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AN L_p ANALYTIC FOURIER-FEYNMAN TRANSFORM ON ABSTRACT WIENER SPACE

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ABSTRACT. In this paper, we establish an L_p analytic Fourier-Feynman transform theory for a class of cylinder functions on an abstract Wiener space. Also we define a convolution product for functions on an abstract Wiener space and then prove that the L_p analytic Fourier-Feynman transform of the convolution product is a product of L_p analytic Fourier-Feynman transforms.

1. Introductory Preliminaries

The concept of an L_1 analytic Fourier-Feynman transform was introduced in 1972 by Brue [2], and it was based on the analytic Wiener and Feynman integral defined on the classical Wiener space [3]. In [5], Cameron and Storvick established an L_2 analytic Fourier-Feynman transform theory. Also Johnson and Skoug [11] developed an L_p analytic Fourier-Feynman transform theory for $1 \leqslant p \leqslant 2$ which extended the results in [2,5], and they gave various relationships between the L_1 and the L_2 theories. Recently, Huffman, Park and Skoug [9] introduced an L_p analytic Fourier-Feynman transform theory for a class of functionals on the classical Wiener space not considered in [2, 5, 11]. In this paper, we establish an L_p analytic Fourier-Feynman transform theory for a class of cylinder functions on an abstract Wiener space. Also we define a convolution product for functions on an abstract Wiener space and then prove that the L_p analytic Fourier-Feynman transform of the convolution product is a product of L_p analytic Fourier-Feynman transforms.

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Let H be a real separable infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Let $||\cdot||_o$ be a measurable norm on H with respect to the Gauss measure μ. Let B denote the completion of H with respect to $||\cdot||_o$. Let i denote the natural injection from H into B. The adjoint operator i^* of i is one-to-one and maps the dual B^* continuously onto a dense subset of H^* . By identifying H with H^* and B^* with i^*B^*, we have a triple $B^* \subset H^* \equiv H \subset B$ and $\langle h, x\rangle = (h, x)$ for all h in H and x in B^*, where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. By a well known result of Gross [8], $\mu\cdot i^{-1}$ has an unique countably additive extension m to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (H, B, m) is called an abstract Wiener space and the Hilbert space H is called the generator of (H, B, m). For more details, see [8, 12, 13, 14, 15].

A subset E of B is said to be scale–invariant measurable provided ρE is Wiener measurable for each $\rho > 0$, and a scale–invariant measurable set N is said to be scale–invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale– invariant null set is said to hold scale–invariant almost everywhere (s–a.e.). If two functionals F and G are equal s–a.e., we write $F \approx G$. For a complete discussion of scale–invariant measurability, see [6].

Throuhgout this paper, let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of complex numbers and complex numbers with positive real part, respectively.

DEFINTION 1.1. Let F be a complex-valued scale–invariant measurable function on B such that the integral

$$J(F;\lambda) = \int_B F(\lambda^{-\frac{1}{2}}x)\,dm(x) \tag{1.1}$$

exists for all real $\lambda > 0$. If there exists an analytic function $J^*(F; z)$ on $\mathbb{C}_+$ such that $J^*(F;\lambda) = J(F;\lambda)$ for all real $\lambda > 0$, then we define $J^*(F; z)$ to be the *analytic Wiener integral* of F over B with parameter z, and for each $z \in \mathbb{C}_+$, we write

$$I^{aw}(F; z) = J^*(F; z). \tag{1.2}$$

Let q be a non-zero real number and let F be a function on B whose analytic Wiener integral exists for each z in $\mathbb{C}_+$. If the following limit exists, then we call it the *analytic Feynman integral* of F over B with parameter q, and we write

$$I^{af}(F;q) = \lim_{z \to -iq} I^{aw}(F;z), \tag{1.3}$$

where z approaches $-iq$ through $\mathbb{C}_+$.

Let $\{e_n\}_{n=1}^{\infty}$ denote a complete orthonormal (C.O.N.) set in H such that the e_n's in B^*. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(\cdot,\cdot)^{\sim}$ between H and B as follows:

$$(h,x)^{\sim} = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (e_j, x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

It is well known [12] that for every $h \in H$, $(h,x)^{\sim}$ exists for m-a.e. $x \in B$ and it is a Borel measurable function on B having a Gaussian distribution with mean zero and variance $|h|^2$. If $\{h_1, \cdots, h_n\}$ is an orthonormal set of elements in H, then $(h_1,x)^{\sim}, \cdots, (h_n,x)^{\sim}$ are independent Gaussian functionals with mean zero and variance one. Note that if both h and x are in H, then $(h,x)^{\sim} = \langle h, x\rangle$.

DEFINITION 1.2. Let (H, B, m) be an abstract Wiener space. A function F is called a *cylinder function* on B if there exists a linearly independent subset $\{h_1, \cdots, h_n\}$ of H such that

$$F(x) = f((h_1,x)^{\sim}, \cdots, (h_n,x)^{\sim}), \tag{1.5}$$

where f is a complex-valued Borel measurable function on $\mathbb{R}^n$.

