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A NOTE ON A FINITE TRIANGULAR OPERATOR MATRIX

EUNGIL KO

ABSTRACT. In this paper we shall characterize a finite triangular operator matrix with M-hyponormal operators on main diagonal. This shows in particular that such an operator is subscalar, i.e., is similar to the restriction to a closed invariant subspace of a scalar operator. As a corollary, we get that every algebraic operator is subscalar.

1. Introduction

In this paper we shall prove that if an operator is a finite triangular operator matrix with M-hyponormal operators on main diagonal, then such an operator is subscalar, i.e., is similar to the restriction to a closed invariant subspace of a (generalized) scalar operator (in the sense of Colojoară-Foias [1]).

Let H be a complex, separable Hilbert space and $\mathcal{L}(H)$ denote the space of all bounded linear operators on H. Recall that $T \in \mathcal{L}(H)$ is called hyponormal if $TT^* \leq T^*T$, or equivalently, if $\|T^*h\| \leq \|Th\|$ for every $h \in H$. A larger class of operators related to hyponormals is the following: $T \in \mathcal{L}(H)$ is called M-hyponormal if there exists a constant $M > 0$ such that $\|(T-z)^*h\| \leq M\|(T-z)h\|$ for all $h \in H$ and all $z \in \mathbf{C}$. There are classical examples of M-hyponormal, non-hyponormal operators, see [4].

A bounded linear operator S on H is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous

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unital morphism of topological algebras

$$\Phi : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(H),$$

such that $\Phi(z) = S$, where z stands for the identity function on $\mathbf{C}$, and $C_0^m(\mathbf{C})$ stands for the space of compactly supported functions on $\mathbf{C}$, continuously differntiable of order m, $0 \leq m \leq \infty$. An operator is subscalar if it is the restriction of a scalar operator to an invariant subspace.

An operator $T \in \mathcal{L}(H)$ is said to satisfy the single valued extension property if for any open subset U in $\mathbf{C}$, the function

$$z - T : \mathcal{O}(U, H) \longrightarrow \mathcal{O}(U, H)$$

defined by the obvious pointwise multiplication is one-to-one where $\mathcal{O}(U, H)$ denotes the space of H-valued analytic functions on U. If, in addition, the above function $z - T$ has closed range on $\mathcal{O}(U, H)$, then T satisfies the Bishop's condition (β). The operators with a rich functional calculus, e.g. the scalar operators, as well as, their restrictions to invariant subspaces have property (β) (see [3], Introduction).

This paper is divided into three sections. Section two deals with some preliminary facts. In section three, we shall prove our main result.

2. Preliminaries

Let z be the coordinate in the complex plane $\mathbf{C}$ and $d\mu(z)$ denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space H and a bounded (connected) open subset U of $\mathbf{C}$. We shall denote by $L^2(U, H)$ the Hilbert space of measurable functions $f : U \rightarrow H$, such that

$$\|f\|_{2,U} = \{\int_U \|f(z)\|^2 d\mu(z)\}^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(U, H)$ which are analytic on U (i.e. $\bar{\partial} f = 0$) is denoted by

$$A^2(U, H) = L^2(U, H) \cap \mathcal{O}(U, H).$$

$A^2(U, H)$ is called the Bergman space for U. Note that $A^2(U, H)$ is complete (i.e. $A^2(U, H)$ is a Hilbert space). We denote by P the orthogonal projection of $L^2(U, H)$ onto $A^2(U, H)$.

Let us define now a special Sobolev type space. Let U be again a bounded open subset of $\mathbf{C}$ and m be a fixed non-negative integer. The vector valued Sobolev space $W^m(U,H)$ with respect to $\bar{\partial}$ and of order m will be the space of those functions $f \in L^2(U,H)$ whose derivatives $\bar{\partial} f, \cdots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U,H)$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U}.$$

$W^m(U,H)$ becomes a Hilbert space contained continuously in $L^2(U,H)$.

Let U be a (connected) bounded open subset of $\mathbf{C}$ and let m be a non-negative integer. The linear operator S of multiplication by z on $W^m(U,H)$ is continuous and it has a spectral distribution of order m, defined by the functional calculus

$$\Phi_S : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(W^m(U,H)), \ \ \Phi_S(f) = S_f.$$

Therefore, S is a scalar operator of order m.

3. Main result

The starting point of this section deals with the basic inequality for the proof of the main result.

