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AN ISOMORPHISM OF THE COUSIN COMPLEXES

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ABSTRACT. Let $\mathcal{C}(\mathcal{F}, M)$ and $\mathcal{C}(S^{-1}\mathcal{F}, S^{-1}M)$ be Cousin complexes for a module M and a module $S^{-1}M$ over a commutative Noetherian ring with respect to a filtration $\mathcal{F}$ and a filtration $S^{-1}\mathcal{F}$ respectively. In this paper, it is shown that there is an isomorphism between the Cousin complexes $S^{-1}\mathcal{C}(\mathcal{F}, M)$ and $\mathcal{C}(S^{-1}\mathcal{F}, S^{-1}M)$.

1. Introduction

Throughout ths paper, we let A denote a commutative Noetherian ring with non-zero identity and M denote an A-module and $\mathcal{C}(A)$ denote the category of A-modules and A-homomorphisms between them.

R.Y.Sharp introduced the concept of the Cousin complex for a module M in [1]. Since then the Cousin complex has been applied to many fields in commutative algebra; for instance a non-zero finitely generated A-module M is Cohen-Macaulay if and only if the Cousin complex $\mathcal{C}(\mathcal{F}, M)$ is exact (See Theorem 2.4 in [2]). Also the Cousin complex provides a natural minimal injective resolution for a Gorenstein ring (See Theorem 5.4 in [1]). The close relation between a Cousin complex and a generalized Hughes complex has been deeply investigated. Our purpose of this paper is to give an alternative proof by using the generalized Hughes complex to obtain the isomorphism between Cousin complexes $S^{-1}\mathcal{C}(\mathcal{F}, M)$ and $\mathcal{C}(S^{-1}\mathcal{F}, S^{-1}M)$ which is the one of the main theorems in [1].

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2. Preliminaries

DEFINITION 2.1. A filtration of $Spec(A)$ is a descending sequence $\mathcal{F} = (F_i)_{i \in \mathbf{N}_0}$ of subsets of $Spec(A)$, so that

$$Spec(A) \supseteq F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots$$

with the property that, for each $i \in \mathbf{N}_0$, each member of $\partial F_i = F_i - F_{i+1}$ is a minimal member of F_i with respect to inclusion. We say that $\mathcal{F}$ admits M if $Supp(M) \subseteq F_0$.

Whenever we have a filtration $\mathcal{F}$ of $Spec(A)$ which admits M, we can construct a Cousin complex $\mathcal{C}(\mathcal{F}, M)$ for M with respect to $\mathcal{F}$. The Cousin complex $\mathcal{C}(\mathcal{F}, M)$ for M with respect to $\mathcal{F}$ has the form

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots$$

where, for each $n \in \mathbf{N}_0$,

$$M^n = \bigoplus_{\mathfrak{p} \in \partial F_n} (Coker\, d^{n-2})_{\mathfrak{p}}.$$

The homomorphisms in this complex have the following property: for each $m \in M$ and $\mathfrak{p} \in \partial F_0$, the component of $d^{-1}(m)$ in $M_{\mathfrak{p}}$ is $m/1$ and for $n > 0$, $x \in M^{n-1}$ and $\mathfrak{q} \in \partial F_n$, the component of $d^{n-1}(x)$ in $(Coker\, d^{n-2})_{\mathfrak{q}}$ is $\pi(x)/1$, where $\pi : M^{n-1} \longrightarrow Coker\, d^{n-2}$ is the canonical epimorphism. The fact that such a complex can be constructed is explained in [3].

DEFINITION 2.2. A system of ideals of A is a non-empty set Φ of ideals of A such that, whenever $I, J \in \Phi$, there exist $K \in \Phi$ such that $K \subseteq IJ$. Such a system of ideals determines a funtor $\Gamma_\Phi : \mathcal{C}(A) \to \mathcal{C}(A)$ for which

$$\Gamma_\Phi(M) = \{m \in M | Im = 0 \quad \text{for some} \quad I \in \Phi\}$$

for each A-module M.

