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ON THE BERWALD CONNECTION OF A FINSLER SPACE WITH A SPECIAL (α, β)-METRIC

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ABSTRACT. In a Finsler space, we introduce a special (α, β)-metric L satisfying $L^2(\alpha, \beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_i are constants. We investigate the Berwald connection in a Finsler space with this special (α, β)-metric.

1. Introduction

The (α, β)-metric is a Finsler metric which is constructed from a Riemannian metric α and a differential 1-form β in an n-dimensional manifold. The concept of the (α, β)-metric was introduced by M. Matsumoto [4] and the Finsler space with the (α, β)-metric have been studied by many authors. The well-known examples of the (α, β)-metric are the Randers metric, the Kropina metric and the slope metric (or Matsumoto metric).

The purpose of the present paper is to introduce a special (α, β)-metric generalizing a Randers metric and investigate the Berwald connection of a Finsler space with this special (α, β)-metric. The concrete form of the Berwald connection in the Finsler space with a special (α, β)-metric is founded in the last section.

Throughout the present paper the terminology and notation are referred to Matsumoto's monograph [5].

2. A special (α, β)-metric

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with (α, β)-metric $L(\alpha, \beta)$. The fundamental function $L(\alpha, \beta)$ is a (1)p-homo-

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geneous of degree one in α and β, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential 1-form in the underlying manifold M^n. The normalized supporting element l_i and the angular metric h_{ij} are given by $l_i = \dot{\partial}_i L, \quad h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ respectively, where $\dot{\partial}_i = \partial/\partial y$. If we put $F = L^2/2$, then the fundamental metric tensor $g_{ij} = \dot{\partial}_i\dot{\partial}_j F$ is written as $g_{ij} = h_{ij} + l_i l_j$ from $\dot{\partial}_i\dot{\partial}_j F = L\dot{\partial}_i\dot{\partial}_j L + (\dot{\partial}_i L)\dot{\partial}_j L$. From the homogeneity of L we have $y_i = Ll_i$ and $y^i = Ll^i$.

Now we shall deal with a general (α,β)-metric $L(\alpha,\beta)$. From $\beta = b_i(x)y^i$ we have $\dot{\partial}_i\beta = b_i, \quad \dot{\partial}_i\dot{\partial}_j\beta = 0$. Putting $\dot{\partial}_i\alpha = \alpha_i, \quad \dot{\partial}_i\dot{\partial}_j\alpha = \alpha_{ij}, \quad \dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\alpha = \alpha_{ijk}$, we get $\alpha_i = Y_i/\alpha, \quad \alpha_{ij} = K_{ij}/\alpha$, where $Y_i = a_{ij}y^j, \quad K_{ij} = a_{ij} - Y_iY_j/\alpha^2$. The tensor K_{ij} is the angular metric tensor of the Riemannian metric a_{ij}. From $\dot{\partial}_k K_{ij} = -(K_{ik}Y_j + K_{jk}Y_i)/\alpha^2$, we have

$$\alpha_{ijk} = -(K_{ij}Y_k + K_{jk}Y_i + K_{ki}Y_j)/\alpha^3.$$

We consider the normalized supporting element $l_i = L_\alpha\alpha_i + L_\beta\beta_i$, which implies $l_i = (L_\alpha/\alpha)Y_i + L_\beta b_i$, where the subscripts α, β denote the partial differentiations by α and β respectively.

Next we shall find the fundamental metric tensor g_{ij}:

$$g_{ij} = F_\alpha\alpha_{ij} + F_{\alpha\alpha}\alpha_i\alpha_j + F_{\alpha\beta}(\alpha_i\beta_j + \alpha_j\beta_i) + F_{\beta\beta}\beta_i\beta_j, \tag{2.1}$$

therefore we get

$$g_{ij} = (F_\alpha/\alpha)K_{ij} + (F_{\alpha\alpha}/\alpha^2)Y_iY_j + (F_{\alpha\beta}/\alpha)(Y_ib_j + Y_jb_i) + F_{\beta\beta}b_ib_j. \tag{2.2}$$

