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A FORMULA RELATED TO FRACTIONAL CALCULUS

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ABSTRACT. We give a formula related to fractional calculus which may be useful in solving some fractional linear differential equations. We also give a brief survey of the history of the fractional calculus.

1. Introduction and Preliminaries

The concept of the differentiation operator $D = d/dx$ is familiar to all who have studied the elementary calculus. And for suitable functions f, the nth derivative of f, namely $D^n f(x) = d^n f(x)/dx^n$ is well-defined provided that n is a positive integer. In 1695, L'Hôpital inquired of Leibniz what meaning could be ascribed to $D^n f$ if n were a fraction. Since that time the fractional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent. But it was not until 1884 that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematician. By then the theory had been extended to include operators D^ν, where ν could be rational or irrational, positive or negative, real or complex. Thus the name fractional calculus became somewhat of a misnomer. A better description might be differentiation and integration to an arbitrary order. However, by tradition this theory is referred to as the fractional calculus. For more and detailed historical survey, see [1], [4], [6].

As noted in [4], in the period 1975 to the present, about 400 papers have been published relating to the fractional calculus. The fractional calculus have been used in many fields of science and engineering. We give some references used in mathematics [2], [3], [5].

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More than one version of the fractional integral exists. We introduce only one version here.

Let $\mathrm{Re}(\nu) > 0$ and let f be piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $[0, \infty)$. Then for $t > 0$ we call

$$ {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi)\, d\xi \tag{1.1}$$

the Riemann-Liouville fractional integral of f of order ν and Γ is the well-known Gamma function. We also denote ${}_0D_t^{-\nu}$ as $D^{-\nu}$. In particular setting $f(t) = e^{at}$ (a constant) in (1.1) yields

$$ D^{-\nu} e^{at} = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} e^{a\xi}\, d\xi, \quad \nu > 0. \tag{1.2}$$

If we make the change of variable $x = t - \xi$, (1.2) becomes

$$ D^{-\nu} e^{at} = \frac{e^{at}}{\Gamma(\nu)} \int_0^t x^{\nu-1} e^{-ax}\, dx, \quad \nu > 0, \tag{1.3}$$

which is not an elementary function. For $\mathrm{Re}(\nu) > 0$ the incomplete gamma function $\gamma^*(\nu, t)$ may be defined as

$$ \gamma^*(\nu, t) = \frac{1}{\Gamma(\nu) t^{\nu}} \int_0^t \xi^{\nu-1} e^{-\xi}\, d\xi. \tag{1.4}$$

Thus we may write (1.2) as

$$ D^{-\nu} e^{at} = t^{\nu} e^{at} \gamma^*(\nu, at). \tag{1.5}$$

Since the right-hand side of (1.5) is the fractional integral of an exponential, it is not surprising that this function frequently arises in the study of the fractional calculus. We shall call it $E_t(\nu, a)$,

$$ E_t(\nu, a) = t^{\nu} e^{at} \gamma^*(\nu, at). \tag{1.6}$$

Now the concept of the fractional derivative is introduced. Let f satisfy conditions of the fractional integral (1.1) and let $\mu > 0$. Let m be the smallest integer that exceeds μ. Then the fractional derivative of f of order μ is defined as

$$D^{\mu} f(t) = D^m[D^{-\nu} f(t)], \ \ \mu > 0, \ \ t > 0 \tag{1.7}$$

(if it exists) where $\nu = m - \mu > 0$.

Suppose that $f(t) = e^{at}$. Then the fractional derivative of e^{at} of order μ is

$$D^{\mu} e^{at} = D^m[D^{-\nu} e^{at}], \tag{1.8}$$

where μ, ν and m have the same meaning as above.

With the help of (1.6), we readily obtain

$$D^m E_t(\nu, a) = E_t(\nu - m, a) = E_t(-\mu, a) \tag{1.9}$$

since $\mu = m - \nu$. Thus we conclude that

$$D^{\mu} e^{at} = E_t(-\mu, a), \ \ t > 0. \tag{1.10}$$

Now we introduce the concept of fractional differential equations. As a first attempt define a fractional differential equation, let $r_m, r_{m-1}, \cdots, r_0$ be a strictly deceasing sequence of nonnegative numbers. Then if $b_1, b_2, \cdots, b_m$ are constants,

$$[D^{r_m} + b_1 D^{r_{m-1}} + \cdots + b_m D^{r_0}]\, y(t) = 0 \tag{1.11}$$

is a candidate. But even this equation is a little too complex. We shall impose the additional requirement that the r_j be rational numbers. Thus if q is the least common multiple of the denominators of the nonzero r_j, we may write (1.11) as

$$\left[D^{n\nu} + a_1 D^{(n-1)\nu} + \cdots + a_n D^0\right] y(t) = 0, \qquad t \geq 0, \tag{1.12}$$

where

$$\nu = \frac{1}{q}. \tag{1.13}$$

If $q = 1$, then $\nu = 1$, and (1.12) is simply an ordinary differential equation. We shall call (1.12) a fractional linear differential equation with constant coefficients of order (n, q), or more briefly, a fractional differential equation of order (n, q).

