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NORMS FOR SCHUR PRODUCTS

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ABSTRACT. We first show that if $\psi : M_n(B(H)) \to M_n(B(H))$ is a $D_n \otimes F(H)$-bimodule map, then there is a matrix $A \in M_n$ such that $\psi = S_A$. Secondly, we show that for an operator space $\mathcal{E}$, $A \in M_n$, the Schur product map $S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E})$ and $\phi_A : M_n(\mathcal{E}) \to \mathcal{E}$, defined by $\phi_A([x_{ij}]) = \sum_{i,j=1}^n a_{ij}x_{ij}$, we have $\|S_A\| = \|S_A\|_{cb} = \|A\|_S$, $\|\phi_A\| = \|\phi_A\|_{cb} = \|A\|_1$ and obtain some characterizations of A for which S_A is contractive.

1. Introduction

Schur products on M_n have been studied in several areas. In particular, Paulsen, Power and Smith [4] proves that for $A \in M_n$, a Hilbert space H and the Schur product map $S_A : M_n \to M_n(B(H))$, $\|S_A\| = \|S_A\|_{cb}$ and obtains a characterization of A for which S_A is contractive.

In this paper, we first show that if $\psi : M_n(B(H)) \to M_n(B(H))$ is a $D_n \otimes F(H)$-bimodule map, then there is a matrix $A \in M_n$ such that $\psi = S_A$. Secondly, we show that for an operator space $\mathcal{E}$, $A \in M_n$, the Schur product map $S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E})$ and $\phi_A : M_n(\mathcal{E}) \to \mathcal{E}$, we have $\|S_A\| = \|S_A\|_{cb} = \|A\|_S$, $\|\phi_A\| = \|\phi_A\|_{cb} = \|A\|_1$, where $\|A\|_1$ is the trace of $|A| = (A^*A)^{\frac{1}{2}}$, and obtain some characterizations of A for which S_A is contractive.

2. Main Results

An operator space is a subspace of $B(H)$ for some Hilbert space and an operator system is a self-adjoint subspace of $B(H)$ containing the

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identity.

For an operator space $\mathcal{E} \subseteq B(H)$ we identify $M_n \otimes \mathcal{E}$ with $M_n(\mathcal{E})$ which is a subspace of $B(H^n)$ where H^n is the n-fold direct sum of copies of H.

If $A = [a_{ij}]$, $B = [b_{ij}]$ are elements of M_n or $M_n(B(H))$, then we denote the Schur product by $A \circ B = [a_{ij}b_{ij}]$. For $A = [a_{ij}] \in M_n$ and an operator space $\mathcal{E}$, let $S_A^{\mathcal{E}} : M_n(\mathcal{E}) \to M_n(\mathcal{E})$ be the Schur product map defined by $S_A^{\mathcal{E}}(x) = A \circ x$, let $\phi_A^{\mathcal{E}} : M_n(\mathcal{E}) \to \mathcal{E}$ be the map defined by $\phi_A^{\mathcal{E}}([x_{ij}]) = \sum_{i,j=1}^n a_{ij}x_{ij}$, and let $\|A\|_S$ denote the norm of the operators on M_n corresponding to Schur multiplication by A. When there is no danger of confusion we let ϕ_A and S_A denote $\phi_A^{\mathcal{E}}$ and $S_A^{\mathcal{E}}$ respectively.

If $\mathcal{E}, \mathcal{F}$ are operator spaces and $\varphi : \mathcal{E} \to \mathcal{F}$ is a linear map, then we can define the linear maps

$$\varphi_n : M_n(\mathcal{E}) \to M_n(\mathcal{F}) \ via \ \varphi_n([x_{ij}]) = [\varphi(x_{ij})].$$

The map φ is called contractive if $\|\varphi\| \leq 1$, completely bounded if $\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in N\}$ is finite and completely contractive if $\|\varphi\|_{cb} \leq 1$.

In the case that $\mathcal{E}, \mathcal{F}$ are operator systems, the map φ is called positive if $\varphi(x)$ is positive for every positive x in $\mathcal{E}$, and completely positive if φ_n is positive for every n.