The angular metric tensor h_{ij} is easily obtained as follows :

$$\begin{aligned} h_{ij} &= (F_\alpha/\alpha)K_{ij} + (F_{\alpha\alpha}/\alpha^2 - L_\alpha^2/\alpha^2)Y_iY_j \\ &\quad + (F_{\alpha\beta}/\alpha - L_\alpha L_\beta/\alpha)(Y_ib_j + Y_jb_i) + (F_{\beta\beta} - L_\beta^2)b_ib_j. \end{aligned} \tag{2.3}$$

Since we have

$$\begin{aligned} F_\alpha = LL_\alpha, \quad F_\beta = LL_\beta, \quad F_{\alpha\alpha} = LL_{\alpha\alpha} + L_\alpha^2, \\ F_{\alpha\beta} = LL_{\alpha\beta} + L_\alpha L_\beta, \quad F_{\beta\beta} = LL_{\beta\beta} + L_\beta^2, \end{aligned} \tag{2.4}$$

the equation (2.3) is rewritten as follows:

(2.5)

$$h_{ij} = (F_\alpha/\alpha)K_{ij} + (LL_{\alpha\alpha}/\alpha^2)Y_iY_j + (LL_{\alpha\beta}/\alpha)(Y_ib_j + Y_jb_i) + LL_{\beta\beta}b_ib_j.$$

From the homogeneity of the fundamental function L, we get

$$L_\alpha \alpha + L_\beta \beta = L, \quad L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = 0, \quad L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = 0. \tag{2.6}$$

Substituting (2.6) in (2.5), we have

$$h_{ij} = (F_\alpha/\alpha)K_{ij} + LL_{\beta\beta}\{b_i b_j - (\beta/\alpha^2)(Y_i b_j + Y_j b_i) + (\beta^2/\alpha^4)Y_i Y_j\}. \tag{2.7}$$

Then, If we put $P_i = b_i - (\beta/\alpha^2)Y_i$, we get

$$P_i P_j = b_i b_j - (\beta/\alpha^2)(Y_i b_j + Y_j b_i) + (\beta^2/\alpha^4)Y_i Y_j.$$

Therefore, we have

$$h_{ij} = (F_\alpha/\alpha)K_{ij} + (F_{\beta\beta} - F_\beta^2/2F)P_i P_j. \tag{2.8}$$

This form of h_{ij} shows immediately $h_{ij}y^j = 0$ from $K_{ij}y^j = 0$ and $P_i y^i = 0$.

Next we shall find the C-tensor $C_{ijk} = \dot\partial_k g_{ij}/2$. From (2.1) we have

$$\begin{aligned} 2C_{ijk} &= (F_{\alpha\alpha}\alpha_k + F_{\alpha\beta}\beta_k)\alpha_{ij} + F_\alpha \alpha_{ijk} \\ &+ (F_{\alpha\alpha\alpha}\alpha_k + F_{\alpha\alpha\beta}\beta_k)\alpha_i\alpha_j + F_{\alpha\alpha}(\alpha_{ik}\alpha_j + \alpha_{jk}\alpha_i) \\ &+ (F_{\alpha\beta\alpha}\alpha_k + F_{\alpha\beta\beta}\beta_k)(\alpha_i\beta_j + \alpha_j\beta_i) \\ &+ F_{\alpha\beta}(\alpha_{ik}\beta_j + \alpha_{jk}\beta_i) + (F_{\beta\beta\alpha}\alpha_k + F_{\beta\beta\beta}\beta_k)\beta_i\beta_j \\ &= F_\alpha\alpha_{ijk} + F_{\alpha\alpha}(\alpha_{ij}\alpha_k + \alpha_{jk}\alpha_i + \alpha_{ki}\alpha_j) \\ &+ F_{\alpha\beta}(\alpha_{ij}\beta_k + \alpha_{jk}\beta_i + \alpha_{ki}\beta_j) \\ &+ F_{\alpha\alpha\alpha}\alpha_i\alpha_j\alpha_k + F_{\alpha\alpha\beta}(\alpha_i\alpha_j\beta_k + \alpha_j\alpha_k\beta_i + \alpha_k\alpha_i\beta_j) \\ &+ F_{\alpha\beta\beta}(\alpha_i\beta_j\beta_k + \alpha_j\beta_k\beta_i + \alpha_k\beta_i\beta_j) + F_{\beta\beta\beta}\beta_i\beta_j\beta_k. \end{aligned} \tag{2.9}$$

