Comm. Korean Math. Soc. **12** (1997), No. 3, pp. 553 - 560

A REMARK ON THE CLASS NUMBER FORMULAS OVER GLOBAL FUNCTION FIELDS

SUNGHAN BAE, PYUNG-LYUN KANG

ABSTRACT. A class number formula over global function field, extending the formula obtained by Shu, is proved

0. Introduction

Let k be a global function field over a finite field $\mathbb{F}_q$. Let ∞ be a fixed place of k with degree δ, and A be the subring of k consisting of those elements which are regular outside ∞. Galovich and Rosen [2] studied cycltomic units when $A = \mathbb{F}_q[T]$ and obtained an analogue of the famous index-class number formula of Kummer and Sinnot. Shu [5] obtained a class number formula for the $\mathfrak{p}$-th cyclotomic function field where $\mathfrak{p}$ is a prime ideal of A when $\delta = 1$, which is the same as Galovich and Rosen's when $A = \mathbb{F}_q[T]$.

In this article we extend Shu's results [5] to general δ and prime power cyclotomic function fields. We use the expression of L-functions by division values of elliptic modules given in Gross and Rosen ([2]) for general δ.

1. Cyclotomic Units

Throughout the paper we fix a sign function $sgn : k_\infty^* \longrightarrow \kappa(\infty)^*$, where k_∞ is the completion of k at ∞ and $\kappa(\infty)$ the residue class field at ∞. Denote by H (resp. $\tilde{H}$) the Hilbert class field of A (resp. normalizing

---

Received February 17, 1997. Revised June 5, 1997.

1991 Mathematics Subject Classification: 11R58, 11G09, 11G15.

Key words and phrases: *sgn*-normalized elliptic module, cyclotomic unit, elliptic unit.

The first author is supported in part by Non Directed Research Fund, Korea Research Foundation, 1996, and the second author by BSRI-96-1427, Ministry of Education.

field of $(A, sgn))$ in the sense of Hayes [3]. Fix a sgn-normalized elliptic A-module ρ associated to the lattice A. Then ρ is defined over $\tilde{H}$ and the map $\sigma \mapsto \rho^\sigma$ sets up a one-to-one correspondence between $Gal(\tilde{H}/k)$ and the set of all the sgn-normalized elliptic A-modules. For an ideal $\mathfrak{a}$ of A, $\rho_\mathfrak{a}$ denotes the monic generator of the left ideal generated by ρ_a with $a \in \mathfrak{a}$, and $\omega_\mathfrak{a}$ the coefficient of the constant term of $\rho_\mathfrak{a}$.

In the following for a finite extension F of k we denote by O_F the integral closure of A in F, h_F the class number of F, and h_{O_F} the class number of the Dedekind domain O_F. For an ideal $\mathfrak{n}$ of A we denote by $k(\mathfrak{n})$ the field extension over $\tilde{H}$ generated by all the $\mathfrak{n}$-torsion points of ρ. Then $k(\mathfrak{n})$ is an abelian extension of k whose decomposition group (= inertia group) G_∞ at ∞ is isomrphic to $\kappa(\infty)^*$. Let $k^+(\mathfrak{n})$ be the fixed field of G_∞ in $k(\mathfrak{n})$. For all the details on elliptic modules and their relations to the class field theory over k, we refer to [3] and [4].

The main concern of this article is the case that $\mathfrak{n} = \mathfrak{p}^\ell$ a power of a prime ideal $\mathfrak{p}$. Throughout the paper we always assume that $\mathfrak{n} = \mathfrak{p}^\ell$. In this case we can see easily that

$$O^*_{k(\mathfrak{n})} = O^*_{k^+(\mathfrak{n})}.$$

Let $\tilde{\rho}[\mathfrak{n}]$ and $\tilde{\rho}[\mathfrak{n}]'$ be the set of all the $\mathfrak{n}$-torsion points of ρ^σ and the set of all primitive $\mathfrak{n}$-torsion points of ρ^σ for any $\sigma \in G(\tilde{H}/k)$, respectively. Let $\mathfrak{J}$ be the $G(k(\mathfrak{n})/k)$-submodule of $k(\mathfrak{n})^*$ generated by $\mathbb{F}^*_{q^\delta}$ and the set $\{\alpha : \alpha \in \tilde{\rho}[\mathfrak{n}] \setminus \{0\}\}$, and let $\mathcal{C}_\mathfrak{n} = \mathfrak{J} \cap O^*_{k(\mathfrak{n})}$. The elements of $\mathcal{C}_\mathfrak{n}$ are called the *cyclotomic units* of $O_{k(\mathfrak{n})}$.

