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ON REFLECTED DIFFUSION WITH DISCONTINUOUS COEFFICIENT

YOUNGMEE KWON

ABSTRACT. Consider a d-dimensional domain D that has finite Lebesque measure and a Dirichlet form which has discontinuous coefficient. Then the stationary Markov process corresponding to the given Dirichlet form is a semimartingale under suitable condition for D and the coefficient.

1. Introduction

Let A and D be bounded and open in R^n with $\overline{A} \subset D$ and let m denote Lebesque measure on D, normalized so that $m(D) = 1$. Consider a piecewise continuous dxd symmetric matrix valued function $a(x) = (a_{ij}(x))$ such that

$$
\begin{aligned}
a(x) &= a^1(x) \text{ on } A \\
&= a^2(x) \text{ on } D \setminus \overline{A} \\
&= I \text{ on } \partial A
\end{aligned} \tag{1}
$$

where a^1 and a^2 are C^1 on A and $D \setminus \overline{A}$ resepectively and there exist constant M_1, M_2 independent of x such that $|a(x)| \le M_1$, $|\nabla .a(x)| \le M_2$ for $x \in D \setminus \partial A$. Here $\nabla .a(x) = (b^1(x), ..., b^d(x))$ where $|\quad|$ denotes the matrix norm and $b^i(x) = \sum_{j=1}^d \frac{\partial}{\partial x_j} a_{ji}(x)$.

(2) $a(x)$ is uniformly positive definite, that is , there is $\lambda > 0$ such that $\sum_{i,j=1}^d a_{ij}(x) y_i y_j \ge \lambda |y|^2$ for all $y \in R^d$ and $x \in D$.

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Now we consider a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ such that

$$(3)\qquad \begin{aligned} \mathcal{E}(f,f) =& \frac{1}{2}\int_D \nabla f.a\nabla f m(dx), \quad f \in \mathcal{D}(\mathcal{E} \\ \mathcal{D}(\mathcal{E}) =& \{f \in L^2(D,dx) \cap H^1(D) : \mathcal{E}(f,f) < \infty.\} \end{aligned}$$

Here $H^1(D) = W^{1,2}(D)$, the Sobolev space of functions $f \in L^2(D)$ that have all distributional derivatives in $L^2(D)$. Then we concern the stationary Markov process X_t associated to $\mathcal{E}$ and show that under some conditions for D, A, and a, it is a semimartingale,that is, a sum of martingale and bounded variation processes. The association between X and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is following: Let (T_t) be the transition semigroup of X and $(f,g) = \int_D fgm(dx)$. Then

$$\mathcal{E}(f,f) = \lim_{t\to 0}\frac{1}{t}(f, f - T_t f), \quad f \in \mathcal{D}(\mathcal{E}).$$

(4) We define that $C \subset R^d$ has the finitely upper Minkowski content if the following holds:

$$\overline{\lim}_{r\downarrow 0}\frac{m\{x \in C; \quad \text{dist}(x, \partial C \le r)\}}{r} < \infty.$$

This is known to be finite when C is a Lipschitz domain. (See [4])

(5) We assume that D and A have the property (4) and $m(\partial A) = 0$.

Pardoux and Williams showed in [3] that under some condition for D, which is more general than having finite upper Minkowski content and the condition that $a(x)$ is locally Lipschitz continuous with condition (2), there exists the associated stationary Markov process X_t to $\mathcal{E}$ and it is a semimartingle (Theorem 6.1 of [3]). We show that they hold when $a(x)$ is piecewise continuous by approximating $a(x)$ with smooth coeffients.

2. Tightness

Let a domain $A \subset R^d$ be given and for $x \in A$, $\theta(x)$ be the distance of x from ∂A. We need $\delta(x)$, the regularized distance function whose existence is guaranteed by Stein ([4], p.171).