In this note we are aiming at getting a formula which can be occurred in solving the differential equation (1.12).

2. Explicit representation of the solution of (1.12)

Let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \tag{2.1}$$

be the corresponding indicial polynomial to (1.12). Let

$$y_1(t) = \mathcal{L}^{-1}\left\{P^{-1}(s^\nu)\right\}, \tag{2.2}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

Then if N is the smallest integer with the property that $N \geq n\nu$,

$$y_1(t),\, y_2(t),\, \dots,\, y_N(t),$$

where

$$y_{j+1}(t) = D^j y_1(t), \qquad j = 0, 1, \dots, N-1$$

are N linearly independent solutions of (1.12) which is well-known.

We introduce the Laplace transform method to solve the equation (1.12). Let $\alpha_1, \dots, \alpha_n$ with $\alpha_i \neq \alpha_j$ for $i \neq j$ be the zeros of $P(x)$ and let

$$A_m^{-1} = DP(\alpha_m), \qquad m = 1, 2, \dots, n. \tag{2.3}$$

But

$$P^{-1}(s^\nu) = \frac{A_1}{s^\nu - \alpha_1} + \frac{A_2}{s^\nu - \alpha_2} + \cdots + \frac{A_n}{s^\nu - \alpha_n}, \tag{2.4}$$

and we can readily obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\nu - a}\right\} = \sum_{j=1}^{q} a^{j-1} E_t(j\nu - 1, a^q). \tag{2.5}$$

If we make the change of dummy index of summation $k = q - j$ in the sum of (2.5),

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\nu - a}\right\} = \sum_{k=0}^{q-1} a^{q-k-1} E_t(-k\nu, a^q). \tag{2.6}$$

Using (2.6) we see the inverse Laplace transform of (2.4) yields

$$y_1(t) = \sum_{m=1}^{n} A_m \sum_{k=0}^{q-1} \alpha_m^{q-k-1} E_t(-k\nu, \alpha_m^q) \tag{2.7}$$

is a solution of (1.12). Hence $y_2, y_3 \ldots , y_N$ of the solutions of (1.12) may be constructed from (2.7).

We begin our method of solving (1.12) by recalling that

$$D^{p\nu} E_t(-k\nu, a) = E_t(-(k+p)\nu, a) \tag{2.8}$$

provided that $k\nu < 1$, which it is for $k = 0, 1, \ldots , q-1$.

If we define $e(t)$ as

$$e(t) = \sum_{k=0}^{q-1} \alpha^{q-k-1} E_t(-k\nu, \alpha^q) \tag{2.9}$$

(where for the moment α is an arbitrary constant), then (2.8) implies that

$$D^{\nu} e(t) = \alpha e(t). \tag{2.10}$$

For p a positive integer greater than 1,

$$D^{p\nu} e(t) = \alpha^p e(t) + \sum_{k=1}^{p-1} \frac{\alpha^{p-1-k} t^{-1-k\nu}}{\Gamma(-k\nu)}. \tag{2.11}$$

Formula (2.11) is also valid for $p = 0$ or 1, since in these cases the sum in (2.11) is vacuous. Thus if we write

$$P(D^{\nu}) e(t) = \left(\sum_{p=0}^{n} a_{n-p} D^{p\nu} \right) e(t), \quad a_0 = 1, \tag{2.12}$$

(2.11) implies that

$$P(D^{\nu}) e(t) = P(\alpha) e(t) + \sum_{p=2}^{n} a_{n-p} \sum_{k=1}^{p-1} \frac{\alpha^{p-1-k} t^{-1-k\nu}}{\Gamma(-k\nu)}. \tag{2.13}$$