LEMMA 1. ([3], Proposition 2.1) *For a bounded open disk D in the complex plane $\mathbf{C}$ there is a constant C_D, such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D,H)$ we have*

$$\|(I-P)f\|_{2,D} \le C_D(\|(z-T)^*\bar{\partial} f\|_{2,D} + \|(z-T)^*\bar{\partial}^2 f\|_{2,D})$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space $A^2(D,H)$ and z denotes zI.

LEMMA 2. *Let D and C_D be as in Lemma 1. If $T \in \mathcal{L}(H)$ is M-hyponormal and $f \in W^{2m}(D,H)$, then for $i = 0, 1, \cdots, 2m-2$,*

$$\|(I-P)\bar{\partial}^i f\|_{2,D} \le MC_D(\|(z-T)\bar{\partial}^{i+1} f\|_{2,D} + \|(z-T)\bar{\partial}^{i+2} f\|_{2,D})$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto $A^2(D,H)$.

PROOF. This follows from Lemma 1 and $\|(z-T)^*h\| \le M\|(z-T)h\|$. □

LEMMA 3. *Let $T \in \mathcal{L}(H)$ be M-hyponormal and let D be a bounded disk which contains $\sigma(T)$. Then the operator $V : H \to H(D)$ defined by $Vh = 1 \otimes h + \overline{(z-T)W^{2m}(D,H)}$ is one-to-one and has closed range where $H(D) = W^{2m}(D,H)/\overline{(z-T)W^{2m}(D,H)}$ for $m = 1, 2, \cdots$ and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h.*

PROOF. Let $h_i \in H$ and $f_i \in W^{2m}(D,H)$ be sequences such that

$$\lim_{i\to\infty} \|(z-T)f_i + 1 \otimes h_i\|_{W^{2m}} = 0. \tag{1}$$

Then by the definition of the norm of Sobolev space (1) implies

$$\lim_{i\to\infty} (\|(z-T)\bar{\partial} f_i\|_{2,D} + \|(z-T)\bar{\partial}^2 f_i\|_{2,D}) = 0.$$

Since T is M-hyponormal, Lemma 1 implies

$$\lim_{i\to\infty} \|(I-P)f_i\|_{2,D} = 0$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto $A^2(D,H)$. Then by (1),

$$\lim_{i\to\infty} \|(z-T)Pf_i + 1 \otimes h_i\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{i\to\infty} \|Pf_i(z) + (z-T)^{-1}(1 \otimes h_i)\| = 0$$

uniformly. Hence, by Riesz functional calculus,

$$\lim_{i\to\infty} \|\frac{1}{2\pi i}\int_\Gamma Pf_i(z)dz + h_i\| = 0.$$

But $\int_\Gamma Pf_i(z)dz = 0$. Hence, $\lim_{i\to\infty} h_i = 0$. □

At this point, we set forth some notations.

Notation. For $t = 1, 2, \cdots, n$,

$$T(t) = \begin{pmatrix} T_{11} & \cdots & \cdots & T_{1t} & 0 & \cdots & 0 \\ 0 & T_{22} & \cdots & T_{2t} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & T_{tt} & 0 & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

where T_{ii} are M-hyponormal for $i = 1, \cdots, t$.

LEMMA 4. *Let* $\oplus_1^n h_i^k \in \oplus_1^n H$ *and* $\oplus_1^n f_i^k \in \oplus_1^n W^{2n}(D, H)$ *be sequences (in* k*) such that*

(2) $$\lim_{k\to\infty} \|(z - T(n)) \oplus_1^n f_i^k + \oplus_1^n (1 \otimes h_i^k)\|_{\oplus_1^n W^{2n}} = 0.$$

Then

$$\lim_{k\to\infty} \|(z - T(t))(f_1^k, \cdots, f_t^k, 0, \cdots, 0)^T + (1 \otimes h_1^k, \cdots, 1 \otimes h_t^k, 0, \cdots, 0)^T\|_{\oplus_1^n W^{2t}} = 0$$

for $t = 1, 2, \cdots, n$, *where* A^T *means the transpose of* A.