The homogeneity of F implies

$$\begin{gathered} F_{\alpha\alpha}\alpha + F_{\alpha\beta}\beta = F_\alpha, \quad F_{\beta\alpha}\alpha + F_{\beta\beta}\beta = F_\beta, \quad F_{\alpha\alpha\alpha}\alpha + F_{\alpha\alpha\beta}\beta = 0, \\ F_{\alpha\beta\alpha}\alpha + F_{\alpha\beta\beta}\beta = 0, \quad F_{\beta\beta\alpha}\alpha + F_{\beta\beta\beta}\beta = 0. \end{gathered} \tag{2.10}$$

Then, from (2.9) and (2.10) we have
(2.11)
$$\begin{aligned}2C_{ijk} = &- (F_\alpha/\alpha^3)(K_{ij}Y_k + K_{jk}Y_i + K_{ki}Y_j)\\ &+ \{(F_\alpha - F_{\alpha\beta}\beta)/\alpha^3\}(K_{ij}Y_k + K_{jk}Y_i + K_{ki}Y_j)\\ &+ (F_{\alpha\beta}/\alpha)(K_{ij}b_k + K_{jk}b_i + K_{ki}b_j)\\ &+ F_{\beta\beta\beta}\{b_ib_jb_k - (\beta/\alpha^2)(Y_ib_jb_k + Y_jb_kb_i + Y_kb_ib_j)\\ &+ (\beta^2/\alpha^4)(Y_iY_jb_k + Y_jY_kb_i + Y_kY_ib_j) - (\beta^3/\alpha^6)Y_iY_jY_k\}.\end{aligned}$$

If we construct $P_iP_jP_k$, then we immediately have the conclusion:

(2.12) $$2C_{ijk} = (F_{\alpha\beta}/\alpha)(K_{ij}P_k + K_{jk}P_i + K_{ki}P_j) + F_{\beta\beta\beta}P_iP_jP_k.$$

In the following we pay attension to (2.12). The C-tensor C_{ijk} is written in the term of the angular metric tensor K_{ij} of the Riemannian metric α as follows

(2.13) $$C_{ijk} = K_{ij}B_k + K_{jk}B_i + K_{ki}B_j$$

for some tensor field B_i, if and only if $F_{\beta\beta\beta} = 0$. From the assumption $F_{\beta\beta\beta} = 0$ and (2.10) we have known that F should be a quadratic function of α, β, that is, L^2 is written in the form

(2.14) $$L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2,$$

where c_1, c_2 and c_3 are constants. Consequently we have

THEOREM 2.1. *The C-tensor is of the form* (2.13), *if and only if the metric function* $L(\alpha,\beta)$ *satisfies* (2.14).

In the case of $c_1 = c_2 = c_3 = 1$ in (2.14), the metric L is a a Randers metric and if $c_1c_3 - c_2^2 = 0$, the metric is a Randers metric also. Thus we may be cosidered that the metric L satisfying (2.14) is a generalization of the Randers metric.

REMARK. If the C-tensor is written in the form

(2.15) $$C_{ijk} = h_{ij}A_k + h_{jk}A_i + h_{ki}A_j$$

for some tensor field A_i, then the space is called C-reducible [4]. It is known that a C-reducible Finsler space with (α,β)-metric is a Randers space or Kropina spce [4], that is, $L(\alpha,\beta) = \alpha + \beta$ or $L(\alpha,\beta) = \alpha^2/\beta$. It is interesting to compare the form (2.13) with the form (2.15).

3. The condition to be the Berwald space

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with an (α, β)-metric given by (2.14). In this section, the Matsumoto's method of [6] will now be applied to find the condition that F^n be a Berwald space. The Riemannian space $R^n = (M^n, \alpha)$ is called the associated Riemannian space with F^n and the Christoffel symbols of $R^n = (M^n, \alpha)$ are indicated by $\gamma_j{}^i{}_k$. Then the Riemannian connection $(\gamma_j{}^i{}_k)$ gives rise to the linear Finsler connection $F\Gamma = (\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0)$, where the subscript 0 means a contraction by y^i.