DEFINITION 1.3. Let (H, B, m) be an abstract Wiener space. Let n be a positive integer, and let $\{h_1, \cdots, h_n\}$ be an orthonormal set of elements in H. For $1 \leqslant p < \infty$, let $\mathcal{F}(n;p)$ denote the class of cylinder functions F on B of the form

$$F(x) = f((h_1,x)^{\sim}, \cdots, (h_n,x)^{\sim}), \tag{1.6}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is in $L_p(\mathbb{R}^n)$, the space of functions whose p-th powers are Lebesgue integrable on $\mathbb{R}^n$. Let $\mathcal{F}(n;\infty)$ denote the class of cylinder functions F of the form (1.6) where $f : \mathbb{R}^n \to \mathbb{C}$ is in $C_0(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ which vanish at infinity.

We finish this section by defining an L_p analytic Fourier-Feynman transform on an abstract Wiener space.

DEFINITION 1.4. For a given number p with $1 < p \leqslant 2$, let $\{F_n\}$ and F be scale-invariant measurable functionals such that for each $\rho > 0$,

$$\lim_{n\to\infty} \int_B |F_n(\rho y) - F(\rho y)|^{p'} dm(y) = 0. \tag{1.7}$$

Then we write

$$\lim_{n\to\infty} (w_s^{p'})(F_n) \approx F \tag{1.8}$$

and we call F the *scale-invariant limit in the mean of order* p', where $1/p + 1/p' = 1$. A similar definition is understood when n is replaced by the continuously varying parameter λ.

DEFINITION 1.5. Let $q \neq 0$ be a real number. For $1 < p \leqslant 2$ and for $\lambda \in \mathbb{C}_+$, the L_p *analytic Fourier-Feynman transform* $T_q^{(p)}(F)$ of F is defined by

$$(T_q^{(p)})(F)(y) = \lim_{\lambda\to -iq} (w_s^{p'})(T_\lambda(F))(y) \tag{1.9}$$

whenever the limit exists. And the L_1 *analytic Fourier-Feynman transform* $(T_q^{(1)}(F))$ of F is defined by

$$(T_q^{(1)}(F))(y) = \lim_{\lambda\to -iq} (T_\lambda(F))(y) \text{ for } s-a.e.\ y \in B \tag{1.10}$$

where $T_\lambda(F)(y) \equiv I^{aw}(F(\cdot + y) : \lambda)$.

Note that for $1 \leqslant p \leqslant 2$, $T_q^{(p)}(F)$ is defined only s–a.e.. Also if $T_q^{(p)}(F_1)$ exists and $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_1) \approx T_q^{(p)}(F_2)$.

2. An L_p Analytic Fourier-Feynman Transform

In this section, we establish the existence of an L_p analytic Fourier-Feynman transform for certain classes of cylinder functions on an abstract Wiener space. We begin this section by showing the existence of analytic Wiener integral $T_\lambda(F)(y) \equiv I^{aw}(F(\cdot + y) : \lambda)$ for $F \in \mathcal{F}(n;p)$ where $1 \leq p \leq \infty$ and $\lambda \in \mathbb{C}_+$.

THEOREM 2.1. *Let (H, B, m) be an abstract Wiener space and let $F \in \mathcal{F}(n:p)$ be given by (1.6) for $1 \le p \le \infty$. Then for $\lambda \in \mathbb{C}_+$, the analytic Wiener integral $(T_\lambda(F))(y)$ exists and has the form*

$$(T_\lambda(F))(y) = (G_\lambda f)((h_1, y)^\sim, \cdots, (h_n, y)^\sim), \tag{2.1}$$

where

$$\begin{aligned}&(G_\lambda f)(\omega_1, \cdots, \omega_n)\\ &=(\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\{-\frac{\lambda}{2}\sum_{j=1}^{n}(u_j - \omega_j)^2\}\, d\vec{u}.\end{aligned} \tag{2.2}$$

PROOF. Since $(h_1, x)^\sim, \cdots, (h_n, x)^\sim$ are independent Gaussian functionals with mean zero and variance one, we have that for $\lambda > 0$,

$$\begin{aligned}
(T_\lambda(F))(y) &= \int_B F(\lambda^{-\frac{1}{2}}x + y)\, dm(x)\\
&= \int_B f(\lambda^{-\frac{1}{2}}(h_1, x)^\sim + (h_1, y)^\sim, \cdots, \lambda^{-\frac{1}{2}}(h_n, x)^\sim + (h_n, y)^\sim)\, dm(x)\\
&= (\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f(v_1 + (h_1, y)^\sim, \cdots, v_n + (h_n, y)^\sim) \exp\{-\frac{\lambda}{2}\sum_{j=1}^{n} v_j^2\}\, d\vec{v}\\
&= (\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \cdot \exp\{-\frac{\lambda}{2}\sum_{j=1}^{n}(u_j - (h_j, y)^\sim)^2\}\, d\vec{u}\\
&= (G_\lambda f)((h_1, y)^\sim, \cdots, (h_n, y)^\sim).
\end{aligned}$$