Let Φ be a system of ideals of A. Since Φ is a direct set with respect to reverse inclusion, it is easy to procuce funtors

$$D_\Phi := \varinjlim_{I \in \Phi} Hom_A(I, \quad), \quad \text{and} \quad H^1_\Phi := \varinjlim_{I \in \Phi} Ext^1_A(A/I, \quad)$$

from $\mathcal{C}(A)$ to itself. We can similarly define a functor

$$\varinjlim_{I \in \Phi} Hom_A(A/I, \quad)$$

from $\mathcal{C}(A)$ to itself which is naturally equivalent to Γ_Φ.

For each A-module M and each $I \in \Phi$, the exact sequence

$$0 \to I \to A \to A/I \to 0$$

induces an exact sequence

$$0 \to Hom_A(A/I, M) \to Hom_A(A, M) \to Hom_A(I, M) \to Ext^1_A(A/I, M) \to 0.$$

It follows that there is an exact sequence

$$0 \to \Gamma_\Phi(M) \to M \overset{\eta_\Phi(M)}{\to} D_\Phi(M) \overset{\xi_\Phi(M)}{\to} H^1_\Phi(M) \to 0$$

in $\mathcal{C}(A)$. Furthermore, $\eta_\Phi(M)$ and $\xi_\Phi(M)$ constitutes morphisms of functors

$$\eta_\Phi : Id \to D_\Phi \quad \text{and} \quad \xi_\Phi : D_\Phi \to H^1_\Phi$$

from $\mathcal{C}(A)$ to itself.

Let $(\Phi_i)_{i \in \mathbf{N}}$ be a family of systems of ideals of A. It will be convenient to write $K^{-2} = 0$, $K^{-1} = M$ and $h^{-2} : K^{-2} \to K^{-1}$ to denote the zero homomorphism. Set $E^0 = M$, and let $\pi_0 : K^{-1} \to E^0$ be the identity mapping of M to itself.

Suppose, inductively, that $n \in \mathbf{N}_0$ and we have constructed a complex

$$0 \to M \overset{h^{-1}}{\to} K^0 \to \cdots \to K^{n-2} \overset{h^{n-2}}{\to} K^{n-1},$$

and A-module E^n and an epimorphism $\pi_n : K^{n-1} \to E^n$ for which the sequence

$$K^{n-2} \overset{h^{n-2}}{\longrightarrow} K^{n-1} \overset{\pi_n}{\to} E^n \to 0$$

is exact. To construct the next term and the next homomorphism we apply the ideas above to the system of ideals Φ_{n+1} and A-module E^n. We obtain an exact sequence

$$E^n \overset{\eta_{\Phi_{n+1}}(E^n)}{\longrightarrow} D_{\Phi_{n+1}}(E^n) \overset{\xi_{\Phi_{n+1}}(E^n)}{\longrightarrow} H^1_{\Phi_{n+1}}(E^n) \to 0.$$

We define $K^n = D_{\Phi_{n+1}(E^n)}$ and $E^{n+1} = H^1_{\Phi_{n+1}}(E^n)$, and we set $h^{n-1} = \eta_{\Phi_{n+1}}(E^n) \circ \pi_n : K^{n-1} \to K^n$ and $\pi_{n+1} = \xi_{\Phi_{n+1}}(E^n) : K^n \to E^{n+1}$. Since $h^{n-1} \circ h^{n-2} = \eta_{\Phi_{n+1}}(E^n) \circ \pi_n \circ h^{n-2} = 0$, and we have an exat sequence $K^{n-1} \overset{h^{n-1}}{\longrightarrow} K^n \overset{\pi_{n+1}}{\longrightarrow} E^{n+1} \to 0$, the inductive step in the construction is complete.

The complex that results from this constrution is called the generalized Hughes complex for M with respect to the family of systems of ideals $(\Phi_n)_{n \in \mathbb{N}}$ and is denoted by $\mathcal{H}\left((\Phi_n)_{n\in\mathbb{N}}, M\right)$.

3. Main Theorem

Let $\mathcal{F} = (F_i)_{i \in \mathbf{N}_0}$ be a filtration of $Spec(A)$, and $V(I)$ denote the variety of I. For each $n \in \mathbf{N}$ the set

$$\Phi_n(\mathcal{F}) = \{I | \text{ I is an ideal of A such that } V(I) \subseteq F_n\}$$

is a system of ideals of A. In this situation Sharp and Schenzel proved the following lemma.