The Berwald connection $B\Gamma = (G_j{}^i{}_k, G_0{}^i{}_j, 0)$ is uniquely determined as the Finsler connection satisfying the following axiomatic system by Okada [7]:

(B1) L-metrical: $L_{|i} = 0$,
(B2) (h)h-torsion tensor $T_j{}^i{}_k = G_j{}^i{}_k - G_k{}^i{}_j = 0$,
(B3) deflection tensor $D^i{}_j = y^k G_k{}^i{}_j - G^i{}_j = 0$,
(B4) (v)hv-torsion tensor $P^i{}_{jk} = \dot{\partial}_k G^i{}_j - G_k{}^i{}_j = 0$,
(B5) (h)hv-torsion tensor $C_j{}^i{}_k = 0$,

where the symbol $(_|)$ in (B1) denotes the h-covariant differentiation with respect to the Finsler connection.

Now, we shall find the Berwald connection $B\Gamma$ in F^n. Putting

$$2G^i = \gamma_0{}^i{}_0 + 2B^i, \tag{3.1}$$

we have from (B2), (B3) and (B4)

$$\begin{aligned} G^i_j &= \dot{\partial}_j G^i = \gamma_0{}^i{}_j + B^i{}_j, \\ G^i_{jk} &= \dot{\partial}_j G^i{}_k = \gamma_j{}^i{}_k + B_j{}^i{}_k, \end{aligned} \tag{3.2}$$

where we put $B^i{}_j = \dot{\partial}_j B^i$ and $B_j{}^i{}_k = \dot{\partial}_k B^i{}_j$.

The axiom (B1): $L_{|i} = \partial_i L - G^r{}_i \dot{\partial}_r L = 0$ is written as

$$L_\alpha B_j{}^k{}_i y^j y_k + \alpha L_\beta (B_j{}^r{}_i b_r - \nabla_i b_j) y^j = 0, \tag{3.3}$$

where $y_k = a_{ki} y^i$ *and* ∇_j is the differentiation with respect to $\gamma_j{}^i{}_k$.

Since the metric function L is given by (2.14), we get

$$LL_\alpha = c_1\alpha + c_2\beta, \quad LL_\beta = c_2\alpha + c_3\beta. \tag{3.4}$$

Substituting (3.4) in (3.3), we have

$$\alpha\{c_1 B_j{}^k{}_i y^j y_k + c_3\beta(B_j{}^k{}_i b_k - \nabla_i b_j)y^j\} + c_2\{\beta B_j{}^k{}_i y^j y_k + \alpha^2(B_j{}^k{}_i b_k - \nabla_i b_j)y^j\} = 0. \tag{3.5}$$

Now, we assume that the Finsler space F^n with (α, β)-metric given by (2.14) is a Berwald space, that is, $G_j{}^i{}_k$ is a function of the position alone. Then we have $B_j{}^k{}_i = B_j{}^k{}_i(x)$, so that the terms in the braces of left-hand side of (3.5) are rational polynomials in (y^i) and α is an irrational polynomial in (y^i). Thus we have

$$\begin{aligned} &1)\quad c_1 B_j{}^k{}_i y^j y_k + c_3\beta(B_j{}^k{}_i b_k - \nabla_i b_j)y^j = 0, \\ &2)\quad \beta B_j{}^k{}_i y^j y_k + \alpha^2(B_j{}^k{}_i b_k - \nabla_i b_j)y^j = 0. \end{aligned} \tag{3.6}$$

From the above two equations, we have

$$\begin{aligned} &1)\quad (c_1\alpha^2 - c_3\beta^2)B_j{}^k{}_i y^j y_k = 0, \\ &2)\quad (c_1\alpha^2 - c_3\beta^2)(B_j{}^k{}_i b_k - \nabla_i b_j)y^j = 0. \end{aligned} \tag{3.7}$$

(I) We suppose that $c_1\alpha^2 - c_3\beta^2 = 0$. This assumption implies $c_1 = 0$ and $c_3 = 0$. In this case, (2.14) becomes to $L^2 = 2c_2\alpha\beta$, that is, the fundamental metric $L(\alpha, \beta)$ is a generalized Kropina metric. The left-hand side of (3.6)2) is a polynomial of three order in (y^i) and shows the existence of function $\lambda_i(x)$ satisfying

$$B_j{}^k{}_i y^j y_k = -\lambda_i(x)\alpha^2, \quad (B_j{}^k{}_i b_k - \nabla_i b_j)y^j = \lambda_i(x)\beta.$$