Let $\{e_{ij}\}_{i,j=1}^n$ be the canonical matrix units for M_n, let $D(x_1, \cdots, x_n)$ be the diagonal operator matrix in $M_n(B(H))$, and let $A = [a_{ij}], B = [b_{ij}], C$ in $M_n(B(H))$ be operator matrices with muturally comutting entries. Then by elementary calculations, we get the following Lemma.

LEMMA 1. $(AB) \circ C = \sum_{k=1}^n D(a_{1k}, \cdots, a_{nk})CD(b_{k1}, \cdots, bkn)$.

It is a well-known theorem that the Schur product of two positive matrices is positive. Using Lemma 1, we give a new elementary proof of a generalization of the above well-known theorem.

PROPOSITION 2. *Let $\mathcal{E}$ be an operator system. If $A \in M_n, B \in M_n(\mathcal{E})$ are positive, then $A \circ B$ is positive.*

PROOF. Let $A^{\frac{1}{2}} = [a_{ij}]$, $D_k = D(a_{1k}, \cdots, a_{nk})$. By Lemma 1, $A \circ B = (A^{\frac{1}{2}}A^{\frac{1}{2}}) \circ B = \sum_{k=1}^n D_k B D_k^*$. Hence $A \circ B$ is positive. □

Let $A \in M_n$ be a positive matrix and $\mathcal{E} = C$. Then S_A and ϕ_A are completely positive. The following shows that it holds for any operator system $\mathcal{E}$.

PROPOSITION 3. *Let $\mathcal{E}$ be an operator system and let $A = [a_{ij}] \in M_n$ be a matrix. For $\phi_A : M_n(\mathcal{E}) \to \mathcal{E}$, $S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E})$, the following are equivalent :*

(1) *A is positive,*
(2) *ϕ_A is positive,*
(3) *ϕ_A is completely positive,*
(4) *$\sum_{ij=1}^n a_{ij}\bar{\alpha}_i\alpha_j \geqq 0$ for any $\alpha_1, \cdots, \alpha_n \in C$,*
(5) *S_A is positive,*
(6) *S_A is completely positive.*

PROOF. (1) $\Rightarrow$ (6) Let $A_k = [A_{ij}] \in M_k(M_n)$ with $A_{ij} = A$. Then $(S_A)_k = S_{(A_k)}$. Since A is positive, A_k is positive. Hence by Proposition 2 $(S_A)_k$ is positive and S_A is completely positive.

(6) $\Rightarrow$ (3) Since $(\phi_A)_k(x) = V(S_A)_k(x)V^*$ for some $V \in M_{k,kn}$, ϕ_A is completely positive.

(2) $\Rightarrow$ (4) Let $x = (\alpha_1 I, \cdots, \alpha_n I) \in M_{1,n}(\mathcal{E})$ with $\alpha_i \in C$. Then $\phi_A(x^*x) = (\sum_{i,j=1}^n \bar{\alpha}_i\alpha_j a_{ij})I$ is positive. Hence $\sum_{ij=1}^n a_{ij}\bar{\alpha}_i\alpha_j \geqq 0$ for any $\alpha_1, \cdots, \alpha_n \in C$.

(3) $\Rightarrow$ (2), (4) $\Rightarrow$ (1), (6) $\Rightarrow$ (5) $\Rightarrow$ (1) Clear. □

Let $\mathcal{E}$ be an operator space. For $A, B, C \in M_n, x \in M_n(\mathcal{E})$, let A^t be the transpose of A, $L_A(x) = Ax, R_A(x) = xA$. Then by elementary calculations, we get the following Lemma.

LEMMA 4. *$\phi_{BAC} = \phi_A L_{B^t} R_{C^t}$. In particular, if $U, V \in M_n$ are unitaries, then $\|(\phi_{UAV})_k\| = \|(\phi_A)_k\|$ for each $k \in N$.*

For an operator space $\mathcal{E}$ and a positive matrix $A \in M_n$, S_A is completely bounded and $\|S_A\|_{cb} = \max\{a_{ii}\}_{i=1}^n$. When A is not positive, it is more difficult to calaulate $\|S_A\|$. But, using Lemma 4, we can easily calculate $\|\phi_A\|_{cb}$. Let $\|A\|_1$ denote the trace norm of the matrix A, i, e, $\|A\|_1$ is the trace of $|A| = (A^*A)^{\frac{1}{2}}$.

