Comm. Korean Math. Soc. **12** (1997), No. 3, pp. 513–519

A NOTE ON WEAKLY PATH-CONNECTED ORTHOMODULAR LATTICES

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ABSTRACT. We show that each orthomodular lattice containing only atomic nonpath-connected blocks is a full subalgebra of an irreducible path-connected orthomodular lattice and there is a path-connected orthomodular lattice L containing a weakly path-connected full subalgebra $\mathbf{C}(x)$ for some element x in L.

1. Preliminaries

It is known that there exists a weakly path-connected orthomodular lattice with finite sites which is not path-connected and there exists a path-connected orthomodular lattices which contains a nonpath-connected full subalgebra [6].

We will prove that every orthomodular lattice L containing only atomic nonpath-connected blocks is a full subalgebra of an irreducible path-connected orthomodular lattice and there exists a path-connected orthomodular lattice L containing a nonpath-connected full subalgebra $\mathbf{C}(x)$ for some $x \in L$.

An *orthomodular lattice*(abbreviated by OML) L is an ortholattice L which satisfies *the orthomodular law*: if $x \leq y$, then $y = x \vee (x' \wedge y)$ $\forall x, y \in L$ [5]. A *Boolean algebra* B is an ortholattice satisfying the *distributive law*: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $\forall x, y, z \in B$.

A *subalgebra* of an OML L is a nonempty subset M of L which is closed under the operations $\vee$, $\wedge$ and $'$. We write $M \leq L$ if M is a subalgebra of L. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then the *relative interval sublattice* $M[a, b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with the

Received October 20, 1996. Revised April 10, 1997.
1991 Mathematics Subject Classification: 06C15.
Key words and phrases: Orthomodular lattices, path-connected, weakly path-connected.

relative orthocomplementation $\sharp$ on $M[a,b]$ given by $c^\sharp = (a \vee c') \wedge b = a \vee (c' \wedge b) \quad \forall c \in M[a,b]$. In particular, $L[a,b]$ will be denoted by $[a,b]$ if there is no ambiguity.

The *commutator of a and b* of an OML L is denoted by $a * b$, and is defined by $a * b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. For any two elements a, b of an OML, we say *a commutes with b*, in symbols $a\,\mathbf{C}\,b$, if $a * b = 0$. If M is a subset of an OML L, the set $\mathbf{C}(M) = \{x \in L \,|\, x\,\mathbf{C}\,m \quad \forall m \in M\}$ is called the *commutant* of M in L and the set $\mathbf{C}en(M) = \mathbf{C}(M) \cap M$ is called the *center* of M. The set $\mathbf{C}(L)$ is called the center of L and then $\mathbf{C}(L) = \bigcap\{\mathbf{C}(a) \,|\, a \in L\}$. An OML L is called *irreducible* if $\mathbf{C}(L) = \{0,1\}$, and L is called *reducible* if it is not irreducible.

A *block* of an OML L is a maximal Boolean subalgebra of L. The set of all blocks of L is denoted by $\mathfrak{A}_L$. Note that $\bigcup \mathfrak{A}_L = L$ and $\bigcap \mathfrak{A}_L = \mathbf{C}(L)$.

For any e in an OML L, the subalgebra $S_e = [0,e'] \cup [e,1]$ is called the *(principal) section generated by e*. Note that for $A, B \in \mathfrak{A}_L$, if $e \in (A \cap B)$ and $A \cap B = S_e \cap (A \cup B)$, then $A \cap B = S_e \cap A = S_e \cap B$.

DEFINITION 1.1. For blocks A, B of an OML L define $A \overset{wk}{\sim} B$ if and only if $A \cap B = S_e \cap (A \cup B)$ for some $e \in A \cap B$; $A \sim B$ if and only if $A \neq B$ and $A \cup B \leq L$; $A \approx B$ if and only if $A \sim B$ and $A \cap B \neq \mathbf{C}(L)$.

A *(weak) path* in L is a finite sequence B_0, B_1, ..., B_n $(n \geq 0)$ in $\mathfrak{A}_L$ satisfying $B_i \sim B_{i+1}$ $(B_i \overset{wk}{\sim} B_{i+1})$ whenever $0 \leq i < n$. The path is said to *join* the blocks B_0 and B_n. A path is said to be *proper* if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. A path is called to be *strictly proper* if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$ [1].

Let A, B be two blocks of an OML L. If $A \sim B$ holds, then there exists a unique element $e \in A \cap B$ satisfying $A \cap B = (A \cup B) \cap S_e$ [1]. Using this element e, we say that A and B are *linked at e* (*strongly linked at e*) if $A \sim B$ $(A \approx B)$, and use the notation $A \sim_e B$ $(A \approx_e B)$. This element e is called a *vertex* of L and it is the commutator of any $x \in A \setminus B$ and $y \in B \setminus A$ [1].