Let $\mathcal{M}'$ be a set of coset representatives of $G(k(\mathfrak{n})/k)/G(k(\mathfrak{n})/k^+(\mathfrak{n}))$ containing the identity and $S' = \{\alpha/\beta : \beta = \alpha^\sigma, \quad \sigma \in \mathcal{M}'\}$, where α is a primitive $\mathfrak{n}$-th root of ρ.

PROPOSITION 1.1. *$S' \subset O^*_{k^+(\mathfrak{n})}$ and S' and $\mathbb{F}^*_{q^\delta}$ generate $\mathcal{C}_\mathfrak{n}$. Therefore we have*

$$rank\, \mathcal{C}_\mathfrak{n} \leq |G(k^+(\mathfrak{n})/k)| - 1.$$

PROOF. It is shown in ([3], Proposition 4.15) that $\alpha^\tau = a\alpha$, for some $a \in \mathbb{F}^*_{q^\delta}$ for each $\tau \in G_\infty = G(k(\mathfrak{n})/k^+(\mathfrak{n}))$. Now the proof is exactly the same as that of [5], Proposition 3.3. □

PROPOSITION 1.2. *Let α be a primitive $\mathfrak{n}$-th root of ρ. Then we have*

$$\prod_{\sigma\in G(k(\mathfrak{n})/\tilde{H})} \alpha^{\sigma} = \omega_{\mathfrak{p}}^{(\tau_{\mathfrak{p}})^{1-\ell}},$$

where $\tau_{\mathfrak{p}}$ is the element of $G(\tilde{H}/k)$ associated to the ideal $\mathfrak{p}$. Thus we conclude that elliptic units are cyclotomic units.

PROOF. We know from [3] that

$$\rho_{\mathfrak{ab}} = (\mathfrak{b} * \rho)_{\mathfrak{a}} \cdot \rho_{\mathfrak{b}},$$

and

$$(\mathfrak{b} * \rho) = \rho^{\tau_{\mathfrak{b}}^{-1}}.$$

Since

$$\rho_{\mathfrak{n}}(X)/\rho_{\mathfrak{n}\mathfrak{p}^{-1}}(X) = \prod_{\sigma\in G(k(\mathfrak{n})/\tilde{H})} (X - \alpha^{\sigma}),$$

we have

$$\prod_{\sigma\in G(k(\mathfrak{n})/\tilde{H})} \alpha^{\sigma} = \omega_{\mathfrak{n}}/\omega_{\mathfrak{n}\mathfrak{p}^{-1}} = \omega_{\mathfrak{p}}^{(\tau_{\mathfrak{n}\mathfrak{p}^{-1}})^{-1}} = \omega_{\mathfrak{p}}^{\tau_{\mathfrak{p}}^{(1-\ell)}}.$$

□

Let N be the subgroup of $G(\tilde{H}/k)$ generated by the Artin symbol $(\mathfrak{n}, \tilde{H}/k)$ and $n = |N|$. Let L be the fixed field of N in $\tilde{H}$. Let $\mathcal{M}$ be a set of coset representatives for $G(k(\mathfrak{p})/k)/G(k(\mathfrak{p})/L)$ containing the identity. Then as in [5] for each $\sigma \in \mathcal{M}$ we have a relation $CR(\sigma)$ among the elements of S';

$$CR(\sigma): \prod_{\tau\in G(k(\mathfrak{n})/L)} \left(\frac{\alpha}{\alpha^{\tau\sigma}}\right) = \prod_{\tau\in G(k(\mathfrak{n})/L)} \left(\frac{\alpha}{\alpha^{\tau}}\right).$$

Since $CR(id)$ is a trivial relation and exactly the same proof would give that other relations are indepedent, we have;

PROPOSITION 1.3. $\operatorname{rank} C_{\mathfrak{n}} \leq |G(k^{+}(\mathfrak{n})/k)| - m$ *where* $m = h_A/n$.