Let $\triangle$ be any rectifiable simple closed curve lying in $\mathbb{C}_+$ and let $\alpha = \sup\{|\lambda| : \lambda \in \triangle\}$ and $\beta = \inf\{\mathrm{Re}\lambda : \lambda \in \triangle\}$. If F belongs to $\mathcal{F}(n; 1)$, then $|\frac{\lambda}{2\pi}|^{\frac{n}{2}} |f(\vec{u})| \exp\{-\frac{\mathrm{Re}\lambda}{2}\sum_{j=1}^{n}(u_j - (h_j, y)^\sim)^2\} \le |\frac{\alpha}{2\pi}|^{\frac{n}{2}} |f(\vec{u})|$ and $|\frac{\alpha}{2\pi}|^{\frac{n}{2}} |f(\vec{u})|$ is integrable. If F belongs to $\mathcal{F}(n; p)(1 < p < \infty)$, then the function $|\frac{\alpha}{2\pi}|^{\frac{n}{2}} |f(\vec{u})| \exp\{-\frac{\beta}{2}\sum_{j=1}^{n}(u_j - (h_j, y)^\sim)^2\}$ dominates $|\frac{\lambda}{2\pi}|^{\frac{n}{2}}\ |f(\vec{u})| \exp\{-\frac{\mathrm{Re}\lambda}{2}\sum_{j=1}^{n}(u_j - (h_j, y)^\sim)^2\}$ and it is integrable on $\mathbb{R}^n$ by Hölder's inequality. If F belongs to $\mathcal{F}(n; \infty)$, then the function $|\frac{\alpha}{2\pi}|^{\frac{n}{2}} |f(\vec{u})|$ exp $\{-\frac{\beta}{2}\sum_{j=1}^{n}(u_j - (h_j, y)^\sim)^2\}$ dominates $|\frac{\lambda}{2\pi}|^{\frac{n}{2}} |f(\vec{u})|$

$\exp\{-\frac{\mathrm{Re}\lambda}{2}\sum_{j=1}^{n}(u_j-(h_j,y)^{\sim})^2\}$ and it is integrable on $\mathbb{R}^n$ as $|f(\vec{u})|$ is bounded. Using the dominated convergence theorem, we know that $(G_\lambda f)(\vec{w})$ is continuous in $\mathbb{C}_+$. Moreover, by the Fubini theorem and the Cauchy theorem , we obtain that for $\lambda \in \mathbb{C}_+$,

$$\int_{\triangle}(G_\lambda f)((h_1,y)^{\sim},\cdots,(h_n,y)^{\sim})\,d\lambda$$

$$=\int_{\mathbb{R}^n}(\int_{\triangle}(\frac{\lambda}{2\pi})^{\frac{n}{2}}f(\vec{u})exp[-\frac{\lambda}{2}\sum_{j=1}^{n}(u_j-(h_j,y)^{\sim})^2]d\lambda)d\vec{u}=0.$$

Therefore $(G_\lambda f)((h_1,y)^{\sim},\cdots,(h_n,y)^{\sim})$ is an analytic function of $\lambda \in \mathbb{C}_+$ by the Morera's theorem ,and hence $(T_\lambda(F))(y)$ exists and equals to $(G_\lambda f)((h_1,y)^{\sim},\cdots,(h_n,y)^{\sim})$ for all $\lambda \in \mathbb{C}_+$. □

COROLLARY 2.2. *Let (H,B,m) be an abstract Wiener space. If $F \in \mathcal{F}(n;1)$, then $(T_\lambda(F))(y) \in \mathcal{F}(n;\infty)$ and if $F \in \mathcal{F}(n:p)(1<p\le 2)$, then $(T_\lambda(F))(y) \in \mathcal{F}(n;p')$ where $\frac{1}{p}+\frac{1}{p'}=1$ and $\lambda \in \mathbb{C}_+$. Moreover, for $p=1$,*

$$||G_\lambda f||_\infty \le |\frac{\lambda}{2\pi}|^{\frac{n}{2}}||f||_1, \tag{2.3}$$

and, for $1<p\le 2$,

$$||G_\lambda f||_{p'} \le |\frac{\lambda}{2\pi}|^{n(\frac{1}{2}-\frac{1}{p'})}||f||_p \tag{2.4}$$

where $(T_\lambda(F))(y) \equiv (G_\lambda f)((h_1,y)^{\sim},\cdots,(h_n,y)^{\sim})$ is given by (2.2).

PROOF. If $p=1$, then $|(G_\lambda f)(\vec{w})| \le |\frac{\lambda}{2\pi}|^{\frac{n}{2}}||f(\vec{u})||_1$. By the dominated convergence theorem, $(G_\lambda f)$ $(\vec{w})$ belongs to $C_o(\mathbb{R}^n)$ for all $\lambda \in \mathbb{C}_+$ as a function of $\vec{w} \in \mathbb{R}^n$. Hence (2.3) holds and $(T_\lambda(F))(y) \in \mathcal{F}(n;\infty)$. Now let $1<p\le 2$. Then by [10, Lemma 1.1, p.98], G_λ is in $L(L_p(\mathbb{R}^n), L_{p'}(\mathbb{R}^n))$, the space of continuous linear operators from $L_p(\mathbb{R}^n)$ to $L_{p'}(\mathbb{R}^n)$ and $||G_\lambda|| \le |\frac{\lambda}{2\pi}|^{n(\frac{1}{2}-\frac{1}{p'})}$. From the definition of the operator norm, it follows that (2.4) holds and $(T_\lambda(F))(y) \in \mathcal{F}(n;p')$. □

Next we show that the L_p analytic Fourier-Feynman transform exists for functions $\mathcal{F}(n;p)$ $(1\le p\le 2)$ on an abstract Wiener space.

THEOREM 2.3. *Let (H, B, m) be an abstract Wiener space and let $F \in \mathcal{F}(n;1)$ be given by (1.6). Then for non-zero real q, the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F exists as an element of $\mathcal{F}(n;\infty)$ and it is given by*

$$(2.5) \qquad (T_q^{(1)}(F))(y) \approx (G_{-iq}f)((h_1,y)^\sim, \cdots, (h_n,y)^\sim) ,$$

where $(G_{-iq}f)(\cdot)$ is given by (2.2).

