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KINEMATIC STRUCTURES OF CERTAIN LOOPS

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ABSTRACT. In this paper, we call a loop F *kinematic* if for $a, b \in F \setminus \{0\}$, the following two conditions are valid: (i) the centralizer $Z(a)$ of a is a commutative group under the induced operation from the loop F, and (ii) $Z(a) = Z(b)$ or $Z(a) \cap Z(b) = \{0\}$, where 0 is the identity of F. Some examples of kinematic loops are given.

1. Introduction

In order to describe sharply 2-transitive permutation groups, H. Karzel introduced in [7] the concept of a near-domain(=Fastbereich) $(F, \oplus, \cdot)$ (cf. [11, 23]). The crucial difficulty of a neardomain is the additive structure $(F, \oplus)$, which need not be associative. A neardomain $(F, \oplus, \cdot)$ with an associative addition is already a nearfield. In many notes, neardomains are investigated, but until now no example of a proper neardomain is known. To obtain partial results, W. Kerby and H. Wefelscheid considered the additive structure $(F, \oplus)$ and called such loops *K-loops*. In the last years, the interest on K-loops is grown since A. A. Ungar showed in [20-3] that the set of admissible velocities $\mathbb{R}_c^3 = \{v \in \mathbb{R}^3 : |v| < c\}$ forms a non-commutative non-associative K-loop with respect to a certain relativistic velocity addition $\oplus$, where c is the speed of light (cf. [17]).

A nonempty set F with a binary operation $+$ is called a *loop* if for all $a, b \in F$, there is exactly one $x \in F$ and one $y \in F$ such that $a + x = b$ and $y + a = b$, and if there is an identity element $0 \in F$. Then for $a, b \in F$ the maps $a^+ : F \to F; x \to a + x$ and $\delta_{a,b} := ((a+b)^+)^{-1} \cdot a^+ \cdot b^+$ are permutations of the set F. Hence the equation $a + (b + c) = (a + b) + \delta_{a,b}(c)$

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holds. The loop $(F,+)$ is called a *weak* $K-loop$ if firstly the equation $a+b=0$ implies $b+a=0$, and secondly $\delta_{a,b}$ is an automorphism of the loop $(F,+)$. Let $-a$ be the right inverse of a. If furthermore the so called automorphic inverse property $(-a)+(-b)=-(a+b)$ holds and if $\delta_{a,b}=\delta_{a,b+a}$ for all $a,b\in F$, then F is called a $K-loop$ (cf. [2, 19]).

A loop $(F,+)$ is called a $Bol-loop$, if for all $a,b,c\in F$ the Bol identity $a+(b+(a+c))=(a+(b+a))+c$ holds. We call a Bol-loop a $Bruck-loop$, if the automorphic inverse property is valid. Every K-loop is a Bruck loop (cf. [15]). A. Kreuzer [15-2, 19] constructed large classes of proper K-loops which comprise finite and infinite examples. Very recently, A. Kreuzer has proved in [18] that Bruck loops and K-loops are the same.

A K-loop $(F,+)$ is called *kinematic* in Kolb and Kreuzer [12] if for $a,x,y\in F^*=F\setminus\{0\}$, $\delta_{a,x}=\delta_{a,y}=id\Longrightarrow\delta_{x,y}=id$. In a loop $(F,+)$, we do not assume the associativity, hence the centralizer $Z(a)=\{x\in F|a+x=x+a\}$ of $a\in F$ is not necessarily a subloop of F. In this paper, generalizing the preceding definition given in [12], we take the following for any loop:

DEFINITION. A loop $(F,+)$ is called kinematic if for $a\in F^*$, the following two conditions are valid:

(i) $Z(a)$ is a commutative group under the induced operation from the loop $(F,+)$.

(ii) $\{Z(a)\setminus\{0\}|a\in F^*\}$ is a partition of F^*.