THEOREM 5. *Let $\mathcal{E}$ be an operator space and let $A \in M_n$ be a matrix. Then we have $\|\phi_A\| = \|\phi_A\|_{cb} = \|A\|_1$.*

PROOF. Note that there is a unitary matrix $U \in M_n$ such that $A = U|A|$. Then by Lemma 4, $\|(\phi_A)_k\| = \|(\phi_{|A|})_k\|$ for each $k \in N$. Clearly

$\|(\phi_{|A|})\| \geqq \|A\|_1$. Let $\mathcal{E} \subseteq B(H)$. Since $\phi_{|A|}^{B(H)}$ is completely positive, $\|\phi_{|A|}^{B(H)}\| = \|\phi_{|A|}^{B(H)}\|_{cb}= \|\phi_{|A|}^{B(H)}(I)\|= \|A\|_1$. Hence $\|\phi_A^{\mathcal{E}}\| = \|\phi_A^{\mathcal{E}}\|_{cb} = \|A\|_1$. □

Let A be a subalgebra of $B(H)$. A linear map $\psi : B(H) \to B(H)$ is called a A-bimodule map if $\psi(axb) = a\psi(x)b$ for all $a, b \in A$ and $x \in B(H)$. Let $F(H)$ be the set of all finite rank operators on H, let D_n be the set of all $n \times n$ diagonal matrices, and let $\{e_{ij}\}_{i,j=1}^n$ be the canonical matrix units for M_n.

THEOREM 6. *If $\psi : M_n(B(H)) \to M_n(B(H))$ is a $D_n \otimes F(H)$-bimodule map, then there is a matrix $A \in M_n$ such that $\psi = S_A$.*

PROOF. For any projection $p \in F(H)$ and fixed i, j

$$\begin{aligned}\psi(e_{ij} \otimes p) &= \psi((e_{ii} \otimes p)(e_{ij} \otimes p)(e_{jj} \otimes p)) \\ &= (e_{ii} \otimes p)\psi(e_{ij} \otimes p)(e_{jj} \otimes p)\end{aligned}$$

Hence $\psi(e_{ij}\otimes I) = e_{ij}\otimes x_{ij}$ for some $x_{ij} \in B(H)$. Since $(e_{ii}\otimes y)(e_{ij}\otimes I)= (e_{ij} \otimes I)(e_{jj} \otimes y)$ for $y \in F(H)$ and ψ is a $D_n \otimes F(H)$-bimodule map

$$\begin{aligned}e_{ij} \otimes yx_{ij} &= (e_{ii} \otimes y)(e_{ij} \otimes x_{ij}) \\ &= \psi((e_{ii} \otimes y)(e_{ij} \otimes I)) \\ &= \psi((e_{ij} \otimes I)(e_{jj} \otimes y)) \\ &= (e_{ij} \otimes x_{ij})(e_{jj} \otimes y) \\ &= e_{ij} \otimes x_{ij}y\end{aligned}$$

for any $y \in F(H)$. Hence $x_{ij} \in F(H)' = CI$ and we can put $x_{ij} = a_{ij}I$ for some $a_{ij} \in C$. Put $A = [a_{ij}] \in M_n$. Then clearly $\psi = S_A$. □

REMARK 7. Let $B(H) = M_2, \mathcal{C} = M_n\otimes I \subseteq M_n(M_2)$ or $\mathcal{C} = D_n\otimes I \subseteq M_n(M_2)$ and let $\psi : M_n(M_2) \to M_n(M_2)$ be defined by $\psi([x_{ij}]) = [x_{ij}^t]$. Then ψ is a $\mathcal{C}$-bimodule map but there is no $A \in M_n$ such that $\psi = S_A$.