Note that $A \approx B$ implies $A \sim B$, and $A \sim B$ implies $A \overset{wk}{\sim} B$. Some authors, for example Greechie, use the phrase "*A and B meet in the* *section* S_e" to describe $A \overset{wk}{\sim} B$ [3].

DEFINITION 1.2. Let L be an OML, and $A, B \in \mathfrak{A}_L$. We will say that A and B are *weakly path-connected*, *path-connected*, *strictly path-connected in* L if A and B are joined by a weak path, a proper path, a strictly proper path, respectively. We will say A and B are *nonpath-connected* if there is no proper path joining A and B, and L is called *nonpath-connected* if there exist two blocks which are nonpath-connected. An OML L is called *weakly path-connected*, *path-connected*, *strictly path-connected in* L if any two blocks in L are joined by a weak path, a proper path, a strictly proper path, respectively. An OML L is called *relatively path-connected* if each $[0, x]$ is path-connected for all $x \in L$.

Let L be an OML, and $A, B, C \in \mathfrak{A}_L$. If A and B are joined with a strictly proper path $A = B_0 \approx B_1 \approx ... \approx B_{m-1} \approx B_m = B$ and if B and C are joined with a strictly proper path $B = C_0 \approx C_1 \approx ... \approx C_{n-1} \approx C_n = C$ then A and C are strictly path-connected by the *concatenated path* $A = B_0 \approx B_1 \approx ... \approx B_{m-1} \approx B \approx C_1 \approx ... \approx C_{n-1} \approx C_n = C$.

The following propositions are well known.

PROPOSITION 1.3. *Every finite direct product of path-connected OMLs is path-connected* [7].

PROPOSITION 1.4. *Every infinite direct product of path-connected OMLs containing infinitely many non-Boolean factors is nonpath-connected* [6, 8].

2. Weakly Path-connected Orthomodular Lattices

A sublattice M of an OML L is said to be a *suborthomodular lattice* of L in case the restriction of the orthocomplementation on L makes M an OML. A suborthomodular lattice M of an OML L is called *subcomplete* in case $N \subset M$ and $\bigvee N$ exists as computed in L implies $\bigvee N$ is in M.

In what follows we assume that $(L_1, \leq_1, ^\sharp)$ and $(L_2, \leq_2, ^+)$ are two disjoint OMLs, that S^i is a proper suborthomodular lattice of L_i ($i = 1, 2$), and that there exists an orthoisomorphism $\theta : S^1 \to S^2$.

DEFINITION 2.1.

(1) Let $L_0 = L_1 \cup L_2$.
(2) Let $P_1 = \{(x, y) \in L_0 \times L_0 \ : \ y = x\theta\}$.
(3) Let $\Delta = \{(x, x) \ : \ x \in L_0\}$.
(4) Let P be the equivalence relation defined by $P = \Delta \cup P_1 \cup {P_1}^{-1}$ where ${P_1}^{-1} = \{(y, x) \ : \ (x, y) \in P_1\}$.
(5) Let $L = L_0/P$.
(6) For $i = 1, 2$, let $R_i = \{([x], [y]) \in L \times L :$ there exist $x_i \in [x]$ and $y_i \in [y]$ such that $x_i <_i y_i\}$;
(7) Let $\leq$ be the relation $(R_1 \cup R_2)^2$.
(8) Define $[0]$ to be $[0_1]$ and $[1]$ to be $[1_1]$ where 0_1 and 1_1 are the zero and unit elements of L_1.
(9) Define $' : L \to L$ by the following prescription: for $[x] \in L$,

$$[x]' = \begin{cases} [{x_1}^{\sharp}], & \text{if there exists } x_1 \in L_1 \text{ such that } x_1 \in [x], \\ [{x_2}^{+}], & \text{if there exists } x_2 \in L_2 \text{ such that } x_2 \in [x]. \end{cases}$$

(10) Two sections S^1 and S^2 are said to be *corresponding sections* of L_1 and L_2 in case there exists $M_i \subset S^i \subset L_i$ $(i = 1, 2)$ such that $M_1\theta = M_2$ and $S^1 = \bigcup\{S_{m^\sharp} \ : \ m \in M_1\}$ and $S^2 = \bigcup\{S_{m^+} \ : \ m \in M_2\}$.

THEOREM 2.2. *Let S^1 and S^2 be corresponding sections of L_1 and L_2. Let L_i be complete and let S^i be subcomplete $(i = 1, 2)$. Then L is a complete OML* [3].