Karzel and Wefelscheid [10] showed that in the Minkowski space time world the future cone $\mathfrak{H}^{++}$ can be turned into a K-loop with respect to the binary operation $A\oplus B=\sqrt{A}B\sqrt{A}$, where $\sqrt{A}=\frac{\sqrt{\det A}E+A}{\sqrt{Tr\,A+2\sqrt{\det A}}}\in\mathfrak{H}^{++}$. And B. Im [I] proved that $\mathfrak{H}^{++}$ is a K-loop with respect to the binary operation $A\boxplus B=\sqrt{AB^2A}$ and moreover that two K-loops $(\mathfrak{H}^{++},\oplus)$ and $(\mathfrak{H}^{++},\boxplus)$ are isomorphic. In Section 2, we discuss centralizers of elements in the future cone $\mathfrak{H}^{++}$ with respect to two different loop operations $\boxplus,\oplus$ as well as ordinary matrix multiplication, and see directly that a loop $\mathfrak{H}^{1+}=\{A\in\mathfrak{H}^{++}|detA=1\}$ is kinematic, but that $\mathfrak{H}^{++}$ is not. In Section 3, we show in the case of a K-loop that our definition of a kinematic loop and Kolb-Kreuzer's in [12] are equivalent, if every $\delta_{a,b}$ is either fixed point free on F^* or the identity. Applying the result of [12, 9], we obtain two different examples of kinematic K-

loops, and show that $(\mathfrak{H}^{1+}, \oplus)$ is a kinematic K-loop correspoinding to a 3-dimensional hyperbolic space.

2. Centralizers of certain hermitian matrices

Let $K = (K, +, \cdot, \leq)$ be a euclidean field with $char(K) \neq 2$, $L = K(i)$ be the quadratic extension of K with $i^2 = -1$, and let $K^+ = \{\lambda \in K | \lambda > 0\}$. Let $\mathfrak{M} = \mathfrak{M}_{2,2}(L)$ be the set of all 2×2 matrices over L, and let $E \in \mathfrak{M}$ be the identity matrix. Then the field L will be considered as a subring of the matrix algebra $\mathfrak{M}$ via the monomorphism $L \longrightarrow \mathfrak{M}$; $x \longrightarrow xE$, i.e., we identify each $x \in L$ with the scalar matrix xE as in [3-3, 8, 10].

PROPOSITION 2.1 ([8]). *Let* $\wedge : \mathfrak{M} \longrightarrow \mathfrak{M}; X \longrightarrow \widehat{X}$ *be the mapping defined by* $\widehat{X} = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$ *for* $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathfrak{M}$. *Then we have the following elementary properties:*

(1) $\wedge$ *is an involutive* L*-antiautomorphism, hence* $\widehat{\widehat{X}} = X$, $\widehat{X+Y} = \widehat{X} + \widehat{Y}$, $\widehat{XY} = \widehat{Y}\widehat{X}$.
(2) $\widehat{X} = X \Longleftrightarrow X \in L$.
(3) $X + \widehat{X} = trX$.
(4) $detX = X\widehat{X} = \widehat{X}X = det\widehat{X}$.

Considering a quadratic form $q : \mathfrak{M} \longrightarrow L; X \longrightarrow X\widehat{X}$ with the corresponding symmetric bilinear form $f : \mathfrak{M} \times \mathfrak{M} \longrightarrow L$; $(X, Y) \longrightarrow f(X,Y) = \frac{1}{2}(X\widehat{Y} + Y\widehat{X})$, we obtain $(\mathfrak{M}, L, q = det)$ as a 4-dimensional metric vector space over L. Let $\mathfrak{H}$ be the set of all hermitian 2×2 matrices over L. Then a 4-dimensional metric vector space $(\mathfrak{H}, K, q)$ over K defines the Minkowski space time world and $\mathfrak{H}^{++} = \{X \in \mathfrak{H} | X\widehat{X} > 0, X + \widehat{X} > 0\}$ is called the future cone (cf. [5, 8, 13-2]).

