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ON THE SEMI-HYPONORMAL OPERATORS ON A HILBERT SPACE

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ABSTRACT. Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we construct a pair of semi-positive definite operators

$$|T|_r = (T^*T)^{\frac{1}{2}} \quad \text{and} \quad |T|_l = (TT^*)^{\frac{1}{2}}.$$

An operator T is called a semi-hyponormal operator if

$$Q_T = |T|_r - |T|_l \geq 0.$$

In this paper, by using a technique introduced by Berberian [1], we show that the approximate point spectrum $\sigma_{ap}(T)$ of a semi-hyponormal operator T is empty.

1. Semi-hyponormal operators

Throughout this paper, the letter $\mathcal{H}$ is used for a complex separable Hilbert space, and the $*$-algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$.

We now consider the polar decomposition of an operator. For $T \in \mathcal{L}(\mathcal{H})$, we construct a pair of semi-positive definite operators

$$|T|_r = (T^*T)^{\frac{1}{2}} \quad \text{and} \quad |T|_l = (TT^*)^{\frac{1}{2}}$$

and define a map V from $|T|_r\mathcal{H}$ to $T\mathcal{H}$ by

$$V(|T|_r y) = Ty.$$

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It is clear that V is an isometry. So V can be uniquely extended to an isometric operator U from the closure of $|T|_r\mathcal{H}$ onto the closure of $T\mathcal{H}$. For $x \in (|T|_r\mathcal{H})^{\perp}$, we set $Ux = 0$, thus U is a partial isometry with the initial space $\overline{|T|_r\mathcal{H}}$ and the final subspace $\overline{T\mathcal{H}}$ and we have $T = U|T|_r$. The representation $T = U|T|_r$ is called the polar decomposition of T. It is evident that $|A|_l^2 = AA^* = U|A|_r^2U^*$. Since U^*U is the projection onto the initial space of U, we have $|A|_l^2 = U|A|_rU^*U|A|_rU^* = (U|A|_rU^*)^2$. By the uniqueness of the square root of a positive operator, we have $|A|_l = U|A|_rU^*$. Hence, $AU^*U = |A|_lU$. It follows that $x \in \overline{|A|_r\mathcal{H}}$, $Ax = |A|_lUx$. By definition, $Ux = 0$ for $x \perp |A|_r\mathcal{H}$. On the other hand, $|A|_rx = 0$, so that $Ax = U|A|_rx = 0$, hence it is still true that $Ax = U|A|_lUx$, for $x \perp |A|_r\mathcal{H}$. Thus we have another polar decomposition $T = |T|_lU$. Therefore, the final subspace and the initial subspace of U are $\overline{T\mathcal{H}} = \overline{|T|_l\mathcal{H}}$ and $\overline{|T|_r\mathcal{H}} = \overline{T^*\mathcal{H}}$, respectively.

For $A, B \in \mathcal{L}(\mathcal{H})$, $[A, B] = AB - BA$ is called the commutator of A and B, and the operator $D_A = [A^*, A]$ is called the self-commutator of A. For $T \in \mathcal{L}(\mathcal{H})$, T is said to be normal if $D_T = 0$ and hyponormal if $D_T \geq 0$. An operator A is said to be a semi-hyponormal operator if

$$Q_A = |A|_r - |A|_l \geq 0,$$

Q_A is called the polar-difference operator of A.

PROPOSITION 1.1. ([5]) *A hyponormal operator must be a semi-hyponormal operator. So we have*

$$\textit{Normal} \subset \textit{Hyponormal} \subset \textit{Semi-hyponormal}.$$

PROPOSITION 1.2. ([5]) (1) *If T is invertible semi-hyponormal, then T^{-1} is also semi-hyponormal.*

(2) *If $T = U|T|_r$ is semi-hyponormal, then the eigenspace of U reduces T.*

By using a technique introduced by Berberian ([1]), one can deal with question about the approximate point spectrum by looking at the situation for the point spectrum.

In [1], he constructed an extension $\mathcal{K}$ of $\mathcal{H}$ by means of bounded sequence in $\mathcal{H}$ and the Banach limits and obtained the faithful $*$-representation ϕ of $\mathcal{L}(\mathcal{H})$ on $\mathcal{K}$. Here we present this technique in a simplified form.