PROOF. Since $F \in \mathcal{F}(n;1)$ and $|f(\vec{u})\exp\{-\frac{\lambda}{2}\sum_{j=1}^{n}(u_j - w_j)^2\}| \leq |f(\vec{u})| \in L_1(\mathbb{R}^n)$, $(G_\lambda f)$ $(\vec{w})$ converges pointwise to $(G_{-iq}f)(\vec{w})$ as $\lambda \to -iq$ in $\mathbb{C}_+$, by the dominated convergence theorem . Now let $\mu \in \mathcal{M}(\mathbb{R}^n)$, the dual of $C_o(\mathbb{R}^n)$. Since $|(G_\lambda f)(\vec{w})| \leq |\frac{\lambda}{2\pi}|^{\frac{n}{2}}||f||_1$, we have

$$\lim_{\lambda \to -iq} \int_{\mathbb{R}^n} (G_\lambda f)(\vec{w})\, d\mu(\vec{w}) = \int_{\mathbb{R}^n} (G_{-iq}f)(\vec{w})\, d\mu(\vec{w})$$

by the dominated convergence theorem. Therefore $(G_\lambda f)(\vec{w})$ converges weakly to $(G_{-iq}f)(\vec{w})$ as elements of C_o $(\mathbb{R}^n)$ as $\lambda \to -iq$ in $\mathbb{C}_+$, and hence $T_q^{(1)}(F)$ exists and it is given by (2.5). □

THEOREM 2.4. *Let (H, B, m) be an abstract Wiener space and let $F \in \mathcal{F}(n : p)$ be given by (1.6) for $(1 < p \leq 2)$. Then for non-zero real q, the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F exists as an element of $\mathcal{F}(n;p')$ and it is given by*

$$(2.6) \qquad (T_q^{(p)}(F))(y) \approx (G_{-iq}f)((h_1,y)^\sim, \cdots, (h_n,y)^\sim),$$

where $(G_{-iq}f)(\cdot)$ is given by (2.2) and $\frac{1}{p} + \frac{1}{p'} = 1$.

PROOF. Using [10, Lemma 1.2, p.100], we obtain that for $f \in L_p(\mathbb{R}^n)$,

$$||(G_\lambda f)(\cdot) - (G_{-iq}f)(\cdot)||_{p'} \to 0,$$

whenever $\lambda \to -iq$ through $\mathbb{C}_+$ and $(G_\lambda f)(\cdot) \in L_{p'}(\mathbb{R}^n)$. Since $(h_1,y)^\sim$, $\cdots$, $(h_n,y)^\sim$ are independent Gaussian functionals with mean zero and variance one, we have that for each $\rho > 0$,

$$\int_B |(G_\lambda f)(\rho(h_1,y)^\sim,\cdots,\rho(h_n,y)^\sim)- (G_{-iq}f)(\rho(h_1,y)^\sim,\cdots,\rho(h_n,y)^\sim)|^{p'}\,dm(y)$$
$$=(2\pi)^{-\frac{n}{2}}\rho^{-n}\int_{\mathbb{R}^n}|(G_\lambda f)(\vec{w})-(G_{-iq})(\vec{w})|^{p'}\exp\{-\frac{1}{2\rho^2}\sum_{j=1}^n w_j^2\}\,d\vec{w}$$
$$\leq(2\pi)^{-\frac{n}{2}}\rho^{-n}||(G_\lambda f)(\cdot)-(G_{-iq}f)(\cdot)||_{p'}^{p'}\to 0,$$

whenever $\lambda \to -iq$ through $\mathbb{C}_+$. Thus $T_q^{(p)}(F)$ exists for $1 < p \leq 2$, it belongs to $\mathcal{F}(n;p')$ and it is given by (2.6). □

We end this section by obtaining an inverse transform theorem for $F \in \mathcal{F}(n:p)$. Using the technique as in the proof of Theorem 1.2 in [11], we have the following property.

THEOREM 2.5. *Let $1 \leqslant p \leqslant 2$ and let $F \in \mathcal{F}(n;p)$ be given by (1.6) and let q be a non-zero real number. Then*

(i) *for each $\rho > 0$,*

$$\lim_{\lambda\to -iq}\int_B |(T_{\bar\lambda}T_\lambda(F))(\rho y) - F(\rho y)|^p dm(y) = 0,$$

(ii) $T_{\bar\lambda}T_\lambda \to F$ *s-a.e, as $\lambda \to -iq$ through $\mathbb{C}_+$,where $\bar\lambda$ is the complex conjugate of λ .*

Note that in the case of $p = 2$, $p' = 2$, and so for $F \in \mathcal{F}(n;2)$, $T_q^{(2)}(F) \in \mathcal{F}(n;2)$ by Theorem 2.4. And hence we have the following.

COROLLARY 2.6. *Let $F \in \mathcal{F}(n;2)$ be given by (1.6). Then for non-zero real q,*

$$T_{-q}^{(2)}(T_q^{(2)}(F)) \approx F\ .$$

3. Convolutions and Transforms of convolutions

In this section, we define a convolution product for functionals on abstract Wiener space and show that the L_p analytic Fourier-Feynman transform of the convolution product is a product of L_p analytic Fourier -Feynman transforms.

