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BOUNDS ON PROBABILITY FOR THE OCCURRENCE OF EXACTLY r, t OUT OF m, n EVENTS

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ABSTRACT. Let $A_1, A_2, \cdots, A_m$ and $B_1, B_2, \cdots, B_n$ be two sequences of events on a given probability space. Let X_m and Y_n, respectively, be the number of those A_i and B_j, which occur we establish new upper and lower bounds on the probability $P(X = r, Y = t)$ which improve upper bounds and classical lower bounds in terms of the bivariate binomial moment $S_{r,t}$, $S_{r+1,t}$, $S_{r,t+1}$ and $S_{r+1,t+1}$.

1. Introdution

Let $A_1, A_2, \cdots, A_m$ and $B_1, B_2, \cdots, B_n$ be two sequences of events on the same probability space. Let $X = X_m(A)$ and $Y = Y_n(B)$, respectively, be the number of those A_i and B_j, which occur. Put $S_{o,o} = 1$ and for integers $r \geq 1$ and $t \geq 1$, set

$$S_{r,t} = \sum P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r} \cap B_{j_1} \cap B_{j_2} \cap \cdots B_{j_t}) \tag{1}$$

where the summation $\sum$ is over all subscripts satisfying $1 \leq i_1 < i_2 < \cdots < i_r \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_t \leq n$. It can easily be proved, by turning to the method of indicators, that $S_{r,t}$, $1 \leq r \leq m$, $1 \leq t \leq n$, is called the binomial moment of the vector (X, Y), namely,

$$S_{r,t} = E\left[\binom{X}{r}\binom{Y}{t}\right]. \tag{2}$$

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Bounds by linear combinations of the binomial moments $S_{i,j}$ on the distribution of the vector (X, Y) are called Bonferroni-Type Inequalilities (B-T-I). There are two reasons for developing B-T-I. It can be utilized for obtaining limit theorems for normalized multivariate order statistics [see Galambos] and for actual computations of the bounds for fixed m and n. But the mentioned inequalities may become impractical for one of two reasons ; either (i) $S_{r,t}$ is known only for a very limited number of the values of r and t, or (ii) $S_{r,t}$ is so large that the successive terms, providing the upper and lower bounds, become trivial (Upper bounds exceed one and lower bounds become negative).

In this paper, we establish new upper and lower bounds on the distribution
$P(X = r, Y = t)$ which improve classical upper bounds and lower bounds in terms of $S_{r,t}$, $S_{r+1,t}$, $S_{r,t+1}$ and $S_{r+1,t+1}$.

2. The Results

Using the notation of the introduction, we shall prove the following results.

THEOREM 1. *For integers r, t , m and n with* $1 \leq r < m$, $1 \leq t < n$,

$$P(X = r, Y = t) \geq min(S_{r,t} - (r + 1)S_{r+1,t} \ , \ S_{r,t} - (t + 1)S_{r,t+1}) \tag{3}$$

It is shown by Meyer that

$$P(X = r, Y = t) \geq S_{r,t} - (r + 1)S_{r+1,t} - (t + 1)S_{r,t+1} \tag{4}$$

Thus, it is obvious that inequality (3) is better than (4).

THEOREM 2. *For integers r, t , m and n with* $1 \leq r < m$, $1 \leq t < n$,

$$\begin{aligned} P(X = r, Y = t) \geq\ & S_{r,t} - (r + 1)S_{r+1,t} - (t + 1)S_{r,t+1} \\ & + \frac{(m + n - r - t - 1)}{(m - r)(n - t)}(r + 1)(t + 1)S_{r+1,t+1}. \end{aligned} \tag{5}$$

Thus, it is obvious that inequality (5) is better than (4) if we know $S_{r+1\,,\,t+1}$.

THEOREM 3. *For integers r, t, m and n with* $1 \le r < m$, $1 \le t < n$,

$$(6)\quad P(X=r,Y=t) \le \min\left(S_{r,t} - \frac{r+1}{m-r} S_{r+1,t} \ , \ S_{r,t} - \frac{t+1}{n-t} S_{r,t+1} \right).$$

It is shown by Galambos and Lee(1994) that

$$(7)\qquad P(X=r,Y=t) \le S_{r,t} - \frac{(r+1)(t+1)}{(m-r)(n-t)} S_{r+1,t+1}.$$

Thus, it is obvious that inequality (6) is better than (7).

