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COSYMPLECTIC CONFORMAL CURVATURE TENSOR AND SPECTRUM OF THE LAPLACIAN IN COSYMPLECTIC MANIFOLDS*

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ABSTRACT. The purpose of this paper is to study the spectrum of the Laplacian and the cosymplectic conformal curvature tensor of cosymplectic manifold.

1. Introduction

Let (M, g) be an m-dimensional compact orientable Riemannian manifold (connected and C^∞) with metric tensor g. We denote by $\triangle$ the Laplacian acting on p-forms on M, $0 \leq p \leq m$. Then we have the spectrum for each p :

$$Spec^p(M, g) := \{0 \leq \lambda_{0,p} \leq \lambda_{1,p} \leq \lambda_{2,p} \leq \cdots \uparrow \infty\},$$

where each eigenvalue $\lambda_{\alpha,p}$ is repeated as many times as its multiplicity indicates. In order to study the relation between $Spec^p(M, g)$ and the geometry of (M, g) we use the Minakshisundaram-Pleijel-Gaffney's formula. J. S. Pak, J.-H. Kwon and K.-H. Cho ([7]) studied the spectrum of the Laplacian and the curvature of cosymplectic manifolds. On the other hand, J.-H. Kwon, J. D. Lee, K.-H. Cho and W.-H. Sohn ([5]) found a relation between the cosymplectic conformal curvature tensor of a cosymplectic manifold and the conformal curvature tensor of the transversal Kaehlerian hypersurface.

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The purpose of the present paper is to study the spectrum of the Laplacian and the cosymplectic conformal curvature tensor of cosymplectic manifold.

We shall be in C^∞-category. The indices $h, i, j, k, s, t, \dots$ run over the range $\{1, 2, \dots, 2n+1\}$. The Einstein summation convention with respect to those system of indices will be used.

2. Preliminaries

By $R = (R_{kji}{}^h)$, $R_1 = (R_{ji})$ and r we denote the Riemannian curvature tnesor, the Ricci curvature tensor and the scalar curvature, respectively. For a tensor field T on M, we denote by $|T|$ the norm of T with respect to g. Then the Minakshisundaram-Pleijel-Gaffney's formula for $Spec^p(M,g)$ is given by

$$\sum_{\alpha=0}^{\infty} exp(-\lambda_{\alpha,p}t) \sim (4\pi t)^{-\frac{m}{2}} \sum_{\alpha=0}^{\infty} a_{\alpha,p}t^{\alpha} \quad \text{as} \quad t \longrightarrow 0^+,$$

where the constants $a_{\alpha,p}$ are spectral invariants. In the present paper we are interested in the case of $p = 0, 1$ or 2. For $p = 0$ we have (cf. [1])

$$a_{0,0} = \int_M dM = Vol(M,g), \tag{2.1}$$

$$a_{1,0} = \frac{1}{6}\int_M rdM, \tag{2.2}$$

$$a_{2,0} = \frac{1}{360}\int_M [2|R|^2 - 2|R_1|^2 + 5r^2]dM, \tag{2.3}$$

where dM denotes the natural volume element of (M,g). For $p = 1$, we have (cf. [11])

$$a_{0,1} = m = Vol(M,g), \tag{2.4}$$

$$a_{1,1} = \frac{m-6}{6} \int_M r dM, \tag{2.5}$$

$$a_{2,1} = \frac{1}{360} \int_M [2(m-15)|R|^2 - 2(m-90)|R_1|^2 + 5(m-12)r^2] dM, \tag{2.6}$$

For $p = 2$, we have (cf. [8], [10], [11])

$$a_{0,2} = \frac{1}{2} m(m-1) Vol(M, g). \tag{2.7}$$

$$a_{1,2} = \frac{1}{12}(m^2 - 13m + 24) \int_M r dM, \tag{2.8}$$

$$\begin{aligned} a_{2,2} = \frac{1}{720} \int_M & [2(m^2 - 31m + 240)|R|^2 \\ & - 2(m^2 - 181m + 1080)|R_1|^2 + 5(m^2 - 25m + 120)r^2] dM. \end{aligned} \tag{2.9}$$

3. Cosymplectic manifolds

Let M be $(2n+1)$-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$ in which there are given a tensor field φ of type $(1,1)$, a vecter field ξ^h and a 1-form η_h satisfying

$$\varphi_t{}^h \varphi_i{}^t = -\delta_i^h + \eta_i \xi^h, \quad \eta_t \varphi_h{}^t = 0, \quad \varphi_t{}^h \xi^t = 0, \quad \eta_t \xi^t = 1. \tag{3.1}$$

Such a set of a tensor field of $(1,1)$, a vecter field and a 1-form is called *almost contact structure* and a manifold with an almost contact structure an *almost contact manifold.*

If, in an almost contact manifold, there is given a Riemannian metric g_{ji} such that

$$g_{ts} \varphi_j{}^t \varphi_i{}^s = g_{ji} - \eta_j \eta_i, \quad \eta_i = g_{it} \xi^t, \tag{3.2}$$

then the manifold is an *almost contact metric manifold.*

If we define $\varphi_{ji} = \varphi_j{}^t g_{ti}$, we see from (3.1) and (3.2) that φ_{ji} is skew-symmetric.

The almost contact structure is said to be *normal* if $[\varphi, \varphi] + d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ denotes the Nijenhuis tensor formed with φ and d the operator of the exterior derivative.

