+1}(((2q+1)/2n)^{\frac{1}{2}}),$$

where $p + q = n - 1$. Then we have a fibration

$$S^1 \longrightarrow M_{2p+1,2q+1} \longrightarrow M_{p,q}^C,$$

compatible with the standard fibration. In the special case $p = 0$, $M_{0,n-1}^C$ is called a *geodesic minimal hypersphere* (see [6]), and is a homogeneous, positively curved manifold diffeomorphic to the sphere (see [4], [6]).

The minimum of the sectional curvature of $M_{0,n-1}^C$ is $1/n$, and that of $M_{p,q}^C (p, q \leq 1)$ is zero.

If M is a compact n-dimensional generic minimal submanifold of CP^m with nonvanishing parallel section in the normal bundle μ. We can assume that $|\mu| = 1$. Since we have $f = 0$, if the minimum of the sectional curvatures of M is $1/n$, then, by Theorem 4.1, we see that M is a real hypersurface of CP^m. We also have $PA_\mu = A_\mu P$. Thus, from a theorem of [5] we see that M is $M^C_{p,q}$. Since the minimum of the sectional curvature of M is $1/n$, we see that M is the geodesic minimal hypersphere. Consequently, we obtain

THEOREM 4.2. *Let M be a compact n-dimensional minimal generic submanifold of CP^m with nonvanishing parallel section μ in the normal bundle. If the minimum of the sectional curvatures of M is $1/n$, then $2m = n + 1$ and M is the geodesic minimal hypersphere $M^C_{0,(n-1)/2}$.*

If the normal connection of M is flat, then we can choose an orthonormal frame $\{e_a\}$ of the normal bundle such that $De_a = 0$ for all a (cf. [1]). Then we have

COROLLARY 4.1([9]). *Let M be a compact n-dimensional minimal generic submanifold of CP^m with flat normal connection. If the minimum of the sectional curvatures of M is $1/n$, then $2m = n + 1$ and M is the geodesic minimal hypersphere $M^C_{0,(n-1)/2}$.*

COROLLARY 4.2([2]). *Let M be a compact real minimal hypersurface. If the minimum of the sectional curvatures of M is $1/(2m-1)$, then M is the geodesic minimal hypersphere $M^C_{0,m-1}$.*

THEOREM 4.3. *Let M be a compact n-dimensional minimal CR submanifold of CP^m with nonvanishing parallel unit normal section μ in the normal bundle such that $f\mu = 0$. If the minimum of the sectional curvatures of M is $(n-q)/n(n-1)$, then we have $|\nabla A|^2 = 2(n-q)$. Moreover, we have $n = q$ or $q = 1$ and $PA_\mu = A_\mu P$.*

PROOF. From Lemma 2.1 and Theorem 3.1 we have

$$\begin{aligned} &- g(\nabla^2 A, A) - 2(n-q) - 5\mathrm{Tr}A_\mu A_{f\mu} P + 3\mathrm{Tr}A^2_{f\mu} \\ &= -\sum g(R(e_i, e_j)A_\mu)e_i, A_\mu e_j) + (n-q) - 3\mathrm{div}(\nabla_{JU} JU) \\ &\quad + 3\sum \{g(A_a t\mu, A_\mu t e_a) - g(A_a t\mu, A_a t\mu)\}. \end{aligned}$$

Therefore, we have

$$0 \leq \int_M \{|\nabla A| - 2(n-q)\} \star 1$$
$$= \int_M \{(n-q) - \sum g(R(e_i, e_j)A_\mu)e_i, A_\mu e_j) - 3\sum g(A_\lambda t\mu, A_\lambda t\mu)\} \star 1.$$

On the other hand, by the similar method in the proof of Theorem 4.1, we see

$$\sum g(R(e_i, e_j)A_\mu)e_i, A_\mu e_j) = \tfrac{1}{2}\sum(\lambda_i - \lambda_j)^2 K_{ij}$$
$$\geq (n-q)/(n-1)\mathrm{Tr}A_\mu^2.$$

Hence we have

$$(n-q) - \sum g(R(e_i, e_j)A_\mu)e_i, A_\mu e_j)$$
$$\leq (n-q)/(n-1)\{(n-1) - \mathrm{Tr}A_\mu^2\}.$$

From this and Lemma 2.1 we have

$$0 \leq \int_M \{|\nabla A|^2 - 2(n-q)\} \star 1$$
$$\leq \int_M [(n-q)/(n-1)\{(n-1) - \mathrm{Tr}A_\mu^2\} - 3\sum g(A_\lambda t\mu, A_\lambda t\mu)] \star 1$$
$$\leq -\tfrac{1}{2}(n-q)/(n-1)\int_M \|[P, A_\mu]\|^2 \star 1.$$

Thus, we have $|\nabla A|^2 = 2(n-q)$, and $n = q$ or $PA_\mu = A_\mu P$.

We suppose that $n \neq q$ and $q \geq 2$. Then, we can take a unit normal vector field V orthogonal to μ. Hence we have

$$\nabla_{JV} t\mu = -PA_\mu tV = -A_\mu PtV = 0.$$

Thus the sectional curvature spanned by $t\mu$ and tV is zero. This contradicts the assumption $K_{ij} \geq (n-q)/n(n-1) > 0$. Therefore, we must have $q = 1$. □

THEOREM 4.4. *Let M be a compact n-dimensional minimal generic submanifold of CP^m with nonvanishing parallel unit normal section μ in the normal bundle such that $f\mu = 0$. If the minimum of the sectional curvatures of M is $(n-p)/n(n-1)$, then M is a totally real submanifold of CP^m, or $2m = n+1$ and M is the geodesic minimal hypersphere $M^C_{0,(n-1)/2}$.*

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