COROLLARY 8. *If $\psi : M_n(B(H)) \to M_n(B(H))$ is a $D_n\otimes F(H)-$bimodule map, then ψ is also a $D_n \otimes B(H)$-bimodule map.*

PROOF. By Theorem 6, $\psi = S_A$ for some $A \in M_n$. It is trivial that $S_A^{B(H)}$ is a $D_n \otimes B(H)-$bimodule map. □

Let $\mathcal{E} \subseteq B(H)$ be an operator space and let

$$V = \left\{ \begin{bmatrix} P & S \\ T^* & Q \end{bmatrix} : P, Q \in D_n \otimes B(H), S, T \in M_n(\mathcal{E}) \right\},$$

$$W = \left\{ \begin{bmatrix} P & S \\ T^* & Q \end{bmatrix} : P, Q \in D_n \otimes I, S, T \in M_n(\mathcal{E}) \right\}$$

and let $P_A = \begin{bmatrix} I & A \\ A^* & I \end{bmatrix} \in M_{2n}$ for $A \in M_n$.

PROPOSITION 9. *Let $\mathcal{E}$ be an operator space and let $A = [a_{ij}] \in M_n$ be a matrix. For $S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E})$, the following are equivalent :*

(1) $S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E})$ *is contractive,*
(2) $S_A : M_n \to M_n$ *is contractive,*
(3) *There exist vectors $v_1, \cdots, v_n,\ w_1, \cdots, w_n$ in C^n of the norm less than or equal to 1 with $a_{ij} = (w_j\ v_i)$,*
(4) $S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E})$ *is completely contractive,*
(5) $S_A : M_n \to M_n$ *is completely contractive,*
(6) $S_{P_A} : V \to V$ *is positive,*
(7) $S_{P_A} : V \to V$ *is completely positive,*
(8) $S_{P_A} : W \to W$ *is completely positive,*
(9) $S_{P_A} : W \to W$ *is positive.*

PROOF. (1) $\Rightarrow$ (2),(4) $\Rightarrow$ (5) By [5, Proposition 2.2], $\|B \otimes x\| = \|B\|\|x\|$ for $B \in M_m, x \in \mathcal{E}$. Hence they are trivial.

(2) $\Rightarrow$ (3) [4, Theorem 3.2].

(3) $\Rightarrow$ (4) Let $P_1 = [(v_j\ v_i)], P_2 = [(w_j\ w_i)] \in M_n$. Then $\alpha = \begin{bmatrix} P_1 & A \\ A^* & P_2 \end{bmatrix} \in M_{2n}$ is positive and by Proposition 3, the map $S_\alpha : M_{2n}(\mathcal{E}) \to M_{2n}(\mathcal{E})$ is completely positive. Hence $\|S_\alpha\|_{cb} = \|S_\alpha(I)\| \leq 1$ and $\|S_A\|_{cb} \leq \|S_\alpha\|_{cb}$.

(2) $\Rightarrow$ (6), (5) $\Rightarrow$ (7,) (9) $\Rightarrow$ (1) Similar to the proof of [4, Lemma 3.1].

(6) $\Rightarrow$ (9), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) Trivial. □

REMARK 10. If $A = [a_{ij}] \in M_n$ with $a_{ij} = 1$, then $S_A : M_n(B(H)) \to M_n(B(H))$ is contractive, so $S_{P_A} : V \to V$ is positive, but $S_{P_A} : M_2(M_n(B(H))) \to M_2(M_n(B(H)))$ is not positive since P_A is not positive.

From Proposition 9, we get the following theorem.

THEOREM 11. *If $\mathcal{E}(\subseteq B(H))$ is an operator space and $A \in M_n$, then we have $\|S_A^{\mathcal{E}}\| = \|S_A^{\mathcal{E}}\|_{cb} = \|A\|_S$.*

PROOF. Clearly $\|A\|_S \leq \|S_A^{\mathcal{E}}\|$ and $\|S_A^{\mathcal{E}}\|_{cb} \leq \|S_A^{B(H)}\|_{cb}$. By Proposition 9, $\|S_A^{B(H)}\| = \|S_A^{B(H)}\|_{cb} = \|A\|_S$. Hence $\|S_A^{\mathcal{E}}\| = \|S_A^{\mathcal{E}}\|_{cb} = \|A\|_S$.
□