Now if $\alpha_1, \dots, \alpha_n$ are the distinct zeros of $P(x)$, and if

$$e_j(t) = \sum_{k=0}^{q-1} \alpha_j^{q-k-1} E_t(-k\nu, \alpha_j^q), \tag{2.14}$$

then certainly, from (2.13),

$$\begin{aligned} P(D^\nu)\left[\sum_{m=1}^{n} c_m e_m(t)\right] &= \sum_{m=1}^{n} c_m P(\alpha_m) e_m(t) \\ &+ \sum_{m=1}^{n} c_m \sum_{p=2}^{n} a_{n-p} \sum_{k=1}^{p-1} \frac{\alpha_m^{p-1-k} t^{-1-k\nu}}{\Gamma(-k\nu)} \end{aligned} \tag{2.15}$$

for any arbitrary constants $c_1, c_2, \dots, c_n$. But since $\alpha_1, \dots, \alpha_n$ are the roots of $P(x) = 0$, the first term on the right-hand side of (2.15) vanishes and
(2.16)

$$P(D^\nu)\left[\sum_{m=1}^{n} c_m e_m(t)\right] = \sum_{p=2}^{n} a_{n-p} \sum_{k=1}^{p-1} \frac{t^{-1-k\nu}}{\Gamma(-k\nu)} \left[\sum_{m=1}^{n} c_m \alpha_m^{p-1-k}\right].$$

Thus if we can choose the c_m such that the right-hand side of (2.16) vanishes, then

$$y_1(t) = \sum_{m=1}^{n} c_m e_m(t) \tag{2.17}$$

will be a solution of (1.12).

Ignoring the trivial solution $c_1 = c_2 = \cdots = c_n = 0$, we see that if we let $c_m = A_m$, where the A_m are given by (2.3), then we know that the sum in brackets in (2.16) is zero. Thus

$$y_1(t) = \sum_{m=1}^{n} A_m e_m(t) \tag{2.18}$$

is a solution of (1.12). But this is precisely (2.7). Note that in the latter method, the role of (2.11) is important.

If the indicial polynomial $P(x)$ in (2.1) has a double zero at α, then $e(t) * e(t)$ is used to solve (1.12), where $e(t) * e(t)$ is the convolution of $e(t)$ with itself. We also have an explicit representation of $e(t) * e(t)$ in terms of the E_t functions, namely,

$$\begin{aligned} e(t) * e(t) =& \mathcal{L}^{-1}\left\{(s^\nu - \alpha)^{-2}\right\} \\ =& \sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \alpha^{2q-j-k-2}\{tE_t(-(j+k)\nu, \alpha^q) \\ &+ (j+k)\nu E_t(1-(j+k)\nu, \alpha^q)\}. \end{aligned} \tag{2.19}$$

Finally we observe that an explicit representation of $D^{p\nu}e(t) * e(t)$ is as important as that of $D^{p\nu}e(t)$ in (2.11) in solving (1.12) if the corresponding indicial polynomial $P(x)$ has a double zero at α. Fortunately we can give an explicit representation of $D^{p\nu}e(t) * e(t)$: For p a positive integer, we have

$$\begin{aligned} D^{p\nu}e(t) * e(t) =& \alpha^p e(t) * e(t) + p\alpha^{p-1}e(t) \\ &+ \sum_{j=0}^{p-1}\sum_{k=0}^{j-1} \alpha^{p-k-2}\frac{t^{-k\nu-1}}{\Gamma(-k\nu)}. \end{aligned} \tag{2.20}$$

3. Proof of (2.20)

First we give some elementary identities needed to our purpose without proof :

$$D^\mu E_t(\lambda, a) = E_t(\lambda - \mu, a). \tag{3.1}$$

$$D^\mu tE_t(\omega, c) = tE_t(\omega - \mu, c) + \mu E_t(\omega - \mu + 1, c), \quad \omega > -2. \tag{3.2}$$

$$E_t(-1, a) = aE_t(0, a). \tag{3.3}$$

$$E_t(\nu, a) = aE_t(\nu + 1, a) + \frac{t^\nu}{\Gamma(\nu + 1)}. \tag{3.4}$$

For convenience, let

$$F(p\nu) = D^{p\nu} e(t) * e(t).$$

With the aid of (3.1) and (3.2) we obtain

$$\begin{aligned} F(p\nu) =& \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} \alpha^{2q-j-k-2} \{D^{p\nu} t E_t(-(j+k)\nu, \alpha^q) \\ &+ (j+k)\nu D^{p\nu} E_t(1-(j+k)\nu, \alpha^q)\} \\ =& \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} \alpha^{2q-j-k-2} \{t E_t(-(j+k+p)\nu, \alpha^q) \\ &+ p\nu E_t(1-(j+k+p)\nu, \alpha^q) \\ &+ (j+k)\nu E_t(1-(j+k+p)\nu, \alpha^q)\}. \end{aligned}$$