2. Class Number Formulas

In this section we express the values of L-functions at 0 using $\mathfrak{n}$-th root of ρ. Let χ be a character of k. If χ is unramified and nontrivial on N, then we have as in [5]

$$L_k(0,\chi) = \frac{1}{(1-\chi(\mathfrak{p}))\delta(q-1)} \sum_{\mathfrak{a}\in Pic\ A} \bar{\chi}(\mathfrak{a})(\log_q |\omega_{\mathfrak{p}}^{\tau_{\mathfrak{a}^{-1}}}| - \deg \mathfrak{p}).$$

Hence

$$\prod_{\chi|_N\neq 1} \sum_{\mathfrak{a}\in Pic\ A} \bar{\chi}(\mathfrak{a})(\log_q |\omega_{\mathfrak{p}}^{\tau_{\mathfrak{a}^{-1}}}|) = n^m(\delta(q-1))^{h-m}\frac{h_H}{h_L}\frac{q^e-1}{d(q^\delta-1)},$$

where q^e is the number of constants in L and $d=\frac{\delta}{e}$.

We set $g = |G(k^+(\mathfrak{n})/k)| - m$ for simplicity. For an ideal $\mathfrak{c}$ of A we denote by $e_{\mathfrak{c}}(x)$ the exponential function associated to the lattice $\mathfrak{c}$, that is,

$$e_{\mathfrak{c}}(x) = x\prod_{b\in\mathfrak{c}\setminus\{0\}}(1-\frac{x}{b}). \tag{2.1}$$

Let $\mathfrak{f}$ be an integral ideal of A. Let $I(\mathfrak{f})$ be the group of fractional ideals in k which are relatively prime to $\mathfrak{f}$, $P(\mathfrak{f})$ be the subgroup of principal ideals (a) with $a\equiv 1\,(mod\,\mathfrak{f})$. Put $G(\mathfrak{f}) = I(\mathfrak{f})/P(\mathfrak{f})$. Let

$$F_{\mathfrak{c}}(x) = \xi(\mathfrak{c})e_{\mathfrak{c}}(x) \tag{2.2}$$

and

$$\phi_{\mathfrak{f}}(\mathfrak{c}) = F_{\mathfrak{f}\mathfrak{c}^{-1}}(1)^{q^\delta-1}. \tag{2.3}$$

Let χ be a character of k of conductor $\mathfrak{f}\neq A$. Then it is shown in [2] that

$$L'_A(0,\chi) = \frac{-1}{q^\delta-1}\sum_{\mathfrak{c}\in G(\mathfrak{f})}\chi(\mathfrak{c})\ln|\phi_{\mathfrak{f}}(\mathfrak{c})|.$$

Hence

$$L_k(0,\chi) = \frac{-1}{\delta(q^\delta - 1)} \sum_{\mathfrak{c}\in G(\mathfrak{f})} \chi(\mathfrak{c}) \log_q |\phi_{\mathfrak{f}}(\mathfrak{c})|. \tag{2.4}$$

Here $|\cdot|$ is an extension of the normalized absolute value on k_∞ to C, the completion of the algebraic closure of k_∞. Now let $\mathfrak{f} = \mathfrak{n} = \mathfrak{p}^\ell$. Then the ray class field $mod\,\mathfrak{n}$ is just $k^+(\mathfrak{n})$ and $G(\mathfrak{n}) \simeq G(k^+(\mathfrak{n})/k)$ via the Artin map. Let $\epsilon = F_{\mathfrak{n}}(1)$ and $\beta = \epsilon^{q^\delta - 1}$. Then $\beta \in k^+(\mathfrak{n})$.

Now the equation (2.4) becomes

$$L_k(0,\chi) = \frac{-1}{\delta(q^\delta - 1)} \sum_{\mathfrak{c}\in G(\mathfrak{n})} \chi(\mathfrak{c}) \log_q |\beta^{\tau_{\mathfrak{c}}}|, \tag{2.5}$$

and

$$L_k(0,\chi) = \frac{-1}{\delta(q^\delta - 1)} \sum_{\sigma\in G(k(\mathfrak{n})/k)} \chi(\sigma) \log_q |\epsilon^{\sigma}|. \tag{2.6}$$