The former is written as

$$B_j{}^k{}_i a_{kh} + B_h{}^k{}_i a_{kj} = -2\lambda_i(x)a_{jh},$$

which implies

$$B_i{}^k{}_j = \lambda^k a_{ij} - \lambda_i\delta_j^k - \lambda_j\delta_i^k, \tag{3.8}$$

where $\lambda^k = a^{ki}\lambda_i$. Therefore, the latter gives

$$\nabla_i b_j = \lambda^k b_k a_{ij} - 2\lambda_i b_j - \lambda_j b_i. \tag{3.9}$$

Conversely, if there exists the vector $\lambda_i(x)$ satisfying (3.9), we have $L_{|i} = 0$ with respect to $G_j{}^i{}_k = \gamma_j{}^i{}_k + B_j{}^i{}_k$, where $B_j{}^i{}_k$ is given by (3.8). Hence, by the well-known Hashiguchi-Ichijyō's theorem [2], the Finsler space is a Berwald space.

(II) We suppose that $c_1\alpha^2 - c_3\beta^2 \neq 0$, that is $c_1 \neq 0, c_3 \neq 0$. Then from (3.7) we have

$$1)\quad B_j{}^k{}_i y^j y_k = 0, \qquad 2)\quad (B_j{}^k{}_i b_k - \nabla_i b_j) y^j = 0, \tag{3.10}$$

which implies

$$1)\quad B_j{}^k{}_i a_{kh} + B_h{}^k{}_i a_{kj} = 0, \qquad 2)\quad B_j{}^k{}_i b_k - \nabla_i b_j = 0. \tag{3.11}$$

The former yields $B_j{}^k{}_i = 0$ and from which $\nabla_i b_j = 0$ immediately.

On the other hand, Hashiguchi and Ichijyō have shown in [2] that if $\nabla_k b_i = 0$, then the Finsler space F^n with an (α, β)-metric is a Berwald space. Thus we have

THEOREM 3.1. *Let F^n be the Finsler space with an (α, β)-metric given by* (2.14) *and the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$ given by* (3.2).

(i) *If $c_1 = c_3 = 0$, then F^n is a Berwald space if and only if there exists the covariant vector $\lambda_i(x)$ satisfying* (3.9), *and the Berwald connection is written as $(\gamma_j{}^i{}_k + B_j{}^i{}_k, \gamma_0{}^i{}_j + B_0{}^i{}_j, 0)$, where $B_j{}^i{}_k$ are given by* (3.8).

(ii) *If $c_1 \neq 0, c_3 \neq 0$, then F^n is a Berwald space if and only if $\nabla_i b_j = 0$ and the Berwald connection is $(\gamma_j{}^i{}_k, \gamma^i_{0j}, 0)$.*

4. Concrete form of the Berwald connection

In this section we will find the concrete form of the Berwald connection in the Finsler space with an (α, β)-metric given by (2.14). The Berwald connection is determined by $B_j{}^i{}_k$ in the equation (3.3) uniquely. We will solve $B_j{}^k{}_i$ concretely. From (3.3) and (3.4) we get

$$(4.1)\qquad (c_1\alpha + c_2\beta)B_j{}^k{}_i y^j y_k + \alpha(c_2\alpha + c_3\beta)(B_j{}^k{}_i b_k - \nabla_i b_j)y^j = 0.$$

By the homogeneity, (4.1) is rewritten as

$$(4.2)\qquad (c_2\alpha + c_3\beta)(\nabla_i b_j)y^j = \{(c_1\alpha + c_2\beta)e_k + (c_2\alpha + c_3\beta)b_k\}B^k{}_i,$$

where $e_k = y_k/\alpha$. We put

$$r_{ij} = (\nabla_j b_i + \nabla_i b_j)/2, \qquad s_{ij} = (\nabla_j b_i - \nabla_i b_j)/2.$$

Transvecting (4.2) by y^i and using the homogeneity, we have

$$(4.3)\qquad (c_2\alpha + c_3\beta)r_{00} = 2\{(c_1\alpha + c_2\beta)e_k + (c_2\alpha + c_3\beta)b_k\}B^k.$$