Let F_1 and F_2 be functionals defined on B. The convolution product of F_1 and F_2 is defined by

$$(F_1 * F_2)_\lambda(y) = \begin{cases} I^{aw}(F_1(\frac{y+\cdot}{\sqrt{2}})F_2(\frac{y-\cdot}{\sqrt{2}});\lambda) & , \ \lambda \in \mathbb{C}_+ \\ I^{af}(F_1(\frac{y+\cdot}{\sqrt{2}})F_2(\frac{y-\cdot}{\sqrt{2}});q) & , \ \lambda = -iq, q \in \mathbb{R} - \{0\} \end{cases}$$

if it exists.

THEOREM 3.1. *Let (H, B, m) be an abstract Wiener space and let $F_1 \in \mathcal{F}(n;1)$ and $F_2 \in \mathcal{F}(n;p)$ for $1 \leq p < \infty$. Then for $\lambda \in \mathbb{C}_+$, the convolution product $(F_1 * F_2)_\lambda$ belongs to $\mathcal{F}(n;p)$ and is given by*

$$(F_1 * F_2)_\lambda(y) = H_\lambda((h_1, y)^\sim, \cdots, (h_n, y)^\sim), \tag{3.1}$$

where

$$\begin{aligned} &H_\lambda(w_1, \cdots, w_n) \\ =&(\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f_1(\frac{\vec{w}+\vec{u}}{\sqrt{2}}) f_2(\frac{\vec{w}-\vec{u}}{\sqrt{2}}) \exp\{-\frac{\lambda}{2}\sum_{j=1}^n u_j^2\}\, d\vec{u}. \end{aligned} \tag{3.2}$$

Moreover, for $p = 1$,

$$||H_\lambda||_1 \leq |\frac{\lambda}{2\pi}|^{\frac{n}{2}} ||f_1||_1 ||f_2||_1, \tag{3.3}$$

and, for $1 < p < \infty$,

$$||H_\lambda||_p \leq |\frac{\lambda}{\pi}|^{\frac{n}{2}} ||f_1||_1 ||f_2||_p. \tag{3.4}$$

PROOF. Since $(h_1,x)^\sim,\cdots,(h_n,x)^\sim$ are independent Gaussian functionals with mean zero and variance one, we have that for $\lambda>0$,

$$\begin{aligned}(F_1*F_2)_\lambda(y) &= \int_B F_1\Big(\frac{y+\lambda^{-\frac12}x}{\sqrt2}\Big)F_2\Big(\frac{y-\lambda^{-\frac12}x}{\sqrt2}\Big)dm(x)\\ &=\Big(\frac{\lambda}{2\pi}\Big)^{\frac n2}\int_{\mathbb{R}^n} f_1\Big(\frac{1}{\sqrt2}((h_1,y)^\sim+u_1),\ldots,\frac{1}{\sqrt2}((h_n,y)^\sim+u_n)\Big)\\ &\cdot f_2\Big(\frac{1}{\sqrt2}((h_1,y)^\sim-u_1),\ldots,\frac{1}{\sqrt2}((h_n,y)^\sim-u_n)\Big)\exp\Big[-\frac{\lambda}{2}\sum_{j=1}^n u_j^2\Big]d\vec{u}\\ &=H_\lambda((h_1,y)^\sim,\cdots,(h_n,y)^\sim),\end{aligned}$$

where H_λ is given by (3.2). Now by analytic continuation in λ, we see that (3.1) holds for all $\lambda\in\mathbb{C}_+$.

Let $p=1$. Using the following transformation

$$\frac{1}{\sqrt2}(\vec{w}+\vec{u})=\vec{v},\ \frac{1}{\sqrt2}(\vec{w}-\vec{u})=\vec{r}, \tag{3.5}$$

we have

$$\begin{aligned}&\int_{\mathbb{R}^n}|H_\lambda(\vec{w})|\,d\vec{w}\\ &\le\Big|\frac{\lambda}{2\pi}\Big|^{\frac n2}\int_{\mathbb{R}^{2n}}\Big|f_1\Big(\frac{\vec{w}+\vec{u}}{\sqrt2}\Big)\Big|\,\Big|f_2\Big(\frac{\vec{w}-\vec{u}}{\sqrt2}\Big)\Big|\,d\vec{u}d\vec{w}\\ &=\Big|\frac{\lambda}{2\pi}\Big|^{\frac n2}\int_{\mathbb{R}^n}|f_1(\vec{v})|\,d\vec{v}\int_{\mathbb{R}^n}|f_2(\vec{r})|\,d\vec{r}=\Big|\frac{\lambda}{2\pi}\Big|^{\frac n2}||f_1||_1\cdot||f_2||_1.\end{aligned}$$

Therefore the convolution product $(F_1*F_2)_\lambda$ belongs to $\mathcal{F}(n;1)$.