For an unramified character χ,

$$\begin{aligned} L_k(0,\chi) &= \frac{1}{\delta(q-1)(1-\chi(\mathfrak{p}))} \sum_{\mathfrak{c}\in PicA} \bar{\chi}(\mathfrak{c}) \log_q |\omega_{\mathfrak{p}}^{\tau_{\mathfrak{c}}}| \\ &= \frac{1}{\delta(q-1)(1-\chi(\mathfrak{p}))} \frac{q-1}{q^\delta - 1} \sum_{\sigma\in G(\tilde{H}/k)} \bar{\chi}(\sigma) \log_q |\omega_{\mathfrak{p}}^{\sigma}| \\ &= \frac{1}{\delta(q^\delta-1)(1-\chi(\mathfrak{p}))} \sum_{\sigma\in G(k(\mathfrak{n})/k)} \bar{\chi}(\sigma) \log_q |\epsilon^{\sigma\tau_{\mathfrak{n}\mathfrak{p}^{-1}}}| \\ &= \frac{-\bar{\chi}(\tau_{\mathfrak{n}\mathfrak{p}^{-1}})}{\delta(q^\delta-1)(1-\chi(\mathfrak{p}))} \sum_{\mathfrak{c}\in G(\mathfrak{n})} \chi(\mathfrak{c}) \log_q |\beta^{\tau_{\mathfrak{c}}}|. \end{aligned} \tag{2.7}$$

For an ideal $\mathfrak{n}' \neq A$ dividing $\mathfrak{n}$, we let $\epsilon' = F_{\mathfrak{n}'}(1)$, and $\beta' = (\epsilon')^{q-1}$. Write $\mathfrak{m} = \frac{\mathfrak{n}}{\mathfrak{n}'}$. Since $\epsilon \in \tilde{\rho}[\mathfrak{n}]'$, $\rho_{\mathfrak{m}}^{\sigma}(\epsilon) \in \tilde{\rho}[\mathfrak{n}']'$ for some $\sigma \in G(\tilde{H}/k)$. Hence there exists an element $\sigma_{\mathfrak{n}'} \in G(k(\mathfrak{n}')/k)$ so that

$$\sigma_{\mathfrak{n}'}(\epsilon') = \rho_{\mathfrak{m}}^{\sigma}(\epsilon).$$

Since $G(k(\mathfrak{n})/k(\mathfrak{n}')) = 1 + \mathfrak{n}/\mathfrak{n}'$,

$$\begin{aligned} N_{k(\mathfrak{n})/k(\mathfrak{n}')}(\epsilon) &= \prod_{n \in \mathfrak{n}' \mod \mathfrak{n}} \rho^{\sigma}_{1+n}(\epsilon) \\ &= \prod_{n \in \mathfrak{n}' \mod \mathfrak{n}} (\epsilon + \rho^{\sigma}_{n}(\epsilon)) \\ &= \prod_{\lambda \in \rho^{\sigma}[\mathfrak{m}]} (\epsilon + \lambda) \\ &= \rho^{\sigma}_{\mathfrak{m}}(\epsilon) \\ &= \sigma_{\mathfrak{n}'}(\epsilon'). \end{aligned}$$

Then, using (2.6), we have for a character χ of k of conductor $\mathfrak{n}'$

$$L_k(0,\chi) = \frac{-\chi(\sigma_{\mathfrak{n}'})}{\delta(q^{\delta}-1)} \sum_{\mathfrak{c} \in G(\mathfrak{n})} \chi(\mathfrak{c}) \log_q |\beta^{\tau_{\mathfrak{c}}}|. \tag{2.8}$$

It is easy to see that $\prod_{\chi} \chi(\sigma_{\mathfrak{n}'}) = \pm 1$.

THEOREM 2.1. *We have*

$$\prod_{\chi} L_k(0,\chi) = n^{-m}(\delta(q^{\delta}-1))^{-g} \prod_{\chi} \sum_{\tau \in G(k^+(\mathfrak{n})/k)} \chi(\tau) \log_q |\beta^{\tau}|,$$

and

$$\frac{(q^e-1)h_{k^+(\mathfrak{n})}}{d(q^{\delta}-1)h_L} = n^{-m}(\delta(q^{\delta}-1))^{-g} \prod_{\chi} \sum_{\tau \in G(k^+(\mathfrak{n})/k)} \chi(\tau) \log_q |\beta^{\tau}|,$$

where χ runs through all the characters which are nontrivial when restricted to the subgroup $G(k^+(\mathfrak{n})/L)$, q^e is the number of constants in L, and $d = \frac{\delta}{e}$.