Setting $j' = j + p$ in the resulting equation and dropping the prime on j', we get

$$\begin{aligned} F(p\nu) =& \sum_{j=p}^{q+p-1} \sum_{k=0}^{q-1} I(p,q,j,k) \\ =& \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} I(p,q,j,k) + \sum_{j=q}^{q+p-1} \sum_{k=0}^{q-1} I(p,q,j,k) \\ &- \sum_{j=0}^{p-1} \sum_{k=0}^{q-1} I(p,q,j,k), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} I(p,q,j,k) =& \alpha^{2q-j-k+p-2} \{t E_t(-(j+k)\nu, \alpha^q) \\ &+ (j+k)\nu E_t(1-(j+k)\nu, \alpha^q)\}. \end{aligned}$$

Setting $j' = j - q$ in the second sum of (3.5) and dropping the prime on j' yields

$$F(p\nu) = \alpha^p e(t) * e(t)$$

(3.6)
$$+ \sum_{j=0}^{p-1}\sum_{k=0}^{q-1} \alpha^{q-j-k+p-2}\{tE_t(-(j+k)\nu - 1, \alpha^q)$$
$$+ ((j+k)\nu + 1)E_t(-(j+k)\nu, \alpha^q) - \alpha^q t E_t(-(j+k)\nu, \alpha^q)$$
$$- \alpha^q (j+k)\nu E_t(1-(j+k)\nu, \alpha^q)\}.$$

Using (3.4) we find that

$$tE_t(-(j+k)\nu - 1, \alpha^q) = \alpha^q t E_t(-(j+k)\nu, \alpha^q) + \frac{t^{-(j+k)\nu}}{\Gamma(-(j+k)\nu)},$$

(3.7)
$$(j+k)\nu E_t(-(j+k)\nu, \alpha^q) = (j+k)\nu\alpha^q E_t(1-(j+k)\nu, \alpha^q) - \frac{t^{-(j+k)\nu}}{\Gamma(-(j+k)\nu)}.$$

Putting (3.7) into (3.6) leads to

(3.8)
$$F(p\nu) = \alpha^q e(t) * e(t) + \sum_{k=0}^{q-1}\sum_{j=0}^{p-1} \alpha^{q-j-k+p-2} E_t(-(j+k)\nu, \alpha^q).$$

Setting $j + k = k'$ in the summation part of (3.8) and dropping the prime on k', we obtain

(3.9)
$$\sum_{j=0}^{p-1}\sum_{k=0}^{q-1} \alpha^{q-j-k+p-2} E_t(-(j+k)\nu, \alpha^q) = \sum_{j=0}^{p-1}\sum_{k=j}^{q+j-1} \alpha^{q-k+p-2} E_t(-k\nu, \alpha^q).$$

Now we also separate the inner sum in the right side of (3.9) into three parts:

(3.10)
$$\sum_{k=j}^{q+j-1} \alpha^{q-k+p-2} E_t(-k\nu, \alpha^q) = \sum_{k=0}^{q-1} \alpha^{q-k+p-2} E_t(-k\nu, \alpha^q)$$
$$+ \sum_{k=q}^{q+j-1} \alpha^{q-k+p-2} E_t(-k\nu, \alpha^q) - \sum_{k=0}^{j-1} \alpha^{q-k+p-2} E_t(-k\nu, \alpha^q).$$

Setting $k-q=k'$ in the second sum of the right side of (3.10) and dropping the prime on k' with the aid of (3.4) and (2.9) yields

$$(3.11)\quad \sum_{k=j}^{q+j-1} \alpha^{q-k+p-2} E_t(-k\nu, \alpha^q) = \alpha^{p-1} e(t) + \sum_{k=0}^{j-1} \alpha^{-k+p-2} \frac{t^{-k\nu-1}}{\Gamma(-k\nu)}.$$

Finally putting (3.11) into (3.9) and considering (3.8) leads to the desired result (2.20).

We conclude this note by giving some special cases of (2.20). Setting $p=1,\ 2,\ 3$ in (2.20) with $1/\Gamma(0)=0$, we obtain

$$(3.12)\quad \begin{aligned} D^{\nu}e(t) * e(t) &= \alpha e(t) * e(t) + e(t), \\ D^{2\nu}e(t) * e(t) &= \alpha^2 e(t) * e(t) + 2\alpha e(t), \\ D^{3\nu}e(t) * e(t) &= \alpha^3 e(t) * e(t) + 3\alpha^2 e(t) + \frac{t^{-\nu-1}}{\Gamma(-\nu)}. \end{aligned}$$

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