LEMMA 1. *There exists a function $\delta(x) = \delta(x, \partial A)$ defined for $x \in A$ such that*

(i) $c_1\theta(x) \leq \delta(x) \leq c_2\theta(x)$ *for all* $x \in A$ *and*

(ii) δ *is* C^∞ *in* A *and for any multi-index* β, *the* β*- th derivatives* $\delta^{(\beta)}$ *satisfies the following inequality:*

$$|\delta^{(\beta)}(x)| \leq b_\beta(\theta(x))^{1-|\beta|} \qquad \text{for all} \quad x \in A.$$

The constants b_β, c_1 *and* c_2 *are independent of* A. *Now we have an important lemma for* $\delta(x)$.

LEMMA 2. *Suppose* A *has the condition (4). Then there is a finite constant* C *such that for each* $i \in \{1, 2, ..., d\}$,

$$\int_A |\frac{\partial}{\partial x_i} q(\delta(x))| dx \leq C$$

for any monotone function q *defined on* $[0, \infty)$, *which is* C^1 *on* $(0,\infty)$ *and satisfies* $q(0) = 0$ *and* $q(\infty) \equiv \lim_{x\to\infty} q(x) = 1$.

PROOF. In Lemma 2.2 of [5], Williams and Zheng showed that the lemma holds with $B_n \cap A$ instead of A and the constant depends on n where $\{B_n, n = 1, 2, ...\}$ denotes some open sets in R^d such that $\overline{A} \subset \cup_n B_n$ and A has their Condition (2.1) which implies (4). Hence we can choose finitely many B_n such that $A \subset \cup_{n=1}^k B_n$ and take C such that $\int_{A\cap B_n} |\frac{\partial}{\partial x_i} q(\delta(x))| dx \leq C$ for any $n = 1, 2, ..., k$ and the lemma holds. □

REMARK 1. We can extend $\delta(x)$ to R^d such that $\delta(x) = 0$ if $x \notin A$. Then $\delta(x)$ is continuous on R^d and C^∞ on $R^d \setminus \partial A$. From now on, we mean $\delta(x)$ as this extension.

When $a(x) = (a_{ij}(x))$ is C^1, symmetric and uniformly positive definite and D has the property (4), Pardoux and Williams ([3]) showed that the stationary Markov process X_t associated to $\mathcal{E}(f, f) = \frac{1}{2}\int_D \nabla f.a \nabla f m(dx)$ with $\mathcal{D}(E) = \{f \in L^2(D, dx) \cap H^1(D) : \mathcal{E}(f, f) < \infty\}$ is a continuous semimartingale. More precisely

THEOREM 1. *Let* $b(x) = (b^1, ..., b^d)(x)$ *such that* $b^i(x) = \frac{1}{2}\sum_{j=1}^{d}(\frac{\partial}{\partial x_j} a_{ji}(x))$ *and* $\{\mathcal{F}_t^X\}$ *be the filtration generated by* X. *Then for* $t \in [0,1]$.

$$X_t = X_0 + M_t + \int_0^t b(X_s)ds + V_t,$$

where M *is a martingale relative to* $\{\mathcal{F}_t^X\}$ *with* $< M_i, M_j >_t = \int_0^t a_{ij}(X_s) ds$ *and* V *is a* $\{\mathcal{F}_t^X\}$ *adapted process of bounded variation such that for each* $v \in C_c^2(R^d, R^d)$, $E[\int_0^1 v(X_t).dV_t] = -\frac{1}{2}\int_D div(av)m(dx)$.

Now we give the main theorem.

THEOREM 2. *Under the condition (1),(2),(4) and (5) for* $a(x)$, D *and* A, X_t, *the stationary Markov process associated to the Dirichlet form in (3) is a semimartingale with the decomposition such that*

$$X_t = X_0 + M_t + \int_0^t b(X_s)(1_{(X_s \notin \partial A)})ds + L_t + V_t$$

where M_t *is a martingale with* $< M_i, M_j >_t \leq c_1 t$ *and* L_t *and* V_t *are bounded variation processes adapted to* $\{\mathcal{F}_t^X\}$ *such that* $L_t = \int_0^t 1_{(X_s \in \partial A)} dL_s$ *and* $E[\int_0^t v(X_s).dV_s] \leq c_2 t$ *for any function* $v \in C_c^2(R^d, R^d)$ *for some constants* c_1, c_2.