PROPOSITION 2.2. *Let* $E^{\perp}$ *be the orthogonal complement of* $E \in \mathfrak{M}$. *Then* $E^{\perp} = \{X \in \mathfrak{M} | X + \widehat{X} = 0\} = \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in L\right\}$ *and for* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$

(1) $\mathfrak{a} \in E^{\perp} \Longleftrightarrow \widehat{\mathfrak{a}} = -\mathfrak{a} \Longrightarrow \mathfrak{a}^2 = -\mathfrak{a}\widehat{\mathfrak{a}} \in L$.
(2) $\mathfrak{a}, \mathfrak{b} \in E^{\perp} \Longrightarrow \mathfrak{a}\mathfrak{b} + \mathfrak{b}\mathfrak{a} \in L$, $\mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a} \in E^{\perp}$.

(3) *Let* $\mathfrak{a}, \mathfrak{b} \in E^{\perp}$, *then* $\mathfrak{b} \in \mathfrak{a}^{\perp} \Longleftrightarrow \mathfrak{a}\mathfrak{b} + \mathfrak{b}\mathfrak{a} = 0$.
(4) $\{X \in \mathfrak{H} | X\widehat{X} = 0\} \cap E^{\perp} = \{0\}$.

THEOREM 2.3. *Let* $A \in \mathfrak{M} \setminus LE$ *and let* $Z(A)$ *be its centralizer with respect to the ordinary matrix multiplication on* $\mathfrak{M}$. *Then* $Z(A)$ *is precisely the 2-dimensional* L*-algebra* $LE + LA$.

THEOREM 2.4. *Let* $A \in \mathfrak{H}^{++} \setminus K^{+}E$ *and let* $Z_{\boxplus}(A)$ *be its centralizer contained in* $\mathfrak{H}^{++}$ *with respect to the loop operation* $\boxplus$ *of the loop* $\mathfrak{H}^{++}$. *Then* $Z_{\boxplus}(A) = \{B \in \mathfrak{H}^{++} | AB^2A = BA^2B\} = (KE + KA) \cap \mathfrak{H}^{++}$.

PROOF. Given $A \in \mathfrak{H}^{++} \setminus K^{+}E$, let $B \in Z_{\boxplus}(A)$. Then $A \boxplus B = B \boxplus A$, hence $AB^2A = BA^2B$. Since $\mathfrak{M} = LE \oplus E^{\perp}$, we can decompose $A = \alpha E + \mathfrak{a}$, where $\alpha E = \frac{1}{2}(A + \widehat{A}) \in KE$, $\mathfrak{a} = \frac{1}{2}(A - \widehat{A}) \in E^{\perp}$. Also, let $B = \beta E + \mathfrak{b}$, where $\beta \in K$, $\mathfrak{b} \in E^{\perp}$. Then we must have $\alpha \neq 0$ and $\beta \neq 0$, for $tr(\mathfrak{a}) = tr(\mathfrak{b}) = 0$, $tr(A) > 0$, and $tr(B) > 0$. So we may assume $A = E + \mathfrak{a}$, $B = E + \mathfrak{b}$ where $\mathfrak{a}$, $\mathfrak{b} \in E^{\perp}$. Since the equation $AB^2A = BA^2B$ is equivalent to $(\mathfrak{b}\mathfrak{a} - \mathfrak{a}\mathfrak{b})(\mathfrak{a} + \mathfrak{b}) = 0$ by Proposition 2.2, we have $(\mathfrak{b}\mathfrak{a} - \mathfrak{a}\mathfrak{b})(\mathfrak{a} + \mathfrak{b})^2 = 0$. But $(\mathfrak{a} + \mathfrak{b})^2$ is a scalar matrix, so we have either $\mathfrak{b}\mathfrak{a} - \mathfrak{a}\mathfrak{b} = 0$ or $(\mathfrak{a} + \mathfrak{b})^2 = 0$. If $\mathfrak{b}\mathfrak{a} - \mathfrak{a}\mathfrak{b} = 0$, then $\mathfrak{b} \in Z(\mathfrak{a}) \cap E^{\perp}$, hence $\mathfrak{b} \in K\mathfrak{a}$ by Theorem 2.3. If $(\mathfrak{a} + \mathfrak{b})^2 = 0$, then $\mathfrak{a} + \mathfrak{b} = 0$ by Proposition 2.2(4), hence $\mathfrak{b} = -\mathfrak{a} \in K\mathfrak{a}$. Therefore we have $B \in (KE + KA) \cap \mathfrak{H}^{++}$, hence obtain the result. □