THEOREM 4. *For integers r, t , m and n with* $1 \le r < m, 1 \le t < n$,

$$(8)\qquad \begin{aligned} P(X=r,Y=t) \le S_{r,t} &- \frac{(r+1)}{(m-r)} S_{r+1,t} - \frac{(t+1)}{(n-t)} S_{r,t+1} \\ &+ \frac{(r+1)(t+1)}{(m-r)(n-t)} S_{r+1,t+1}. \end{aligned}$$

This inequality is same as that of Galambos and Xu(1995).

3. proofs

PROOF OF THEOREM 1. We use the method of indicators. Let

$$I(X=r,Y=t) = I(X=r)I(Y=t) = \begin{cases} 1 & \text{if } X=r \text{ and } Y=t \\ 0 & \text{if otherwise} \end{cases}.$$

By using binomial moment of (2) and the method of indicators, the right hand side of (3) becomes

$$\begin{aligned} \min \mathrm{E}\Big[\binom{X}{r}\binom{Y}{t} - (r+1)\binom{X}{r+1}\binom{Y}{t} \ , \ \binom{X}{r}\binom{Y}{t} \\ - (t+1)\binom{X}{r}\binom{Y}{t+1} \Big] \\ = \min \mathrm{E}\Big[\binom{X}{r}\binom{Y}{t} (1+r-X \ , \ 1+t-Y) \Big] \end{aligned}$$

Since $\mathrm{E}[I(X=r,Y=t)] = P(X=r,Y=t)$, and thus in order to prove (3), it suffices to show that

$$(9) \qquad I(X=r)I(Y=t) \geq \min\left[\binom{X}{r}\binom{Y}{t}(1+r-X\ ,\ 1+t-Y)\right]$$

Note that both sides of (9) are zero if either X or Y is less than r or t, respectively. Also, both sides of (9) are one if X and Y equal r and t, respectively. Thus, we have to prove that
(10) $f(X,Y) =$ (the right hand side of (9))is non-negative for $X=r$ and $Y \geq t+1$, $X \geq r+1$ and $Y=t$, $X \geq r+1$ and $Y \geq t+1$.

(i) First case. For positive integers r, t with $X=r$ and $Y \geq t+1$; that is, there are the events that exactly r A_i and at least t+1 B_j which occur. Then

$$\begin{aligned} f(r,t+p) &= \min\left[\binom{t+p}{t}\left(1,\ 1+t-(t+p)\right)\right] \\ &= \binom{t+p}{t}(1-p) \leq 0 \text{ for integer } p \text{ with } \ 1 \leq p \leq n-t. \end{aligned}$$

(ii) Second case. For positive integers r, t with $X \geq r+1$ and $Y=t$; that is, there are the event that exactly t B_j and at least $r+1$ A_i which occur. Then

$$f(r+q,t) = \min\left[\binom{r+q}{r}\left(1+r-(r+q),1\right)\right] = \binom{r+q}{r}(1-q) \leq 0$$

for integer q with $1 \leq q \leq n-r$.

(iii) Third case. For positive integers r, t, X and Y with $r+1 \leq X$ and $t+1 \leq Y$; that is, there are the events that at least r+1 A_i and at least t+1 B_j which occur. Then

$$f(r+q,t+p) = \min\left[\binom{r+q}{r}\binom{t+p}{t}\left(1+r-(r+p),1+t-(t+p)\right)\right] \leq 0$$

for integers p, q with $1 \leq p \leq n-t$, $1 \leq q \leq m-r$. Hence, we get (10). This completes the proof. □

PROOF OF THEOREM 2. We use the method of indicators. Let

$$I(X=r,Y=t)=I(X=r)I(Y=t)=\begin{cases}1 & \text{if } X=r \text{ and } Y=t\\ 0 & \text{if otherwise.}\end{cases}$$