PROOF. We consider regularized distance functions $\delta_1(x)$ and $\delta_2(x)$ for A and $D \setminus \overline{A}$. Then $\delta_1(x) = 0$ if $x \notin A$ and $\delta_2(x) = 0$ if $x \notin D \setminus \overline{A}$. Take increaing functions on $[0, \infty)$, $\{f_n\}$ such that for each n, $f_n \in C^\infty$ on $[0,\infty)$, $f_n(0) = 0$, $f_n(r) = 1$ if $r \geq \frac{1}{n}$. Here the derivative of f_n at 0 means $f_n'(0+)$. We can extend $a^1(x)$ and $a^2(x)$ to D so that they satisfy the condition (1) and (2) on D. Now let

$$c^n(x) = a^1(x)(1 - f_n(\delta_2(x))) + a^2(x)(1 - f_n(\delta_1(x)))$$

where a^1 and a^2 are above extensions. Then c^n is differentiable on D satisfying the condition (2). Let $\mathcal{E}_n(f,f) = \frac{1}{2}\int_D \nabla f.c^n \nabla f m(dx)$ on $\mathcal{D}(\mathcal{E}_n) = \{f \in L^2(D, dx) \cap H^1(D) : \mathcal{E}_n(f,f) < \infty\}$. Let $X_s^{(n)}$ be the stationary Markov process associated to $\mathcal{E}_n$. Then by Theorem 1,

$X_t^{(n)} = X_0^{(n)} + M_t^{(n)} + \int_0^t b^n(X_s^{(n)})ds + V_t^{(n)}$ where $< M_i^{(n)}, M_j^{(n)} >_t = \int_0^t c_{ij}^n(X_s)ds$, $b^{(n)i}(x) = \frac{1}{2}\sum_{j=1}^d \frac{\partial}{\partial x_j} c_{ji}^{(n)}(x)$ and

$$E[\int_0^t v(X_t^{(n)}).dV_t^{(n)}] = -\frac{1}{2}\int_D div(c^n v)m(dx).$$

Here $c^n(x) \to a(x)$ in matrix norm except ∂A. Hence by Lyons and Zheng ([1]), $X_t^{(n)}$ converges weakly to X_t associated to $\mathcal{E}(f,f) = \frac{1}{2}\int_D \nabla f.a\nabla f m(dx)$ since $m(\partial A) = 0$. In fact, Theorem in [1] is with respect to $\mathcal{E}(f,f) = \frac{1}{2}\int_{R^d} \nabla f.a\nabla f dx$ instead of D. But the proof goes also in case of D only if $c^n(x) \to a(x)$ except some measure zero set. Hence for all large n,

$$E[< M_i^{(n)}, M_j^{(n)} >_t] = E[\int_0^t c_{ij}^{(n)}(X_s^{(n)})ds] \le \sup_{x\in D}|a(x)|m(D)t \le c_1 t.$$

Therefore $M_t^{(n)}$ has a subsequence converging to M_t in L^2 norm and M_t is a continuous martingale with $< M_i, M_j >_t \le c_1 t$. Now

$$\begin{aligned} b^{(n)i}(x) &= \sum_{j=1}^d (\frac{\partial}{\partial x_j} c_{ji}^{(n)}(x)) \\ &= \sum_{j=1}^d [\frac{\partial}{\partial x_j} a_{ji}^1(x)(1 - f_n(\delta_2(x))) + \frac{\partial}{\partial x_j}(a_{ji}^2(x)(1 - f_n(\delta_1(x)))] \\ &= \sum_{k=1}^2 \sum_{j=1}^d [(\frac{\partial}{\partial x_j}(a_{ji}^k(x))(1 - f_n(\delta_l(x))) + a_{ji}^k(x)\frac{\partial}{\partial x_j}(1 - f_n(\delta_l(x)))]. \end{aligned}$$