THEOREM 2.5. *For each* $A \in \mathfrak{H}^{++} \setminus K^{+}E$, *let* $Z_{\oplus}(A)$ *be its centralizer contained in* $\mathfrak{H}^{++}$ *with respect to the loop operation* $\oplus$ *of the loop* $\mathfrak{H}^{++}$. *Then* $Z_{\oplus}(A) = Z_{\boxplus}(A) = (KE + KA) \cap \mathfrak{H}^{++} = Z(A) \cap \mathfrak{H}^{++}$.

PROOF. Let A, $B \in \mathfrak{H}^{++} \setminus K^{+}E$. Then $B \in Z_{\boxplus}(A) = Z_{\boxplus}(\sqrt{A}) \Longleftrightarrow B^2 \in Z_{\oplus}(A) \Longleftrightarrow BAB = \sqrt{A}B^2\sqrt{A} \Longleftrightarrow \sqrt{A}BAB\sqrt{A} = AB^2A \Longleftrightarrow \sqrt{A}B\sqrt{A} = \sqrt{AB^2A} \Longleftrightarrow A \oplus B = A \boxplus B$. So we obtain $Z_{\boxplus}(A) \subseteq Z_{\oplus}(A)$. If $B \in Z_{\oplus}(A)$, then $\sqrt{B} \in Z_{\boxplus}(\sqrt{A}) = Z_{\boxplus}(A)$, hence $B \in Z_{\boxplus}(A)$ by Theorem 2.4. □

By Theorem 2.5 and Definition given in the introduction, the future cone $\mathfrak{H}^{++}$ is not a kinematic loop, because two distinct centralizers contain not just E but $K^{+}E$. However, we obtain the following:

THEOREM 2.6. *Let* $\mathfrak{H}^{1+} = \{A \in \mathfrak{H}^{++} | det A = 1\}$. *Then for each* $A \in \mathfrak{H}^{1+} \setminus E$, $\{X \in \mathfrak{H}^{1+} | A \oplus X = X \oplus A\} = (KE + KA) \cap \mathfrak{H}^{1+}$, *hence* $(\mathfrak{H}^{1+}, \oplus)$ *is a kinematic loop.*

3. Kinematic structures of certain K-loops

Let $(F, +)$ be a loop with identity 0. Even though the loop operation is not commutative, we use the additive notation for the convenience. In order to show abstractly that the centralizer of an element of F is closed under the loop operation $+$, we need some kind of associativity. Let $Z(a) = \{x \in F | a + x = x + a\}$ be the centralizer of $a \in F^*$ as usual and let $[a] = \{x \in F | \delta_{a,x} = id\}$, $[a^+] = \{x \in F | a^+ \circ x^+ = x^+ \circ a^+\}$. Then note that $0, a \in Z(a) \cap [a^+]$ and $0 \in [a]$ in any loop. From the Bol identity $a + (0 + (a + c)) = (a + (0 + a)) + c$, however, we obtain $\delta_{a,a} = id$, hence $a \in [a]$ in a Bol loop or K-loop.

PROPOSITION 3.1. ([19 (2.9)]) Let $(F, +)$ be a weak K-loop with the automorphic inverse property and let $a, b \in F$. Then

(1) $\delta_{a,b}(b + a) = a + b$,
(2) $\delta_{a,b}^{-1} = \delta_{b,a}$.