DEFINITION 2.3. An OML L is said to be obtained by *pasting two OMLs L_1 and L_2 along the sections S^1 and S^2* if all the conditions of 2.2 are satisfied, and we write $L = P(L_1, L_2; S^1, S^2; \theta)$.

Let $X = \{a_1, a_2, a_3, ...\}$, and let $\wp(X)$ be the power set of X. Then the Boolean algebra B consists of all finite and cofinite elements of the power set $\wp(X)$ of X is denoted by $B =< a_1, a_2, a_3, ... >$. The pasting of two disjoint OMLs L_1 and L_2 along the principal sections $S_{c_1} \leq L_1$ and $S_{c_2} \leq L_2$ generated by c_1, c_2 respectively is denoted by $L = P(L_1, L_2; S_{c_1}, S_{c_2}; \theta)$ (see definition 2.3). We may omit the isomorphism θ if there is no difficulty.

Let L be an OML. A subalgebra S of L is said to be a *full subalgebra* if every block of S is a block of L. Note that each $\mathbf{C}(x)$ is a full subalgebra of L for all $x \in L$ since $\mathfrak{A}_{\mathbf{C}(X)} = \{B \in \mathfrak{A}_L | x \in B\}$.

THEOREM 2.4. *If L is an OML such that each pair of nonpath-connected blocks A, B of L have atoms $a \in A$ and $b \in B$, then L is a full subalgebra of an irreducible path-connected OML.*

PROOF. Let $\mathcal{S}$ be the set of all nonpath-connected pairs of blocks of the given OML L, and let $\{A, B\} \in \mathcal{S}$. Then, for all $\{A, B\} \in \mathcal{S}$, $A \neq B$ and there exist two atoms a, b such that $a \in A$ and $b \in B$ by the given hypothesis (it may be that $a = b$) . Let $C =< a, c, d >$ with $c \neq d$ and $c, d \notin L$. Let $L_1 = P(L, C; S^L_{a'}, S^C_{a'})$. Then L_1 is an OML by theorem 2.2. Let $D =< d, e, f >$ with $e \neq f$ and $e, f \notin L_1$. Let $L_2 = P(L_1, D; S^{L_1}_{d'}, S^D_{d'})$. Then L_2 is an OML by theorem 2.2. Let $E =< b, g, h >$ with $g \neq h$ and $g, h \notin L_2$. Let $L_3 = P(L_2, E; S^{L_2}_{b'}, S^E_{b'})$. Then L_3 is an OML by theorem 2.2. Let $F =< h, m, n >$ with $m \neq n$ and $m, n \notin L_3$. Let $L_4 = P(L_3, F; S^{L_3}_{h'}, S^F_{h'})$. Then L_4 is an OML by theorem 2.2. Let $G =< f, p, n >$ with $p \neq f$, $p \neq n$ and $p \notin L_4$. Let $L_5 = L_4 \cup G$ where the operations and ordering are the union of those in L_5 and G. Then L_5 is an OML since $x \vee y = 1$ where $x \in G$, $y \in \bigcup(\mathfrak{A}_{L_4} \setminus \{C, D, E, F\})$ and $x \vee z \in L_5$ $\forall z \in C \cup D \cup E \cup F$.

Moreover, $A \approx_{a'} C \approx_{d'} D \approx_{f'} G \approx_{n'} F \approx_{h'} E \approx_{b'} B$ since $\mathbf{C}(L_5) = D \cap F = \{0, 1\}$, $a \in A \cap C$, $d \in C \cap D$, $f \in D \cap G$, $n \in G \cap F$, $h \in F \cap E$ and $b \in E \cap B$.

We add pairwise disjoint paths $C_\alpha \approx_{d'_\alpha} D_\alpha \approx_{f'_\alpha} G_\alpha \approx_{n'_\alpha} F_\alpha \approx_{h'_\alpha} E_\alpha$ with $A_\alpha \approx_{a'_\alpha} C_\alpha$ and $E_\alpha \approx_{b'_\alpha} B_\alpha$ to L for each nonpath-connected pair of blocks $\{A_\alpha, B_\alpha\}$ in $\mathcal{S}$ by the similar process which is given in the first part of this proof, where $d_\alpha, f_\alpha, n_\alpha, h_\alpha$ are distinct atoms not in L and that $d_\alpha \neq d_\beta$, $f_\alpha \neq f_\beta$, $n_\alpha \neq n_\beta$ and $h_\alpha \neq h_\beta$ for all two distinct pairs of blocks $\{A_\alpha, B_\alpha\}$, $\{A_\beta, B_\beta\}$ of $\mathcal{S}$. Then the resulting OML Γ contains at least one path between each $\{A_\alpha, B_\alpha\}$ in $\mathcal{S}$ and L is a subalgebra of Γ. Also, $\forall \alpha \neq \beta$ any distinct blocks U, V in Γ with $U \in \{C_\alpha, D_\alpha, G_\alpha, F_\alpha, E_\alpha\}$ and $V \in \{C_\beta, D_\beta, G_\beta, F_\beta, E_\beta\}$ are path-connected by a concatenated path since each pair of blocks $\{U, A_\alpha\}$, $\{A_\alpha, B_\beta\}$, $\{B_\beta, V\}$ are joined with strictly proper paths. Thus Γ is path-connected. Furthermore, Γ is irreducible and L is a full subalgebra of Γ. This completes the proof. □