By using binomial moment of (2) and the method of indicators, the right hand side of (5) becomes

$$\begin{aligned}\mathrm{E}\Big[&\binom{X}{r}\binom{Y}{t}-(r+1)\binom{X}{r+1}\binom{Y}{t}-(t+1)\binom{X}{r}\binom{Y}{t+1}\\&+\frac{(m+n-r-t-1)}{(m-r)(n-t)}(r+1)(t+1)\binom{X}{r+1}\binom{Y}{t+1}\Big]\\=\mathrm{E}\Big[&\binom{X}{r}\binom{Y}{t}\Big[1+r+t-X-Y\\&+\frac{(m+n-r-t-1)}{(m-r)(n-t)}(X-r)(Y-t)\Big]\Big].\end{aligned}$$

Then $\mathrm{E}\big[I(X=r,Y=t)\big]=P(X=r,Y=t)$, and thus in order to prove (5), it suffices to show that

$$I(X=r)I(Y=t)\geq\binom{X}{r}\binom{Y}{t}\Big[1+r+t-X-Y+\frac{(m+n-r-t-1)}{(m-r)(n-t)}(X-r)(Y-t)\Big] \tag{11}$$

Note that both sides of (11) are zero if either X or Y is less than r or t, respectively. Also, both sides of (11) are one if X and Y equal r and t, respectively. Thus, we have to prove that

(12) $f(X,Y)=$ (the right hand side of (11) is non-negative) for $X=r$ and $Y\geq t+1$, $X\geq r+1$ and $Y=t$, $X\geq r+1$ and $Y\geq t+1$.

(i) First case. For positive integers r, t with $X=r$ and $Y\geq t+1$; that is, there are exactly r A_i and at least $t+1$ B_j which occur. Then

$$f(r,t+p)=\binom{t+p}{t}\big[1+t-(t+p)\big]\leq 0$$

for integer P with $1\leq p\leq n-t$. Hence, we get (12).

(ii) Second case. For positive integers r,t with $X \geq r+1$ and $Y = t$; that is, there are at least $r+1$ A_i and exactly t B_j which occur. Then

$$f(r+q,t) = \binom{r+q}{r}\left[1 + r - (r+q)\right] \leq 0$$

for integer q with $1 \leq q \leq m - r$. Hence, we get (12).

(iii) Third case. For positive integers r, t, X and Y with $r+1 \leq X$ and $t+1 \leq Y$; that is, there are at least $r+1$ A_i and at least $t+1$ B_j which occur. Then

$$f(r+q,t+p) = \binom{r+q}{r}\binom{t+p}{t}\left[1 - q - p + \frac{(m+n-r-t-1)}{(m-r)(n-t)}pq\right]$$

for integers p q with $1 \leq p \leq n-t,\ \ 1 \leq q \leq m-r$. Let $g(q,p) = 1 - q - p + \frac{(m+n-r-t-1)}{(m-r)(n-t)}pq$. Then $g(q,p),\ 1 \leq q \leq m-r,\ \ 1 \leq p \leq n-t$, attains its maximum at end point, yielding $g(m-r, n-t) = 0$. Thus, $f(r+q,t+p)$ is less than zero and equals zero. Hence, we get (12). This completes the proof. □

PROOF OF THEOREM 3. We have to prove that the following inequalities hold ; that is,

$$P(X = r, Y = t) \leq S_{r,t} - \frac{r+1}{m-r}S_{r+1,t} \text{ and} \tag{13}$$

$$P(X = r, Y = t) \leq S_{r,t} - \frac{t+1}{n-t}S_{r,t+1} \tag{13'}$$

By using the method of indicators and binomial moment of (2), they become

$$I(X = r)I(Y = t) \leq \binom{X}{r}\binom{Y}{t} - \frac{r+1}{m-r}\binom{X}{r+1}\binom{Y}{t} \quad \text{and} \tag{14}$$

$$I(X = r)I(Y = t) \leq \binom{X}{r}\binom{Y}{t} - \frac{t+1}{n-t}\binom{X}{r}\binom{Y}{t+1} \tag{14'}$$

Thus, by proving (14) and (14′) we obtain (13) and (13′) by taking expectations. First, note that both (14) and (14′) are valid if either X or Y is less than r or t, respectively, and both sides of (14) and (14′) are one if X and Y equal r and t, respectively. Hence, for the sequel, we assume that $X \geq r+1$ and $Y \geq t$, $X \geq r$ and $Y \geq t+1$, $X \geq r+1$ and $Y \geq t+1$. Thus, (14) becomes

$$0 = I(X=r)I(Y=t) \leq \binom{X}{r}\binom{Y}{t}\left[1 - \frac{X-r}{m-r}\right]$$

Now, $h(X) = \binom{X}{r}\binom{Y}{t}\left[1 - \frac{X-r}{m-r}\right]$, $r \leq X \leq m$, attains its minimum at $x = m$, yielding $h(m) = 0$. Hence, we get (14).