Let $\{e_{ij}\}_{i,j=1}^n$ be the canonical matrix units for M_n, let $c_{ij} = I - e_{ii} - e_{jj} + e_{ij} + e_{ji}$, $d_i(\lambda) = I + (\lambda - 1)e_{ii}$ for $|\lambda| = 1$, let G be the multiplicative group generated by $\{c_{ij}, d_i(\lambda) : 1 \leq i, j \leq n, |\lambda| = 1\}$ and let

$$R = \{B \in M_n : \|A\|_S = \|AB\|_S \text{ for all } A \in M_n\}$$

$$L = \{B \in M_n : \|A\|_S = \|BA\|_S \text{ for all } A \in M_n\}$$

$$LR = \{B \in M_n : \|A\|_S = \|B^*AB\|_S \text{ for all } A \in M_n\}$$

By elementary calculations we get $[(Ac_{ij}) \circ (xc_{ij})]c_{ij} = c_{ij}[(c_{ij}A) \circ (c_{ij}x)] = [Ad_i(\lambda) \circ x]d_i(\bar{\lambda}) = d_i(\bar{\lambda})[(d_i(\lambda)A) \circ x] = A \circ x$ for $A \in M_n, x \in M_n(\mathcal{E})$ where $\mathcal{E}$ is an operator space and $|\lambda| = 1$. Hence $\|c_{ij}A\|_S = \|Ac_{ij}\|_S = \|A\|_S$ and $\|Ad_i(\lambda)\|_S = \|d_i(\lambda)A\|_S = \|A\|_S$ for $A \in M_n, |\lambda| = 1$. Clearly $L \cdot L = L, R \cdot R = R, (LR) \cdot (LR) = LR$. So $G \subseteq L \cap R \cap LR$.

For $B = [b_{ij}] \in R$, $\|e_{kk}\|_S = \|e_{kk}B\|_S = \max\{|b_{k1}|, \cdots, |b_{kn}|\}$ by Lemma 1. That is, for $1 \leq k \leq n$

$$\max\{|b_{k1}|, \cdots, |b_{kn}|\} = 1 \tag{1}$$

Choose λ_{ij} with $|\lambda_{ij}| = 1, \lambda_{ij}b_{ij} = |b_{ij}|$ and put $A_k = \sum_{i=1}^n \lambda_{ki}e_{ki}$. Then $\|A_kB\|_S = \|A_k\|_S = 1$ and the $(k,\ k)$ entry of A_kB is $\sum_{i=1}^n |b_{ik}|$. Hence for $1 \leq k \leq n$

$$\sum_{i=1}^{n} |b_{ik}| \leq 1 \tag{2}$$

By (1),(2), each column and each row of B have exactly one entry whose absolute value is 1 and the others are 0. Therefore $B \in G$ and $R = G$. Similarly $L = G$.

For $B = [b_{ij}] \in LR$, $\|B^* e_{kk} B\|_S = \max\{|b_{k1}^2|, \cdots, |b_{kn}^2|\}$ by Lemma 1. Hence $\max\{|b_{k1}^2|, \cdots, |b_{kn}^2|\} = 1$, that is, $\max\{|b_{k1}|, \cdots, |b_{kn}|\} = 1$.

Since $\|B^* B\|_S = 1$ and the $(k,\ k)$ entry of $B^* B$ is $\sum_{i=1}^n |b_{ik}^2|$, $\sum_{i=1}^n |b_{ik}^2| \leq 1$. Hence each column and each row of B have exactly one entry whose absolute value is 1 and the others are 0. Therefore $B \in G$ and $LR = G$.

By the above, we get the following Proposition.

PROPOSITION 12. $L = R = LR = G$.

References

[1] E. Effros and Z.-J. Ruan, *On matricially normed spaces*, Pacific J. Math. **132** (1988), 243–264.

[2] C. K. Fong, Heydar Radjavi and Peter Rosenthal, *Norms for matrices and operators*, J. Operator Theory **18** (1987), 99–113.

[3] V. I. Paulsen, *Completely Bounded Maps and Dilations,*, Pitman Research Notes in Math. **146** Longman, London, 1986.

[4] V. I. Paulsen, S. C. Power and R. R. Smith, *Schur products and matrix completions*, J. Func. Anal. **85** (1989), 151–178.

[5] D. Y. Shin, S. G. Lee, C. H. Byun and S. O. Kim, *On matrix normed spaces*, Bull. Korean Math. Soc. **27** (1990), 103–112.

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