The following corollary follows.

COROLLARY 2.5. *Every OML L containing only atomic nonpath-connected blocks is a full subalgebra of an irreducible path-connected OML.*

An OML L is called the *horizontal sum* of a family $(L_i)_{i\in I}$ (denoted by $\circ(L_i)_{i\in I}$) of at least two subalgebras, if $\bigcup L_i = L$, and $L_i \cap L_j = \{0,1\}$ whenever $i \neq j$, and one of the following equivalent conditions is satisfied:
(1) if $x \in L_i \setminus L_j$ and $y \in L_j \setminus L_i$, then $x \vee y = 1$;
(2) every block of L belongs to some L_i;
(3) if S_i is a subalgebra of L_i, then $\bigcup S_i$ is a subalgebra of L [2].

THEOREM 2.6. *There exists a path-connected orthomodular lattice L containing a nonpath-connected full subalgebra $\mathbf{C}(x)$ for some $x \in L$.*

PROOF. Let $MO2$ be the horizontal sum of two Boolean algebras 2^2 and 2^2 with four elements. Let $L = \prod_{n\geq 1} L_n$ where $L_n = MO2 \quad \forall n \geq 1$. Let A_n, B_n be the two distinct blocks in L_n, $A = \prod_{n\geq 1} A_n$ and $B = \prod_{n\geq 1} B_n$. Then A and B are nonpath-connected by proposition 1.4 and L does not contain a nonatomic block since each block of L is an infinite direct product of 2^2. Let Γ be the path-connected extention of L which is constructed by the same method in theorem 2.4. Then L is a full subalgebra of Γ. Choose $x = (y, 0, 0, ...) \in L \subseteq \Gamma$ where $y \in A_1$ and $y \notin \{0,1\}$, and $0 \in A_n$, $\forall n \geq 2$. Then $\mathbf{C}(x) = \{A_1 \times (\prod_{n\geq 2} L_n)\} \cup \{B \mid x \in B \in (\mathfrak{A}_\Gamma \setminus \mathfrak{A}_L)\}$ is a full subalgebra of Γ, but $\mathbf{C}(x)$ is not path-connected since two blocks A and $C = A_1 \times (\prod_{n\geq 2} B_n)$ are nonpath-connected in Γ by proposition 1.4. □

The following corollary follows by the given OML $L = \prod_{n\geq 1} L_n$ in theorem 2.6 which is nonpath-connected by proposition 1.4 and weakly path-connected by our constructive method. Hence $\mathbf{C}(x)$ satisfies the following conclusion.

COROLLARY 2.7. *There is a path-connected OML with a weakly path-connected full subalgebra $\mathbf{C}(x)$ for some $x \in L$.*

References

[1] Bruns, G., *Block-finite Orthomodular Lattices*, Can. J. Math. **31** (1979), 961-985.

[2] Bruns, G. and Greechie, R., *Blocks and Commutators in Orthomodular Lattices*, Algebra Universalis **27** (1990), 1-9.
[3] Greechie, R., *On the Structure of Orthomodular Lattices Satisfying the Chain Condition*, J. of Combinatorial Theory **4** (1968), 210-218.
[4] Greechie, R. and Herman, L., *Commutator-finite Orthomodular Lattices*, Order **1** (1985), 277-284.
[5] Kalmbach, G., *Orthomodular Lattices*, Academic Press Inc. (London) Ltd. (1983).
[6] Park, E., *Path-connected Orthomodular Lattices*, Kansas State University Ph. D. Thesis (1989).
[7] ———, *A Note on Relatively Path-connected Orthomodular Lattices*, Bull. of Korean Math. Soc. **31** (1994), 61-72.
[8] ———, *Note on Nonpath-connected Orthomodular Lattices*, Comm. of Korean Math. Soc. **10** (1995), 285-292.

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