PROPOSITION 3.2. *Let* $(F, +)$ *be a weak* K*-loop with the automorphic inverse property and let* $a \in F^*$. *Then*

(1) $[a^+] = \{x \in Z(a) | \delta_{a,x} = \delta_{x,a}\}$,
(2) $[a] \subseteq [a^+] \subseteq Z(a)$.

PROOF. (1) Note that $a^+ \circ x^+ = x^+ \circ a^+ \Longleftrightarrow a + (x + c) = x + (a + c), \forall c \Longleftrightarrow (a + x) + \delta_{a,x}(c) = (x + a) + \delta_{x,a}(c), \forall c$. Hence $\{x \in Z(a) | \delta_{a,x} = \delta_{x,a}\} \subseteq [a^+]$. Let $x \in [a^+]$ and if we take $c = 0$ in the above note, then $x \in Z(a)$, hence $\delta_{a,x} = \delta_{x,a}$.

(2) Let $x \in [a]$. Then by the above proposition we have $\delta_{a,x} = \delta_{x,a}(= id)$ and $\delta_{a,x}(x + a) = a + x = x + a$, hence $[a] \subseteq [a^+]$ by (1). □

PROPOSITION 3.3. *Let* $(F, +)$ *be a* K*-loop and let* $a, b \in F$. *If* $\delta_{a,b}$ *is either fixed point free on* F^* *or the identity, then* $Z(a) = [a^+] = [a]$.

PROOF. By the above proposition, it is enough to show that $Z(a) \subseteq [a]$. Let $x \in Z(a)$ and suppose contrarily that $\delta_{a,x} \neq id$. Then the equation $\delta_{a,x}(x+a) = a+x$ implies $x+a = a+x = 0$ since $\delta_{a,x}$ fixes only 0. So $\delta_{a,x} = \delta_{a,x+a} = \delta_{a,0} = id$, since F is a K-loop. □

Kolb and Kreuzer calls a K-loop $(F,+)$ *kinematic* in [12] if for $a, x, y \in F^* = F \setminus \{0\}$, $\delta_{a,x} = \delta_{a,y} = id \Longrightarrow \delta_{x,y} = id$, and proves the following: $x, y \in [a] \Longrightarrow y \in [x]$ if and only if either $[a] \cap [x] = \{0\}$ or $[a] = [x]$, which imply that $[a]$ is a commutative group under the induced operation from a loop F. Therefore, by the above proposition, in the case of a K-loop, our definition of a kinematic loop F and Kolb-Kreuzer's are equivalent, if every $\delta_{a,b}$ is either fixed point free on F^* or the identity. Therefore we obtain the following theorem due to [9, Theorem 3.1-3.4]:

THEOREM 3.4. *Let $(R,+,\cdot)$ be an integral domain and J be the Jacobson radical of R. Let $^-$ be an involutive automorphism of R and fix some $\epsilon \in J$ with $\bar{\epsilon} = \epsilon$. Then R is a kinematic K-loop with respect to a binary operation defined by $x \oplus y = \frac{x+y}{1+\epsilon\bar{x}y}$, and the automorphism $\delta_{a,b}$ is given by $\delta_{a,b}(x) = \frac{1+\epsilon a\bar{b}}{1+\epsilon\bar{a}b} \cdot x$.*

As an example, let R be the power series ring in t over the complex number field. Then R is a local ring and its Jacobson radical J is the maximal ideal and $R \setminus J$ is the group of units. Choose for $^-$ the natural extension of the complex conjugation and set $\epsilon = t$. Then by the above theorem it follows that $(R, \oplus)$ is a non-commutative and non-associative kinematic K-loop.