where $l = 2$ if $k = 1$ and $l = 1$ if $k = 2$. $|a_{ji}^k(x)|$ is uniformly bounded for i, j, k by the condition (1). Since $X_t^{(n)}$ is stationary,

$$E[|\int_0^t b^{(n)i}(X_s^{(n)})ds|] \le t\int_D |b^{(n)i}(x)|dx.$$

Now we show that for all n, i, $\int_D |b^{(n)i}(x)|dx \le c_2$ which shows $E[|\int_0^t b^{(n)} (X_s^{(n)})ds|] \le c_2 t$. Let

$$I = \int_D |(\frac{\partial}{\partial x_j}(a^k_{ji}(x)))(1 - f_n(\delta_l(x)))|dx$$

and

$$II = \sum_{k=1}^{2} \int_D |\frac{\partial}{\partial x_j} f_n(\delta_k(x))|dx.$$

Then for all i, j, k, $|\frac{\partial}{\partial x_j}(a^k_{ji}(x))| \le M_2$ by the condition (1), hence $I \le m(D)M_2$. Now for II, for all n, j,

$$\begin{aligned} II \le & \int_D |\frac{\partial}{\partial x_j} f_n(\delta_1(x))|dx + \int_D |\frac{\partial}{\partial x_j} f_n(\delta_2(x))|dx \\ = & \int_A |\frac{\partial}{\partial x_j} f_n(\delta_1(x))|dx + \int_{D\setminus A} |\frac{\partial}{\partial x_j} f_n(\delta_2(x))|dx \\ \le & Cm(D) \end{aligned}$$

by Lemma 2. Simiarly for $v \in C^2_c(R^d, R^d)$, $E[\int_0^t v(X^{(n)}_s).dV^{(n)}_s] = \int_D div(c^n v)m(dx)$ is uniformly bounded. Hence by Meyer-Zheng condition ([2]), $\{X^{(n)}_t\}$ is tight and we have a subsequence for $\{X^{(n)}_t\}$ converging to a semimartingale. But we already know $X^{(n)}_t \to X_t$ in distribution. Hence X_t must be a semimartingale such that $X_t = X_0 + M_t + A_t + V_t$ where M_t is a martingale with $< M_i, M_j >_t \le c_1 t$, A_t and V_t are the weak limits of $\int_0^t b^{(n)}(X^{(n)}_s)ds$ and $V^{(n)}_t$ respectively and $A_t = \int_0^t b(X_s)1_{(X_s \notin \partial A)}ds + L_t$ where $b(x) = \frac{1}{2}(\nabla.a)(x)$ on $D \setminus \partial A$ and L_t varies only when X_t is on ∂A. □

REMARK 2. It is true that $E[\int_0^t 1_{(X^{(n)}_s \in \partial A)}ds] = 0$ but still we do not know $E[\int_0^t 1_{(X_s \in \partial A)}ds] = 0$. If it is true, we have $L_t = 0$.

References

[1] T. Lyons and T. S. Zheng, *Note on convergence of Dirichlet process*, Bull. London Math. Soc **25** (1993), 353-356.

[2] P. A. Meyer and W. Zheng, *Tightness criteria for laws of semimartingales*, Ann. Inst. Henri Poincare **20** (1984), 357-372.

[3] E. Pardoux and R. J. Williams, *Symmetric reflected diffusions*, Ann. Inst. Henri Poincare **30** (1994), 13-62.

[4] E. M. Stein, *Singular Integrals and Differentiability properties of functions*, Princeton University press, 1970.

[5] R. J. Williams and W. Zheng, *On reflecting Brownian motion-a weak convergence Approach*, Ann. Inst. Henri Poincare **26** (1990), 461-488.

Hansung University
389 2 ga Samsun-dong Sungbuk-gu
Seoul 136-792, Korea