PROOF. We prove Lemma 4 by induction. We assume that Lemma 4 holds for some given $t = 2, \cdots, n$. In fact, for $t = 2, \cdots, n$,

$$\begin{cases} \lim_{k\to\infty} \|(z - T_{11})f_1^k - T_{12}f_2^k - \cdots - T_{1t}f_t^k + 1 \otimes h_1^k\|_{W^{2t}} = 0 & (3,1) \\ \vdots \\ \lim_{k\to\infty} \|(z - T_{jj})f_j^k - T_{jj+1}f_{j+1}^k - \cdots - T_{jt}f_t^k + 1 \otimes h_j^k\|_{W^{2t}} = 0 & (3,j) \\ \vdots \\ \lim_{k\to\infty} \|(z - T_{tt})f_t^k + 1 \otimes h_t^k\|_{W^{2t}} = 0 & (3,t) \end{cases}$$

We only need to verify that

$$\begin{cases} \lim_{k\to\infty} \|(z - T_{11})f_1^k - T_{12}f_2^k - \cdots - T_{1t-1}f_{t-1}^k + 1 \otimes h_1^k\|_{W^{2(t-1)}} = 0 \\ \vdots \\ \vdots \\ \lim_{k\to\infty} \|(z - T_{t-1t-1})f_{t-1}^k + 1 \otimes h_{t-1}^k\|_{W^{2(t-1)}} = 0 \end{cases}$$

However, one can easily see that this result follows directly from $(3, 1), \cdots, (3, t)$ provided $\lim_{k\to\infty} \|\bar{\partial}^i f_t^k\|_{2,D} = 0$ for $i = 0, 1, \cdots, 2(t-1)$. So this will be shown to be true.

The proof of Lemma 3 with $T = T_{tt}$ shows that

(4) $$\lim_{k\to\infty} h_t^k = 0.$$

By (4) and the equation (3,t),

$$\lim_{k\to\infty} \|(z - T_{tt})f_t^k\|_{W^{2t}} = 0.$$

Then we can apply Lemma 2 with $T = T_{tt}$. Therefore, for $i = 0, 1, \cdots, 2t-2$,

$$\|(I-P)\bar{\partial}^i f_t^k\|_{2,D} \leq MC_D(\|(z-T_{tt})\bar{\partial}^{i+1} f_t^k\|_{2,D} + \|(z-T_{tt})\bar{\partial}^{i+2} f_t^k\|_{2,D})$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto $A^2(D,H)$. Hence from the equation (3,t),

$$\lim_{k\to\infty} \|(I-P)\bar{\partial}^i f_t^k\|_{2,D} = 0 \tag{5}$$

for $i = 0, 1, \cdots, 2(t-1)$. By (5),

$$\lim_{k\to\infty} \|(z-T_{tt})\bar{\partial}^i f_t^k - (z-T_{tt})P\bar{\partial}^i f_t^k\|_{2,D} = 0$$

for $i = 0, 1, \cdots, 2(t-1)$. Since $\lim_{k\to\infty} \|(z-T_{tt})f_t^k\|_{W^{2t}} = 0$ by (4) and the equation (3,t),

$$\lim_{k\to\infty} \|(z-T_{tt})P\bar{\partial}^i f_t^k\|_{2,D} = 0$$

for $i = 0, 1 \cdots, 2(t-1)$. Since every M-hyponormal operator has property (β), for $i = 0, 1, \cdots, 2(t-1)$, $P\bar{\partial}^i f_t^k \to 0$ uniformly on compact subsets of D.

Consider $\sigma(T) \subset B(0,r) \subset \overline{B(0,r)} \subset D$. For $i = 0, 1, \cdots, 2(t-1)$,

$$\begin{aligned}\|P\bar{\partial}^i f_t^k\|_{2,D}^2 &= \int_D \|P\bar{\partial}^i f_t^k(z)\|^2 d\mu(z) \\ &= \int_{\overline{B(0,r)}} \|P\bar{\partial}^i f_t^k(z)\|^2 d\mu(z) + \int_{D\setminus\overline{B(0,r)}} \|P\bar{\partial}^i f_t^k(z)\|^2 d\mu(z).\end{aligned}$$

By property (β), the first integral converges to 0. Since $z - T$ is invertible on $D \setminus \overline{B(0,r)}$, the second integral also converges to 0. Therefore,

$$\lim_{k\to\infty} \|P\bar{\partial}^i f_t^k\|_{2,D} = 0.$$

From (5), we get $\lim_{k\to\infty} \|\bar{\partial}^i f_t^k\|_{2,D} = 0$ for $i = 0, 1, \cdots, 2(t-1)$. So this completes the proof of Lemma 4. □

Now we state and prove the main theorem.

THEOREM 5. *Let an operator $T \in \mathcal{L}(\oplus_1^n H)$ be a finite triangular operator matrix of the form*

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & \cdots & T_{12} \\ 0 & T_{22} & \cdots & \cdots & T_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & T_{nn} \end{pmatrix}$$

where T_{ii} are M-hyponormal for $i = 1, 2 \cdots, n$. Then T is a subscalar operator of order $2n$.