Now suppose that $1<p<\infty$. If p' is the conjugate exponent to p, (i.e, $\frac1p+\frac1{p'}=1$), then, by using the Hölder inequality and the transformation (3.5),

$$\begin{aligned}&\int_{\mathbb{R}^n}|H_\lambda(\vec{w})|^p\,d\vec{w}\\ &\le\Big\{\Big|\frac{\lambda}{2\pi}\Big|^{\frac n2}\Big\}^p\int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^n}\Big|f_1\Big(\frac{\vec{w}+\vec{u}}{\sqrt2}\Big)\Big|\cdot\Big|f_2\Big(\frac{\vec{w}-\vec{u}}{\sqrt2}\Big)\Big|\,d\vec{u}\Big]^p\,d\vec{w}\\ &=\Big\{\Big|\frac{\lambda}{2\pi}\Big|^{\frac n2}\Big\}^p\int_{\mathbb{R}^n}\Big[\int_{\mathbb{R}^n}\Big|f_1\Big(\frac{\vec{w}+\vec{u}}{\sqrt2}\Big)\Big|^{\frac1{p'}}\cdot\Big|f_1\Big(\frac{\vec{w}+\vec{u}}{\sqrt2}\Big)\Big|^{\frac1p}\cdot\Big|f_2\Big(\frac{\vec{w}-\vec{u}}{\sqrt2}\Big)\Big|\,d\vec{u}\Big]^p\,d\vec{w}\end{aligned}$$

$$\leq \{|\frac{\lambda}{2\pi}|^{\frac{n}{2}}\}^p \int_{\mathbb{R}^n} \{[\int_{\mathbb{R}^n} |f_1(\frac{\vec{w}+\vec{u}}{\sqrt{2}})|\, d\vec{u}]^{\frac{1}{p'}} \cdot [\int_{\mathbb{R}^n} |f_1(\frac{\vec{w}+\vec{u}}{\sqrt{2}})| \cdot$$
$$\cdot |f_2(\frac{\vec{w}-\vec{u}}{\sqrt{2}})|^p\, d\vec{u}]^{\frac{1}{p}}\}^p\, d\vec{w}$$
$$\leq \{|\frac{\lambda}{2\pi}|^{\frac{n}{2}}\}^p \cdot ((\sqrt{2})^n ||f_1||_1)^{\frac{p}{p'}} \cdot \int_{\mathbb{R}^{2n}} |f_1(\vec{v})| \cdot |f_2(\vec{r})|^p\, d\vec{v} d\vec{r}$$
$$= (\sqrt{2})^{-n} |\frac{\lambda}{\pi}|^{\frac{np}{2}} ||f_1||_1^{\frac{p}{p'}+1} ||f_2||_p^p.$$

Taking the p-th root in the both sides of the above inequality , we have

$$||H_\lambda||_p \leq (\sqrt{2})^{-\frac{n}{p}} |\frac{\lambda}{\pi}|^{\frac{n}{2}} ||f_1||_1 ||f_2||_p \leq |\frac{\lambda}{\pi}|^{\frac{n}{2}} ||f_1||_1 ||f_2||_p.$$

Therefore the convolution product $(F_1 * F_2)_\lambda$ belongs to $\mathcal{F}(n;p)$. □

COROLLARY 3.2. *Let (H,B,m) be an abstract Wiener space and let* $F_1 \in \mathcal{F}(n;1)$ *and* $F_2 \in \mathcal{F}(n;1) \bigcap \mathcal{F}(n;p)$ *for* $1 \leqq p < \infty$. *Then the convolution product* $(F_1 * F_2)_\lambda$ *belongs to* $\mathcal{F}(n;1) \bigcap \mathcal{F}(n;p)$ *for all* $\lambda \in \mathbb{C}_+$.

THEOREM 3.3. *Let (H,B,m) be an abstract Wiener space and let* $F_k \in \bigcup_{1\leq p\leq\infty} \mathcal{F}(n:p)$ *for* $k = 1,2$ *be given by (1.6). Then for all* $\lambda \in \mathbb{C}_+$,

$$(3.6) \qquad (T_\lambda(F_1 * F_2)_\lambda)(z) = (T_\lambda(F_1))(\frac{z}{\sqrt{2}})(T_\lambda(F_2))(\frac{z}{\sqrt{2}}).$$

PROOF. Since $(h_1,x)^\sim, \cdots, (h_n,x)^\sim$ are independent Gaussian functionals with mean zero and variance one, we have that for $\lambda > 0$,

$$(T_\lambda(F_1 * F_2)_\lambda)(z)$$
$$= \int_B (F_1 * F_2)_\lambda(\lambda^{-\frac{1}{2}}x + z)dm(x)$$
$$= \int_B H_\lambda((h_1, \lambda^{-\frac{1}{2}}x + z)^\sim, ..., (h_n, \lambda^{-\frac{1}{2}}x + z)^\sim)dm(x)$$
$$=(\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} H_\lambda(v_1 + (h_1,z)^\sim, ..., v_n + (h_n,z)^\sim) exp[-\frac{\lambda}{2}\sum_{j=1}^n v_j^2] d\vec{v}$$

$$=(\frac{\lambda}{2\pi})^n \int_{\mathbb{R}^{2n}} f_1(\frac{1}{\sqrt{2}}(v_1+u_1+(h_1,z)^{\sim}),...,\frac{1}{\sqrt{2}}(v_n+u_n+(h_n,z)^{\sim}))$$
$$\cdot f_2(\frac{1}{\sqrt{2}}(v_1-u_1+(h_1,z)^{\sim}),...,\frac{1}{\sqrt{2}}(v_n-u_n+(h_n,z)^{\sim}))$$
$$exp[-\frac{\lambda}{2}\sum_{j=1}^{n}(u_j^2+v_j^2)]d\vec{u}d\vec{v}.$$