Let Γ be a group, let $J = \{\gamma \in \Gamma | \gamma^2 = 1, \gamma \neq 1\}$ and let $D \subseteq J$. Then for any $a, b \in D$ with $a \neq b$, $\overline{ab} = \{x \in D | abx \in J\}$ is called a line. Then (Γ, D) is called a reflection group with midpoints in [9], if the following axioms are valid:

(i) If x, y and z are contained in a line, then $xyz \in D$.

(ii) For any $a, b \in D$, there exists the unique $m \in D$ such that $b = mam$.

We call such m the midpoint of a and b. For examples of a reflection group with midpoints, refer to [9]. Now we can apply the result in [9, Theorem 3.3, 4.2, 4.3] to obtain the following theorem:

THEOREM 3.5. *Let* (Γ, D) *be a reflection group with midpoints, let* $o \in D$ *be fixed. Then* $(D, \oplus)$ *is a* K*-loop with respect to a binary operation defined by* $a \oplus b = a'oboa'$*, where* $a' \in D$ *is the midpoint of* o *and* a*. Moreover, if for* $a, b, c \in D$*,* $acbcbca = bcacacb \Longrightarrow abc \in J$*, then a* K*-loop* $(D, \oplus)$ *is kinematic.*

Let $(P, \mathfrak{L}, \equiv, \alpha)$ be a hyperbolic space of arbitrary dimension characterized for instance by the system of axioms given in [8]. Then $(P, \mathfrak{L}, \equiv, \alpha)$ belongs to the class of arbitrary absolute spaces and to each point $a \in P$, there is exactly one reflection $\tilde{a}$ in the point a. If $D = \{\tilde{a} | a \in P\}$ and if Γ denotes the motion group of $(P, \mathfrak{L}, \equiv, \alpha)$, then by [KKo, Theorem 7.2(2)] the pair (Γ, D) is a reflection group with midpoints satisfying also the condition to become a kinematic loop required in Theorem 3.5. Therefore, if we fix a point $o \in P$ and denote by a_m the midpoint of o and a, i.e., $\widetilde{a_m}(o) = a$ and let $a^+ = \widetilde{a_m} \cdot \tilde{o}$, then $(P, +)$ with $a + b = a^+(b)$ becomes a kinematic K-loop. Now, we restrict ourselves to the case that the hyperbolic space has the dimension 3 and obtain the following theorem:

THEOREM 3.6. *Let* $(P, \mathfrak{L}, \equiv, \alpha)$ *be a 3-dimensional hyperbolic space. Then the correspoinding kinematic* K*-loop* $(\mathfrak{H}^{1+}, \oplus)$ *has the algebraic representation* $A \oplus B = \widetilde{\sqrt{A}}(\widetilde{E}(B)) = \sqrt{A}B\sqrt{A}$*, if* E *corresponds to the fixed point* $o \in P$*. And for each* $A \in \mathfrak{H}^{1+} \setminus E$*, the centralizer of* A *is precisely* $(KE + KA) \cap \mathfrak{H}^{1+}$*.*

PROOF. Let $(P, \mathfrak{L}, \equiv, \alpha)$ be a 3-dimensional hyperbolic space. Then by [8, VII 2.4], there is a Euclidean field $(K, +, \cdot)$ and its quadratic extension $L = K(i)$ such that the point set P can be identified with the set $\mathfrak{H}^{1+} = \{A \in \mathfrak{H}^{++} | detA = 1\}$ which contains the identity matrix E. The reflection in the point E is the restriction of the map $\wedge : \mathfrak{M} \longrightarrow \mathfrak{M}$ in Proposition 2.1 onto $\mathfrak{H}^{1+}$, i.e., $\widetilde{E} = \wedge|_{\mathfrak{H}^{1+}}$, and for $A \in \mathfrak{H}^{1+}$, the reflection $\widetilde{A}$ in the point A is defined by $\widetilde{A}(X) = A\widehat{X}A$, and the midpoint A_m of E and A is $\sqrt{A}$. Therefore, by the remark followed by Theorem 3.5 and by Theorem 2.6 we obtain the result. □

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