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# ON A FIBER SPACE OVER A CURVE

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ABSTRACT. Let $X$ be a smooth projective threefold. Let $C$ be a smooth projective curve and let $f : X \to C$ be a fiber space with connected fiber $S$. Assume that $q_1(S) = 0$. Then we have $-\chi(\mathcal{O}_C)\,\chi(\mathcal{O}_S) \leq -\chi(\mathcal{O}_X)$.

Throughout this note, we are working over the complex number field $\mathbb{C}$.

Let $S$ be a smooth projective surface and let $C$ be a smooth projective curve. In some other place, we have investigated the fiber space $f : S \to C$ with connected fibers. We had the result $\chi(\mathcal{O}_C)\,\chi(\mathcal{O}_F) \leq \chi(\mathcal{O}_S)$. In this note, we have tried to extend to a threefold the technique which we have used on a surface. But we can not have the same type result, because we don't have any tools to control the higher direct image of sheaf $R^i f_* \mathcal{O}_X(K_X)$. We add an extra condition about the fiber of $f$ which often appears during investigating the rational map associated with the complete linear system. Then we have the result $-\chi(\mathcal{O}_C)\,\chi(\mathcal{O}_S) \leq -\chi(\mathcal{O}_X)$ like the case of a surface. See Theorem A for the detail matters about our main result.

Now, we are going to fix our notations to use throughout in this note.

Let $X$ be a smooth projective variety. Denote by $K_X$ its canonical divisor. Denote the dimension of $H^i(X, \mathcal{O}_X(D))$ by $h^i(X, \mathcal{O}_X(D))$. Let's denote the genus of $X$ by $p_g(X)$ and for $i = 1,\ 2$, $h^i(X, \mathcal{O}_X)$ by $q_i(X)$ (or simply $p_g$ and $q_i$ if there is no possible confusion). Denote by $\chi(\mathcal{O}_X)$ the Euler characteristic of $X$.

Let's begin our story with the following known theorems.

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THEOREM 1. (Hirzebruch-Riemann-Roch theorem) *For a locally free sheaf $\mathcal{E}$ of rank $r$ on $X$ of dimension $n$, the Euler characteristic $\chi(\mathcal{E})$ of $\mathcal{E}$ is given as follows:*

$$\chi(\mathcal{E}) = deg(ch(\mathcal{E}) \cdot td(\mathcal{T}))_n,$$

*where $ch(\mathcal{E})$ is the Chern character of $\mathcal{E}$, $td(\mathcal{T})$ is the Todd class of the tangent sheaf $\mathcal{T}$ of $X$ and $(\ )_n$ means the component of degree $n$.*

For the detail matters, see Hartshorne [1].

THEOREM 2. *Let $X$ be a smooth projective threefold and let $C$ be a smooth projective curve. Let $f : X \to C$ be a surjective map with connected fibres. Then $R^2 f_* \mathcal{O}_X(K_X)$ is isomorphic to $\mathcal{O}_C(K_C)$.*

For a proof, see Kollár [3].

THEOREM A. *Let $X$ be a smooth projective threefold. Let $C$ be a smooth projective curve. Let $f : X \to C$ be a fiber space with connected fibers and let $S$ be a general fiber of $f$. Assume that $q_1(S) = 0$. Then we have*

$$-\chi(\mathcal{O}_C)\,\chi(\mathcal{O}_S) \leq -\chi(\mathcal{O}_X).$$

PROOF. We have the fiber space $f : X \to C$ with connected fiber $S$. By the spectral sequence, we have

$$\begin{aligned}
p_g &= h^0(X, \mathcal{O}_X(K_X)) = h^0(C, f_*\mathcal{O}_X(K_X)) \\
q_2 &= h^1(X, \mathcal{O}_X(K_X)) = h^1(C, f_*\mathcal{O}_X(K_X)) + h^0(C, R^1 f_*\mathcal{O}_X(K_X)) \\
q_1 &= h^2(X, \mathcal{O}_X(K_X)) = h^2(C, f_*\mathcal{O}_X(K_X)) + h^1(C, R^1 f_*\mathcal{O}_X(K_X)) \\
&\qquad + h^0(C, R^2 f_*\mathcal{O}_X(K_X))
\end{aligned}$$

We have $R^1 f_* \mathcal{O}_X(K_X) = 0$ since $q_1(S) = h^1(S, \mathcal{O}_S(K_S)) = 0$. So, we have $h^i(C, R^1 f_* \mathcal{O}_X(K_X)) = 0$. By Theorem 2, $R^2 f_* \mathcal{O}_X(K_X)$ is isomorphic to $\mathcal{O}_C(K_C)$. Since $C$ is a curve, $h^2(C, f_*\mathcal{O}_X(K_X)) = 0$ clearly. Hence we have

$$\begin{aligned}
p_g &= h^0(C, f_*\mathcal{O}_X(K_X)) \\
q_2 &= h^1(C, f_*\mathcal{O}_X(K_X)) \\
q_1 &= h^0(C, \mathcal{O}_C(K_C)).
\end{aligned}$$

See that $q_1 = p_g(C)$. It is known that $f_*K_{X/C} \stackrel{def}{=} f_*(\mathcal{O}_X(K_X) \otimes f^*\mathcal{O}_C(K_C)^{-1})$ is semipositive and locally free of rank $p_g(S)$. So $\deg f_*(\mathcal{O}_X(K_{X/C}) \geq 0$. (See Kawamata [2] and Ueno [4].) By Theorem 1, we have

$$\begin{aligned} h^0(C, f_*\mathcal{O}_X(K_X)) - h^1(C, f_*\mathcal{O}_X(K_X)) \\ &= \deg f_*\mathcal{O}_X(K_X) + p_g(S)(1 - p_g(C)) \\ &= \deg f_*K_{X/C} + p_g(S)(p_g(C) - 1) \\ &\geq p_g(S)(p_g(C) - 1). \end{aligned}$$

So $p_g - q_2 \geq p_g(S)(p_g(C) - 1)$. Hence we have

$$\begin{aligned} -\chi(\mathcal{O}_X) &= p_g - q_2 + q_1 - 1 \\ &\geq p_g(S)(p_g(C) - 1) + q_1 - 1 \\ &= p_g(S)(p_g(C) - 1) + p_g(C) - 1 \\ &= (p_g(S) + 1)(p_g(C) - 1). \end{aligned}$$

Since $q_1(S) = 0$, $\chi(\mathcal{O}_S) = p_g(S) + 1$. Therefore, we have $-\chi(\mathcal{O}_C)\chi(\mathcal{O}_S) \leq -\chi(\mathcal{O}_X)$. □

REMARK. Actually, $\chi(\mathcal{O}_X) = (-1)^n\chi(\mathcal{O}_X(K_X))$, where $n$ is the dimension of $X$. So our result can be reformulated as $\chi(\mathcal{O}_C(K_C))\,\chi(\mathcal{O}_S(K_S)) \leq \chi(\mathcal{O}_X(K_X))$.

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