PROOF. Consider an arbitrary bounded open disk D in the complex plane $\mathbf{C}$ which contains $\sigma(T)$ and the quotient space

$$H(D) = \oplus_1^n W^{2n}(D, H)/\overline{(z-T)(\oplus_1^n W^{2n}(D, H))}$$

endowed with the Hilbert space norm. Let $S(= S_z)$ be the multiplication operator by z on $W^{2n}(D, H)$. Then $\oplus_1^n S$ is a scalar operator of order $2n$ and its spectral distribution is

$$\Phi : \oplus_1^n C_0^{2n}(\mathbf{C}) \longrightarrow \mathcal{L}(\oplus_1^n W^{2n}(D, H)), \quad \Phi(\oplus_1^n f_i) = \oplus_1^n S_{f_i}.$$

where S_{f_i} is the multiplication operator with f_i. Since $\oplus_1^n S$ commutes with $z - T$, $\widetilde{\oplus_1^n S}$ on $H(D)$ is still a scalar operator of order $2n$, with $\tilde{\Phi}$ as a spectral distribution.

Let V be the operator

$$V(\oplus_1^n h_i) = (1 \otimes h_1, \cdots, 1 \otimes h_n)^T + \overline{(z-T) \oplus_1^n W^{2n}(D, H)},$$

from $\oplus_1^n H$ into $H(D)$, denoting by $(1 \otimes h_1, \cdots, 1 \otimes h_n)^T$ the constant function $\oplus_1^n h_i$. Then

$$VT = (\widetilde{\oplus_1^n S})V.$$

In particular ran V is an invariant subspace for $\widetilde{\oplus_1^n S}$.

In order to conclude the proof of this theorem, it is enough to show the following claim.

CLAIM 1. *Let D be an bounded open disk in $\mathbf{C}$ which contains $\sigma(T)$. Then the operator $V : \oplus_1^n H \to H(D)$ is one-to-one and has closed range.*

PROOF OF CLAIM. Let $\oplus_1^n h_i^k \in \oplus_1^n H$ and $\oplus_1^n f_i^k \in \oplus_1^n W^{2n}(D, H)$ be sequences in k such that

$$\lim_{k \to \infty} \|(z-T) \oplus_1^n f_i^k + \oplus_1^n (1 \otimes h_i^k)\|_{\oplus_1^n W^{2n}} = 0.$$

It suffices to show that $\lim_{k\to\infty} \oplus_1^n h_i^k = 0$. By Lemma 4, we get the following equation.

$$\lim_{k\to\infty} \|(z - T_{11})f_1^k + 1 \otimes h_1^k\|_{W^2} = 0.$$

By the application of Lemma 2 with $T = T_{11}$, we get $\lim_{k\to\infty} h_1^k = 0$. Since $\lim_{k\to\infty} h_j^k = 0$ for $j = 1, 2, \cdots, n$, $\lim_{k\to\infty} h^k = 0$ where $h^k = (h_1^k, \cdots, h_n^k)^T$. Thus V is one-to-one and has closed range. □

This also concludes the proof of Theorem 5, because ran V is a closed invariant subspace for the scalar operator $\widetilde{\oplus_1^n S}$.

Next, we shall prove that every algebraic operator is subscalar. Recall that an operator $T \in \mathcal{L}(H)$ is algebraic if there is a non-zero polynomial p such that $p(T) = 0$. An interesting characterization of algebraic operators was given by P.R. Halmos.

LEMMA 6. ([2]) *If T is an algebraic operator and p is a polynomial of minimal degree n such that $p(T) = 0$, then T is unitarily equivalent to a finite triangular operator matrix of the form*

$$\begin{pmatrix} \alpha_1 & T_{12} & \cdots & \cdots & \cdots & T_{1n} \\ 0 & \alpha_2 & T_{23} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \alpha_{n-1} & T_{n-1,n} \\ 0 & 0 & \cdots & \cdots & \cdots & \alpha_n \end{pmatrix}$$

where α_i are the roots of the polynomial p.

COROLLARY 7. *If T is an algebraic operator, then T is a subscalar operator.*

PROOF. It is clear from Theorem 5 and Lemma 6. □

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Department of Mathematics
Ewha Women's University
Seoul 120-750, Korea
E-mail: eikomm.ewha.ac.kr