Conversely differentiating (4.3) by y^i, we obtain

$$(4.4)\qquad \begin{aligned}&(c_2e_i + c_3b_i)r_{00} + 2(c_2\alpha + c_3\beta)r_{0i}\\ &= 2\{(c_1e_i + c_2b_i)e_k + (c_1\alpha + c_2\beta)(a_{ki} - e_ke_i)/\alpha\\ &+ (c_2e_i + c_3b_i)b_k\}B^k + 2\{(c_1\alpha + c_2\beta)e_k + (c_2\alpha + c_3\beta)b_k\}B^k{}_i\end{aligned}$$

by virtue of $\dot\partial_i\alpha = e_i$, $\dot\partial_i e_k = (a_{ki} - e_ke_i)/\alpha$. From (4.2), (4.3) and (4.4) we have

$$(4.5)\qquad \begin{aligned}2a_{ki}\{(c_1\alpha+c_2\beta)/\alpha\}B^k &= 2(c_2\alpha + c_3\beta)s_{i0} + (c_2e_i + c_3b_i)r_{00}\\ &- 2(c_1e_i + c_2b_i)e_kB^k + 2\{(c_1\alpha + c_2\beta)/\alpha\}e_ie_kB^k\\ &- 2(c_2e_i + c_3b_i)b_kB^k,\end{aligned}$$

where $s^i{}_0 = a^{ij}s_{j0}$. The equation (4.5) is written as the following form

$$(4.6)\qquad B^i = P_1e^i + P_2s^i{}_0 + P_3b^i,$$

where putting $E = e_kB^k$ and $D = b_kB^k$, we have

$$(4.7)\qquad \begin{aligned}P_1 &= E + \alpha\{c_2r_{00} - 2(c_1E + c_2D)\}/2(c_1\alpha + c_2\beta),\\ P_2 &= \alpha(c_2\alpha + c_3\beta)/(c_1\alpha + c_2\beta),\\ P_3 &= \alpha\{c_3r_{00} - 2(c_2E + c_3D)\}/2(c_1\alpha + c_2\beta).\end{aligned}$$

To find E and D, we put $b_ib^i = b^2$, $s_0 = s^i{}_0b_i$. From (4.3) we get

$$(4.8) \qquad (c_2\alpha + c_3\beta)r_{00} = 2(c_1\alpha + c_2\beta)E + 2(c_2\alpha + c_3\beta)D.$$

Transvecting (4.6) by b_i , we have

$$(4.9) \qquad \begin{aligned} &(c_1\alpha + 2c_2\beta + c_3\alpha b^2)D \\ &\quad = c_2(\beta^2 - \alpha^2b^2)E + \alpha^2(c_2\alpha + c_3\beta)s_0 + \alpha r_{00}(c_2\beta + c_3\alpha b^2)/2 \end{aligned}$$

by virtue of $b_ie^i = \beta/\alpha$. Therefore (4.8) and (4.9) give E and D.
Thus we have

THEOREM 4.1. *The vector field $B^i(x,y)$ in* (3.1) *is given by* (4.6) *and* (4.7), *where quantities E and D are determined by* (4.8) *and* (4.9).

EXAMPLE. In an (α,β)-metric given by (2.14), if $c_1 = c_2 = c_3 = 1$, the metric $L(\alpha,\beta)$ is a Randers metric. For the Randers space, from (4.8) and (4.9) the quantities E and D are determined by the following two equations

$$(4.10) \qquad r_{00} = 2(E + D),$$

$$(4.11) \quad \alpha(\alpha+2\beta+\alpha b^2)D = (\beta^2-\alpha^2b^2)E+\alpha^2(\alpha+\beta)s_0+r_{00}(\alpha\beta+\alpha^2b^2)/2,$$

from which we get

$$(4.12.) \quad E = \alpha(r_{00} - 2\alpha s_0)/2(\alpha+\beta), \quad D = 2\alpha^2 s_0 - r_{00}(\alpha-\beta)/2(\alpha+\beta)$$

From (4.7), (4.8) and (4.12) we get

$$P_1 = E = \alpha(r_{00} - 2\alpha s_0)/2(\alpha + \beta), \quad P_2 = \alpha, \quad P_3 = 0.$$

Thus, in a Randers space, the vector field $B^i(x,y)$ in (3.1) is given as follows:

$$B^i = \alpha(r_{00} - 2\alpha s_0)e^i/2(\alpha + \beta) + \alpha s^i_0.$$

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