LEMMA 3.1. *Let $\mathcal{F} = (F_i)_{i \in \mathbf{N}_0}$ be a filtration of $Spec(A)$ and let $(\Phi_n(\mathcal{F}))_{n\in\mathbf{N}}$ be the family of systems of ideals of A. Then the Cousin complex $C(\mathcal{F}, M)$ for a module M with respect to $\mathcal{F}$ is isomorphic to the generalized Hughes complex $\mathcal{H}\left((\Phi_n(\mathcal{F}))_{n\in\mathbf{N}}, M\right)$ for a module M with respect to $(\Phi_n(\mathcal{F}))_{n\in\mathbf{N}}$.*

PROOF. See Theorem 2.3 in [4]. □

LEMMA 3.2. *Let $f : A \to B$ be a flat homomorphism of rings and Φ a system of ideals of A. Then two functors $D_\Phi(\quad) \otimes_A B$ and $D_{\Phi^f}(\quad \otimes B)$ are natually equivalent, where $\Phi^f = \{I^e | I \in \Phi\}$.*

PROOF. Let M be an A-module. For every two ideals $I, J \in \Phi$ with $I \leq J$, the inclusion map $\lambda_I^J : J \to I$ gives rise to a diret system

$$(Hom_A(I, M) \otimes_A B)_{I\in\Phi} \quad \text{and} \quad \left(Hom_A(\lambda_I^J, id_M) \otimes id_B\right)_{(I,J)\in\Phi\times\Phi}$$

of B-modules and B-homomorphisms over the direct set Φ. Hence there exists $\varinjlim_{I\in\Phi} (Hom_A(I, M) \otimes_A B)$. For any $I, J \in \Phi$ with $I \leq J$, the diagram

$$\begin{array}{ccc} Hom_A(I, M) \otimes_A B & \xrightarrow{\simeq} & Hom_B(I^e, M \otimes_A B) \\ \downarrow & & \downarrow \\ Hom_A(J, M) \otimes_A B & \xrightarrow{\simeq} & Hom_B(J^e, M \otimes_A B) \end{array}$$

commutes. Therefore we obtain a B-isomorphism

$$\alpha_M : \varinjlim_{I\in\Phi} (Hom_A(I, M) \otimes_A B) \to \varinjlim_{I\in\Phi} Hom_B(I^e, M \otimes_A B).$$

By the universal property of direct limits, there exists a natural B-homomorphism

$$\beta_M : \varinjlim_{I\in\Phi} (Hom_A(I, M) \otimes_A B) \to \left(\varinjlim_{I\in\Phi} Hom_A(I, M)\right) \otimes_A B.$$

Since direct limit commutes with a tensor product, β_M is indeed a B-isomorphism. Hence we have a B-isomorphism

$$\alpha_M\beta_M^{-1} : \left(\varinjlim_{I\in\Phi} Hom_A(I, M)\right) \otimes_A B \to \varinjlim_{I\in\Phi} Hom_B(I^e, M \otimes_A B).$$

On the other hand, we have a direct system

$$(Hom_B(I^e, M \otimes B))_{I^e\in\Phi^f} \quad \text{and} \quad \left(Hom_B(\lambda_{I^e}^{J^e}, id_{M\otimes B})\right)_{(I^e,J^e)\in\Phi^f\times\Phi^f}$$

of B-modules and B-homomorphisms over the direct set Φ^f. Hence there exists $\varinjlim_{I^e \in \Phi^f} (Hom_B(I^e, M \otimes B))$. Now for any $I \leq J$ in Φ, the commutative diagram

$$\begin{array}{ccc} Hom_B(I^e, M \otimes_A B) & \xrightarrow{natural} & Hom_B(J^e, M \otimes_A B) \\ & \searrow & \downarrow \\ & & \varinjlim_{I^e \in \Phi_f} Hom_B(I^e, M \otimes B) \end{array}$$

implies that there exists a natural B-homomorphism

$$\gamma : \varinjlim_{I \in \Phi} Hom_B(I^e, M \otimes B) \to \varinjlim_{I^e \in \Phi^f} Hom_B(I^e, M \otimes_A B).$$