The spectrum, the point spectrum and the approximate point spectrum of an operator T are denoted by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$, respectively.

THEOREM 1.3. ([1], [4]) *Let $\mathcal{H}$ be a seperable Hilbert space. Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and an unital faithful $*$-representation ϕ of $\mathcal{L}(\mathcal{H})$ on $\mathcal{K}$ with the following properties :*

(1) $\|\phi(T)\| = \|T\|$ *and* $\phi(A) \leq \phi(B)$ *whenever* $A \leq B$,
(2) $\sigma(T) = \sigma(\phi(T))$, $\sigma_{ap}(T) = \sigma_{ap}(\phi(T)) = \sigma_p(\phi(T))$.

An operator is said to be reducible if it has a nontrivial reducing subspace. If an operator is not reducible, then it is called irreducible.

PROPOSITION 1.4. *If T is an irreducible operator, then $\phi(T)$ is an irreducible operator.*

PROOF. Suppose that $\phi(T)$ is reducible. Then there exists a proper subspace $\mathcal{M}$ of $\mathcal{K}$ such that $\phi(T)\mathcal{M} \subset \mathcal{M}$ and $\phi(T^*)\mathcal{M} \subset \mathcal{M}$, that is, $\phi(T)s' = \{Tx_n\}' \in \mathcal{M}$ and $\phi(T)^*s' = \{T^*x_n\}' \in \mathcal{M}$ for all $s' = \{x_n\} + \mathcal{N} \in \mathcal{M}$, where $\mathcal{N} = \{\{x_n\} | x_n \in \mathcal{H}, LIM\{\|x_n\|\} = 0\}$ and LIM is the Banach limit. Put $\mathcal{M}_1 = \{x_n | \{x_n\}' \in \mathcal{M}\}$. If $\mathcal{M}_1 = \mathcal{H}$, then obviously $\mathcal{M} = \mathcal{K}$ by the construction of $\mathcal{K}$ which contradicts that $\phi(T)$ is reducible. Thus, $\mathcal{M}_1$ is the subspace of $\mathcal{H}$. Hence, for all $x_n \in \mathcal{M}_1$, Tx_n and T^*x_n are in $\mathcal{M}_1$ which contradicts that T is irreducible. Therefore, $\phi(T)$ is irreducible. □

PROPOSITON 1.5. ([3]) *If $T \in \mathcal{L}(\mathcal{H})$, then there is a reducing subspace $\mathcal{H}_0$ for T such that*

(1) $T_0 = T|_{\mathcal{H}_0}$ *is normal,*
(2) $T_1 = T|_{\mathcal{H}_0^\perp}$ *has no reducing subspace on which it is normal.*

An operator T is pure if it has no reducing subspace on which it is normal. In other words, T is pure if the space $\mathcal{H}_0$ in Propositon 1.5 is $\{0\}$.

PROPOSITION 1.6. *If T is a pure operator, then $\phi(T)$ is a pure operator.*

PROOF. If $\phi(T)$ is not pure, then there exist a proper subspace $\mathcal{M}$ of $\mathcal{K}$ and the restriction of $\phi(T)$ to $\mathcal{M}$ is normal. Put $\mathcal{M}_1 = \{x_n | \{x_n\}' \in \mathcal{M}\}$. Then since ϕ is an unital faithful $*$-representation of $\mathcal{L}(\mathcal{H})$ on $\mathcal{K}$, the restriction of T to $\mathcal{M}_1$ is normal which contradicts that T is pure. □

PROPOSITION 1.7. *If $T \in \mathcal{L}(\mathcal{H})$, then $\phi(|T|_r) = |\phi(T)|_r$ and $\phi(|T|_l) = |\phi(T)|_l$.*

PROOF. It is clear that $|\phi(T)|_r^2 = \phi(T)^*\phi(T) = \phi(T^*T) = \phi(|T|_r^2) = \phi(|T|_r)^2$. By the uniqueness of the square root of a positive operator, we have $\phi(|T|_r) = |\phi(T)|_r$. Similarly, $\phi(|T|_l) = |\phi(T)|_l$. □

By Proposition 1.7, we obviously have the following results.