The proof of (14′) is identical to the above argument. The proof is completed. □

Proof of Theorem 4. We also use the method of indicators.

Let $I(X=r, Y=t) = I(X=r)I(Y=t) = \begin{cases} 1 & \text{if } X=r \text{ and } Y=t \\ 0 & \text{if otherwise.} \end{cases}$

By using binomial moment of (2) and the method of indicators, the right hand side of (8) becomes

$$\begin{aligned} \mathrm{E}\Bigg[&\binom{X}{r}\binom{Y}{t} - \frac{(r+1)}{(m-r)}\binom{X}{r+1}\binom{Y}{t} - \frac{(t+1)}{(n-t)}\binom{X}{r}\binom{Y}{t+1} \\ &+ \frac{(r+1)(t+1)}{(m-r)(n-t)}\binom{X}{r+1}\binom{Y}{t+1}\Bigg] \\ =\mathrm{E}&\left[\binom{X}{r}\binom{Y}{t}\left(1 - \frac{(X-r)}{(m-r)} - \frac{(Y-t)}{(n-t)} + \frac{(X-r)(Y-t)}{(m-r)(n-t)}\right)\right] \end{aligned}$$

Then $\mathrm{E}I(X=r, Y=t) = P(X=r, Y=t)$, and thus in order to prove (8), it suffices to show that

$$\begin{aligned} &I(X=r)I(Y=t) \\ (15)\qquad &\leq\left[\binom{X}{r}\binom{Y}{t}\left(1 - \frac{(X-r)}{(m-r)} - \frac{(Y-t)}{(n-t)} + \frac{(X-r)(Y-t)}{(m-r)(n-t)}\right)\right]. \end{aligned}$$

Note that both sides of (15) are zero if either X or Y is less than r or t, respectively. Also, both sides of (15) are one if X and Y equal r and t, respectively. Thus, we have to prove that

(16) $f(X,Y)$ = the right hand side of (15) ≥ 0 for $X = r$ and $Y \geq t+1$, $X \geq r+1$ and $Y = t$, $X \geq r+1$ and $Y \geq t+1$.

(i) First case. For positive integers r,t with $X = r$ and $Y \geq t+1$; that is, there are exactly r A_i and at least $t+1$ B_j which occur. Then

$$f(r,t+p) = \binom{t+p}{t}\left[1 - \frac{p}{(n-t)}\right] \geq 0$$

for integer P with $1 \leq p \leq n-t$. Hence, we get (16).

(ii) Second case. For positive integers r,t with $X \geq r+1$ and $Y = t$; that is, there are at least $r+1$ A_i and exactly t B_j which occur. Then

$$f(r+q,t) = \binom{r+q}{r}\left[1 - \frac{q}{(m-r)}\right] \geq 0$$

for integer q with $1 \leq q \leq m-r$. Hence, we get (16).

(iii) Third case. For positive integers r, t, X and Y with $r+1 \leq X$ and $t+1 \leq Y$; that is, there are at least $r+1$ A_i and at least $t+1$ B_j which occur. Then

$$f(r+q,t+p) = \binom{r+q}{r}\binom{t+p}{t}\left[1 - \frac{q}{(m-r)} - \frac{p}{(n-t)} + \frac{qp}{(m-r)(n-t)}\right]$$

for integers p,q with $1 \leq p \leq n-t$, $1 \leq q \leq m-r$. Let

$$g(q,p) = \left[1 - \frac{q}{(m-r)} - \frac{p}{(n-t)} + \frac{qp}{(m-r)(n-t)}\right]$$

Then

$$g(q,p) = \frac{(m-r-q)(n-t-p)}{(m-r)(n-t)} \geq 0$$

for $1 \leq q \leq m-r$, $1 \leq p \leq n-t$. Thus, $f(r+q,t+p)$ is greater than zero and equals zero. Hence, we get (16). This completes the proof. □

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