Using the transformation (3.5), we obtain that for $\lambda > 0$,

$$(T_\lambda(F_1 * F_2)_\lambda)(z)$$
$$=(\frac{\lambda}{2\pi})^n \int_{\mathbb{R}^{2n}} f_1(w_1+\frac{1}{\sqrt{2}}(h_1,z)^{\sim},...,w_n+\frac{1}{\sqrt{2}}(h_n,z)^{\sim})\exp[-\frac{\lambda}{2}\sum_{j=1}^{n}$$
$$w_j^2]\cdot f_2(r_1+\frac{1}{\sqrt{2}}(h_1,z)^{\sim},...,r_n+\frac{1}{\sqrt{2}}(h_n,z)^{\sim})\exp[-\frac{\lambda}{2}\sum_{j=1}^{n}r_j^2]d\vec{w}d\vec{r}$$
$$=(\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f_1(\vec{w})\exp[-\frac{\lambda}{2}\sum_{j=1}^{n}(w_j-\frac{1}{\sqrt{2}}(h_j,z)^{\sim})^2]d\vec{w}$$
$$\cdot(\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f_2(\vec{r})\exp[-\frac{\lambda}{2}\sum_{j=1}^{n}(r_j-\frac{1}{\sqrt{2}}(h_j,z)^{\sim})^2]d\vec{r}$$
$$=(T_\lambda(F_1))(\frac{z}{\sqrt{2}})(T_\lambda(F_2))(\frac{z}{\sqrt{2}}).$$

By analytic continuation in λ, (3.6) holds through $\mathbb{C}_+$. □

THEOREM 3.4. *Let (H,B,m) be an abstract Wiener space and let $F_1 \in \mathcal{F}(n;1)$ and $F_2 \in \mathcal{F}(n;p)$ for $1 \leqslant p \leqslant 2$. Then the analytic Wiener integral $(T_\lambda(F_1 * F_2)_\lambda)(y)$ belongs to $\mathcal{F}(n;\infty)$ for $p=1$ and it belongs to $\mathcal{F}(n;p')$ for $1 < p \leq 2$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $\lambda \in \mathbb{C}_+$. Moreover, for $p=1$,*

$$\|K_\lambda\|_\infty \leqslant |\frac{\lambda}{2\pi}|^n\|f_1\|_1\|f_2\|_1, \tag{3.7}$$

and, for $1 < p \leqslant 2$,

$$\|K_\lambda\|_{p'} \leqslant |\frac{\lambda}{\pi}|^{\frac{n}{p}} \|f_1\|_1 \|f_2\|_p \ , \tag{3.8}$$

where K_λ *is given by*

$$K_\lambda((h_1, z)^\sim, ..., (h_n, z)^\sim) = (T_\lambda(F_1 * F_2)_\lambda)(z). \tag{3.9}$$

PROOF. In the case of $p = 1$, $(F_1 * F_2)_\lambda(y)$ belongs to $\mathcal{F}(n;1)$ by Theorem 3.1 , and so by Corollary 2.2, $(T_\lambda(F_1 * F_2)_\lambda)(y)$ belongs to $\mathcal{F}(n;\infty)$. And also, using Theorem 2.1 and 3.3, we have

$$\begin{aligned}
&K_\lambda((h_1, z)^\sim, ..., (h_n, z)^\sim)\\
=&(\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f_1(\vec{w}) \exp[-\frac{\lambda}{2}\sum_{j=1}^{n}(w_j - \frac{1}{\sqrt{2}}(h_j, z)^\sim)^2] d\vec{w}\\
&\cdot (\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f_2(\vec{r}) \exp[-\frac{\lambda}{2}\sum_{j=1}^{n}(r_j - \frac{1}{\sqrt{2}}(h_j, z)^\sim)^2] d\vec{r}\\
=&(G_\lambda f_1)(\frac{1}{\sqrt{2}}(h_1, z)^\sim, ..., \frac{1}{\sqrt{2}}(h_n, z)^\sim)\\
&\cdot (G_\lambda f_2)(\frac{1}{\sqrt{2}}(h_1, z)^\sim, ..., \frac{1}{\sqrt{2}}(h_n, z)^\sim).
\end{aligned}$$

Thus from Corollary 2.2, it follows that

$$\begin{aligned}
|K_\lambda(\vec{w})| &= |(G_\lambda f_1)(\frac{1}{\sqrt{2}}\vec{w})||(G_\lambda f_2)(\frac{1}{\sqrt{2}}\vec{w})|\\
&\leqslant |\frac{\lambda}{2\pi}|^{\frac{n}{2}} \|f_1\|_1 |\frac{\lambda}{2\pi}|^{\frac{n}{2}} \|f_2\|_1 = |\frac{\lambda}{2\pi}|^{n} \|f_1\|_1 \|f_2\|_1.
\end{aligned}$$