PROPOSITION 1.8. *If T is a semi-hyponormal operator, then $\phi(T)$ is a semi-hyponormal operator.*

2. Spectra of semi-hyponormal operators

For every operator $T \in \mathcal{L}(\mathcal{H})$, there is a Cartesian decomposition $T = X + iY$ of T, where $X = \frac{1}{2}(T + T^*)$ and $Y = \frac{1}{2i}(T - T^*)$. The operator X and Y are called the real and imaginary parts of T, respectively.

The joint point spectrum $\sigma_{jp}(T)$ of $T = X + iY$ is the set of all complex numbers $z = x + iy$ (x and y are real numbers) such that there exists a common eigenvector $f(\neq 0)$ of X and Y such that

$$Xf = xf \quad \text{and} \quad Yf = yf.$$

In addition, $z \in \sigma_{jp}(T)$ if and only if there exists a non-zero vector f such that

$$Tf = zf \quad \text{and} \quad T^*f = \bar{z}f.$$

It is evident that $\sigma_{jp}(T) \subset \sigma_p(T)$, and moreover for a normal operator T, $\sigma_{jp}(T) = \sigma_p(T)$.

PROPOSITION 2.1. ([2], [5]) *For any $T \in \mathcal{L}(\mathcal{H})$,*

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T^*)^*.$$

PROPOSITION 2.2. ([5]) *If T is a semi-hyponormal operator, then* $\sigma_{jp}(T) = \sigma_p(T)$.

The joint approximate point spectrum, $\sigma_{ja}(T)$ of $T = X + iY$ is the set of all complex numbers $z = x + iy$ which there exists a sequence $\{f_n\}$ of unit vectors such that

$$\lim_{n\to\infty} \|(X - xI)f_n\| = \lim_{n\to\infty} \|(Y - yI)f_n\| = 0.$$

It is also evident that $\sigma_{ja}(T) \subset \sigma_{ap}(T)$, and moreover for a normal operator T

$$\sigma_{ja}(T) = \sigma_{ap}(T) = \sigma(T).$$

PROPOSITION 2.3. ([5]) *If T is a semi-hyponormal operator, then* $\sigma_{ja}(T) = \sigma_{ap}(T)$ *and* $\sigma(T) = \sigma_{ap}(T^*)^*$.

PROPOSITION 2.4. *If T is a pure semi-hyponormal operator, then* $\sigma_p(T) = \emptyset$.

PROOF. Suppose that there exists a $\lambda \in \sigma_p(T)$. Since $\sigma_{jp}(T) = \sigma_p(T)$ by Proposition 2.2, there is a vector $f \neq 0$ such that $Tf = \lambda f$ and $T^* f = \bar{\lambda} f$. Thus the one dimensional vector space

$$W = \{\mu f : \mu \in \mathbb{C}\}$$

reduces T, and $TT^*(f) = T(\bar{\lambda} f) = \bar{\lambda} T(f) = |\lambda|^2 f$ and $T^*T(f) = T^*(\lambda f) = \lambda T^*(f) = |\lambda|^2 f$. Hence the restriction of T to W is normal, which contradicts the assumption that T is pure. Therefore, $\sigma_p(T) = \emptyset$. □

PROPOSITION 2.5. *If T is a pure semi-hyponormal operator, then* $\sigma_{ap}(T) = \emptyset$.

PROOF. Suppose that T is a pure semi-hyponormal operator. Then by Propositions 1.6 and 1.7, $\phi(T)$ is a pure semi-hyponormal operator. Thus, $\sigma_p(\phi(T)) = \emptyset$. Therefore, by Theorem 1.3, since $\sigma_{ap}(T) = \sigma_{ap}(\phi(T)) = \sigma_p(\phi(T))$, $\sigma_{ap}(T) = \emptyset$. □

References

[1] S. K. Berberian, *Approximate proper vectors*, Proc. Amer. Math. Soc. **13** (1962), 111- 114.

[2] ———, *Lectures in Functional Analysis and Operator Theory*, Springer-Verlag, New York, 1974.

[3] J. B. Conway, *Subnormal Operators, Research Notes in Math. vol.51*, Pitman Advanced Pub. Program, Boston, 1981.

[4] J. G. Stampfli, *Compact Perturbations, Normal Eigenvalues and a Problem of Salinas*, J. London Math. Soc.(2) **9** (1974), 165-175.

[5] Daoxing Xia, *Spectral Theory of Hyponormal Operators*, Birkhäuser Verlag, Basel, Boston, 1983.

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