It is easy to see that γ is a B-isomorphism. Therefore we have

$$\begin{aligned} D_\Phi(M) \otimes_A B &= \left(\varinjlim_{I \in \Phi} Hom_A(I, M) \right) \otimes_A B \\ &\simeq \varinjlim_{I \in \Phi} Hom_B(I^e, M \otimes_A B) \\ &\simeq \varinjlim_{I^e \in \Phi^f} Hom_B(I^e, M \otimes_A B) \\ &= D_{\Phi^f}(M \otimes_A B). \end{aligned}$$

Moreover, if N is a second A-module and $g : M \to N$ is an A-homomorphism, the following diagram

$$\begin{array}{ccc} D_\Phi(M) \otimes_A B & \xrightarrow{\sim} & D_{\Phi^f}(M \otimes_A B) \\ {\scriptstyle D_\Phi(g) \otimes id_B} \downarrow & & \downarrow {\scriptstyle D_{\Phi^f}(g \otimes id_B)} \\ D_\Phi(N) \otimes_A B & \xrightarrow{\sim} & D_{\Phi^f}(N \otimes_A B) \end{array}$$

commutes. The proof is now complete. □

If we set, for each $i \in \mathbf{N}_0$,

$$F_i = \{\mathfrak{p} \in Supp(M) | ht_M \mathfrak{p} \geq i\},$$

$\mathcal{F} = (F_i)_{i\in\mathbf{N}_0}$ is a filtration of $Spec(A)$ which admits M, so that we can construct the Cousin complex $\mathcal{C}(\mathcal{F}, M)$. Now let S be a multiplicative closed subset of A. Since

$$Supp_{S^{-1}A}(S^{-1}M) = \{S^{-1}\mathfrak{p} | \mathfrak{p} \in Supp_A(M) \text{ and } \mathfrak{p} \cap S = \emptyset\}$$

it follows that if $\mathfrak{p} \in Supp_A(M)$ with $\mathfrak{p} \cap S = \emptyset$,then $ht_{S^{-1}M}S^{-1}\mathfrak{p} = ht_M\mathfrak{p}$. Here, for $\mathfrak{p} \in Supp(M)$, the notation $ht_M\mathfrak{p}$ denotes the M-height of $\mathfrak{p}$, that is the dimension of the $A_\mathfrak{p}$ – module $M_\mathfrak{p}$. Therefore if we set $S^{-1}F_i = \{S^{-1}\mathfrak{p} | \mathfrak{p} \in F_i \text{ and } \mathfrak{p} \cap S = \emptyset\}$, $S^{-1}\mathcal{F} = (S^{-1}F_i)_{i\in\mathbf{N}_0}$ is a filtration of $Spec(S^{-1}A)$, so we can construt the Cousin complex $\mathcal{C}_{S^{-1}A}(S^{-1}\mathcal{F}, S^{-1}M)$. In this situation, we have the following theorem.

THEOREM 3.3. *There is an isomorphism of complexes of $S^{-1}A$-modules and $S^{-1}A$-homomorphisms*

$$\Psi = \psi^n{}_{n\geq -1} : S^{-1}\mathcal{C}(\mathcal{F}, M) \to \mathcal{C}(S^{-1}\mathcal{F}, S^{-1}M)$$

which is such that $\psi^{-1} : S^{-1}M \to S^{-1}M$ is the identity.

PROOF.

$$\begin{aligned} S^{-1}(\mathcal{C}(\mathcal{F}, M)) &\simeq \mathcal{C}(\mathcal{F}, M) \otimes_A S^{-1}A \\ &\simeq \mathcal{H}((\Phi_n(\mathcal{F})), M) \otimes_A S^{-1}A \quad \text{by Lemma 3.1} \\ &\simeq \mathcal{H}((\Phi_n^f(\mathcal{F})), S^{-1}M) \quad \text{by Lemma 3.2} \\ &\simeq \mathcal{C}(S^{-1}\mathcal{F}, S^{-1}M.) \end{aligned}$$

□

References

[1] R. Y. Sharp, *The Cousin complex for a module over a commutative Noetherian ring*, Math. Zeit. **112** (1969), 340-356.
[2] ———, *Gorenstein modules*, Math. Zeit. **115** (1970), 117-139.
[3] ———, *A Cousin complex characterization of balanced big Cohen-Macaulay modules*, Quart. J. Math. Oxford **33** (1982), 471-485.
[4] ———and P.Schenzel, *Cousin complexes and generalized Hughes complexes*, Proc. London Math. Soc. **68** (1994), 499-517.

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