PROOF. The first formula follows from (2.7), (2.8), and the fact that $\prod_{i=1}^{n-1}(1-\xi_n^i) = n$, where ξ_n is a primitive nth root of unity. The second formula follows from the first and the fact that for function field extension K/L,

$$\lim_{s \to 0} \frac{\zeta_K(s)}{\zeta_L(s)} = \frac{(q_K-1)h_K}{d(q_L-1)h_L},$$

where d is the degree of the extension $\mathbb{F}_{q_K}$ over $\mathbb{F}_{q_L}$. □

For a finite extension F of k we denote by S_F^0 the set of divisors of degree zero which are supported on the infinite places of F. Let E be any subset of F, we use $\tilde{E}$ to denote the set consisting of all the divisors associated to any elements in E. It is well-known that $[S_F^0 : \tilde{O}_F^*] = R_F$, the regulator of F, and $fh_F = R_F h_{O_F}$, where f is the greatest common divisor of $deg_F\, \bar{\infty}$'s for $\bar{\infty}|\infty$. Thus $h_{k+(\mathrm{n})} = R_{k+(\mathrm{n})} h_{O_{k+(\mathrm{n})}}$ and $dh_L = R_L h_{O_L}$. Exactly the same process as in [5] would give

PROPOSITION 2.2. *Let q^e be the number of constants in L. Then we have*

$$[\tilde{O}_{k+(\mathrm{n})}^* : \tilde{C}_{\mathrm{n}}^{q^\delta - 1} + \tilde{O}_L^*] = n^m (\delta(q^\delta - 1))^g \frac{(q^e - 1) h_{O_{k+(\mathrm{n})}}}{(q^\delta - 1) h_{O_L}}.$$

REMARK. The formula (4.7) in [5] should not contain the factor $(q-1)^{-g}$.

COROLLARY 2.3. *$rank\, C_{\mathrm{n}} = g$.*

We now have the following generalization of Main Theorem 2 of Shu[5].

THEOREM 2.4. *We have*

$$[O_{k+(\mathrm{n})}^* : C_{\mathrm{n}} O_L^*] = n^m (\delta)^g \frac{(q^e - 1) h_{O_k^+(\mathrm{n})}}{(q^\delta - 1) h_{O_L}}.$$

PROOF. Since $\tilde{C}_{\mathrm{n}}$ and S_L^0 are free abelian groups, $\tilde{C}_{\mathrm{n}} \cap S_L^0 = \{1\}$ by Proposition 2.2. Thus

$$[\tilde{C}_{\mathrm{n}} + S_L^0 : \tilde{C}_{\mathrm{n}}^{q^\delta - 1} + S_L^0] = (q^\delta - 1)^g$$

by Corollary 2.3. Then

$$\begin{aligned}[O_{k+(\mathrm{n})}^* : C_{\mathrm{n}} O_L^*] &= [\tilde{O}_{k+(\mathrm{n})}^* : \tilde{C}_{\mathrm{n}} + \tilde{O}_L^*] \\ &= \frac{1}{(q^\delta - 1)^g} [\tilde{O}_{k+(\mathrm{n})}^* : \tilde{C}_{\mathrm{n}}^{q^\delta - 1} + \tilde{O}_L^*] \\ &= n^m \delta^g \frac{(q^e - 1) h_{O_{k+(\mathrm{n})}}}{(q^\delta - 1) h_{O_L}},\end{aligned}$$

by Proposition 2.2. □

References

[1] Galovich, S. and Rosen, M., *The Class Number of Cyclotomic Function Fields*, J. Number Th. **13** (1981), 363-375.

[2] Gross, B. and Rosen, M., *Fourier Series and the Special Values of L-Functions*, Advances in Math. **69** (1988), 1-31.

[3] Hayes, D., *Explicit Class Field Theory in Global Function Fields*, Studies in Algebra and Number Theory, G. C. Rota (ed.) (1979), Academic Press, New York.

[4] ———, *Stickelberger Elements in Function Fields*, Compositio Math. **55** (1985), 209-235.

[5] Shu, L., *Class Number Formulas over Global Function Fields*, J. Number Th. **48** (1994), 133-161.

Sunghan Bae
Department of Mathematics
KAIST
Taejon, 305-701 Korea

Pyung-Lyun Kang
Department of Mathematics
ChoongNam National University
Taejon, 305-764 Korea