Let $1 < p \leqslant 2$. Using Theorem 3.1 and Corollary 2.2, we know that $(F_1 * F_2)_\lambda(y)$ belongs to $\mathcal{F}(n;p)$ and so $(T_\lambda(F_1 * F_2)_\lambda)(y)$ belongs to $\mathcal{F}(n;p')$. And also by Theorem 3.3, we have

$$
\begin{aligned}
||K_\lambda||_{p'}^{p'} &\equiv \int_{\mathbb{R}^n} |K_\lambda(\vec{r})|^{p'} d\vec{r} \\
&= \int_{\mathbb{R}^n} |(G_\lambda f_1)(\frac{\vec{r}}{\sqrt{2}})|^{p'} |(G_\lambda f_2)(\frac{\vec{r}}{\sqrt{2}})|^{p'} d\vec{r} \\
&\leq ||(G_\lambda f_1))||_\infty^{p'} \int_{\mathbb{R}^n} |(G_\lambda f_2)(\frac{\vec{r}}{\sqrt{2}})|^{p'} d\vec{r} \\
&\leq [|\frac{\lambda}{2\pi}|^{\frac{n}{2}} ||f_1||_1]^{p'} ||G_\lambda f_2||_{p'}^{p'} (\sqrt{2})^n \\
&\leq [|\frac{\lambda}{2\pi}|^{\frac{n}{2}} ||f_1||_1]^{p'} [|\frac{\lambda}{2\pi}|^{n(\frac{1}{2}-\frac{1}{p'})} ||f_2||_p]^{p'} (\sqrt{2})^n \\
&= [|\frac{\lambda}{2\pi}|^{\frac{n}{2}} ||f_1||_1 |\frac{\lambda}{2\pi}|^{n(\frac{1}{2}-\frac{1}{p'})} ||f_2||_p (\sqrt{2})^{\frac{n}{p'}}]^{p'} \\
&= [|\frac{\lambda}{\pi}|^{\frac{n}{p}} ||f_1||_1 ||f_2||_p \cdot 2^{n(-1+\frac{3}{2p'})}]^{p'} \leq [|\frac{\lambda}{\pi}|^{\frac{n}{p}} ||f_1||_1 ||f_2||_p]^{p'},
\end{aligned}
$$

because $\frac{3}{2p'} - 1 = \frac{1}{2} - \frac{3}{2p} \leq -\frac{1}{4}$ for $1 < p \leq 2$ and $G_\lambda f_1 \in C_o(\mathbb{R}^n)$ and $G_\lambda f_2 \in L_{p'}(\mathbb{R}^n)$. Taking the p'-th root in the both sides of the above inequality, we establish our inequality (3.8). □

Next we show that the L_p analytic Fourier Feynman transform of the convolution product is a product of L_p analytic Fourier-Feynman transforms on an abstract Wiener space.

THEOREM 3.5. *Let (H,B,m) be an abstract Wiener space and let $F_1 \in \mathcal{F}(n;1)$ and $F_2 \in \mathcal{F}(n;p)$ for $1 \leqq p \leqq 2$. Then for non-zero real q,*

$$
(3.10) \qquad (T_q^{(p)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(\frac{z}{\sqrt{2}})(T_q^{(p)}(F_2))(\frac{z}{\sqrt{2}}).
$$

PROOF. From Theorem 3.1 and 2.4, it follows that all of the transforms on both sides of (3.10) exist. And hence (3.10) holds immediately by Theorem 3.4. □

COROLLARY 3.6. *Let (H,B,m) be an abstract Wiener space and let* $F_1 \in \mathcal{F}(n;1)$ *and* $F_2 \in \mathcal{F}(n;1) \cap \mathcal{F}(n;p)$ *for* $1 \leq p \leq 2$. *Then for non-zero real* q,

$$(3.11) \qquad (T_q^{(p)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(\frac{z}{\sqrt{2}})(T_q^{(p)}(F_2))(\frac{z}{\sqrt{2}}).$$

REMARK 3.7. Let $C_o[0,T]$ be the Banach space of continuous functions x on $[0,T]$ which vanish at 0 with the uniform norm. Let $(C_o[0,T], \mathcal{B}(C_o[0,T]), m_o)$ denote the classical Wiener space where m_o is the Wiener measure on the Borel σ-algebra $\mathcal{B}(C_o[0,T])$ of $C_o[0,T]$, and let H_o be the space of absolutely continuous functions γ which vanish at 0 and whose derivative $D\gamma$ is in $L_2[0,T]$. The inner product on H_o is given by

$$< \gamma, \beta >= \int_0^T (D\gamma)(s)(D\beta)(s)ds.$$

Then H_o is a real separable Hilbert space and $(H_o, C_o[0,T], m_o)$ is an example of an abstract Wiener space.

(a) Let $0 = s_o < s_1 < \cdots < s_n = T$ be a partition of $[0,T]$ and let $h_j(s) = \int_0^s \chi_{[0,t_j]}(t)dt$ for $j = 1, \cdots, n$. Then $\{h_1, \cdots, h_n\}$ is clearly a linearly independent set in H_o and for $x \in C_o[0,T]$, $(h_j, x)^\sim = x(t_j)$ for $j = 1, \cdots, n$. In this case, Theorem 1.1, 1.2 and 1.3 in [11] are corollaries of our results in Section 2.

(b) Let $\{h_j\}$ be a C.O.N. set in H_o. Then $\{Dh_j\}$ is also a C.O.N. set in $L_2[0,T]$ and $(h_j, x)^\sim = \int_0^T (Dh_j)(s)\tilde{d}x(s)$ for s-a.e. $x \in C_o[0,T]$. In this case, most results in [9] are corollaries of ours in Section 2 and 3.

Throughout this paper, we assume that $\{h_1, \cdots, h_n\}$ is an orthonormal set in H. However, all of our results hold provided that $\{h_1, \cdots, h_n\}$ is a linearly independent set in H .

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