A normal almost contact metric structure is said to be *cosymplectic* (cf. [2], [3], [4], [6], [7]) if the 2-form φ_{ji} and the 1-form η_i are both closed. A manifold with a cosymplectic structure is called a *cosymplectic manifold.* It is known in [2] that the cosymplectic structure is characterized by

$$\nabla_k \varphi_j{}^i = 0 \text{ and } \nabla_k \eta^i = 0, \tag{3.3}$$

where ∇_k denotes the operator of covariant differentiation with respect to g_{ji}.

If we denote the curvature tensor, Ricci tensor and scalar curvature of a cosymplectic manifold M by $R_{kji}{}^h$, R_{ji} and r respectively, then we have

$$\begin{aligned} &R_{kjit}\xi^t = 0, \quad R_{kjts}\varphi_i{}^t\varphi_h{}^s = R_{kjih}, \\ &R_{tjis}\varphi^{ts} = -R_{jt}\varphi_i{}^t, \quad R_{jt}\varphi_i{}^t = -R_{it}\varphi_j{}^t, \\ &R_{kjts}\varphi^{ts} = 2R_{kt}\varphi_j{}^t, \quad R_{jt}\xi^t = 0, \quad R_{ts}\varphi_j{}^t\varphi_i{}^s = R_{ji}, \end{aligned} \tag{3.4}$$

where $\varphi^{ji} = \varphi_t{}^i g^{jt}$, and $R_{kjih} = R_{kji}{}^t g_{th}$.

In a cosymplectic manifold M, we call a sectional curvature

$$k = -\frac{g(R(\varphi X, X)\varphi X, X)}{g(X, X)g(\varphi X, \varphi X)}$$

determined by two orthogonal vectors X and φX the *φ-holomorphic sectional curvature* with respect to the vector X orthogonal to ξ of M. If the φ-holomorphic sectional curvature is always constant with respect to any vector at every point of the manifold M, then we call the manifold M a *manifold of constant φ-holomorphic sectional curvature.* If a cosymplectic manifold has a constant φ-holomorphic sectional curvature

k at every point, then the components of the curvature tensor of the manifold are of the form (cf. [3], [6])

$$R_{kjih} = \frac{k}{4}(g_{kh}g_{ji} - g_{ki}g_{jh} + \varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih} - g_{kh}\eta_j\eta_i + g_{ki}\eta_j\eta_h - \eta_k\eta_h g_{ji} + \eta_k\eta_i g_{jh}),$$

where $k = \frac{r}{n(n+1)}$.

A tensor field $H = (H_{kjih})$ on M is defined by

$$H_{kjih} = R_{kjih} - \frac{r}{4n(n+1)}(g_{kh}g_{ji} - g_{ki}g_{jh} + \varphi_{kh}\varphi_{ji} - \varphi_{ki}\varphi_{jh} - 2\varphi_{kj}\varphi_{ih} - g_{kh}\eta_j\eta_i + g_{ki}\eta_j\eta_h - \eta_k\eta_h g_{ji} + \eta_k\eta_i g_{jh}). \tag{3.5}$$

By using (3.4) and (3.5), we can easily verify that

$$|H|^2 = |R|^2 - \frac{2}{n(n+1)}r^2. \tag{3.6}$$

A cosymplectic manifold is of constant φ-holomorphic sectional curvature if and only if $H = 0$, provided $n \geq 2$.

A tensor field $Q = (Q_{ji})$ on M is defined by

$$Q_{ji} = R_{ji} - \frac{r}{2n}g_{ji} + \frac{r}{2n}\eta_j\eta_i.$$

By a direct calculation, in which we use (3.4), it follows

$$|Q|^2 = |R_1|^2 - \frac{1}{2n}r^2. \tag{3.7}$$

A cosymplectic manifold is said to be *η-Einstein* if $Q = 0$. For any η-Einstein cosymplectic manifold, r is constant, porvied $n \geq 2$.

We also consider the so-called *cosymplectic conformal curvature tensor*

$\overline{B}_0 = (\overline{B}_{0,kjih})$ defiend on M by (cf. [5])

$$\begin{aligned}\overline{B}_{0,kjih} =& R_{kjih} + \frac{1}{2n}(g_{kh}R_{ji} - g_{jh}R_{ki} + R_{kh}g_{ji} - R_{jh}g_{ki} \\ & - \varphi_{kh}S_{ji} + \varphi_{jh}S_{ki} - S_{kh}\varphi_{ji} + S_{jh}\varphi_{ki} + 2\varphi_{kj}S_{ih} \\ & + 2S_{kj}\varphi_{ih} - \eta_k\eta_h R_{ji} + \eta_j\eta_h R_{ki} - R_{kh}\eta_j\eta_i + R_{jh}\eta_k\eta_i) \\ & + \frac{(n+2)r}{4n^2(n+1)}(\varphi_{kh}\varphi_{ji} - \varphi_{jh}\varphi_{ki} - 2\varphi_{kj}\varphi_{ih}) \\ & - \frac{(3n+2)r}{4n^2(n+1)}(g_{kh}g_{ji} - g_{jh}g_{ki} - g_{kh}\eta_j\eta_i + g_{jh}\eta_k\eta_i \\ & - g_{ji}\eta_k\eta_h + g_{ki}\eta_j\eta_h),\end{aligned} \tag{3.8}$$

where $S_{ji} = \varphi_j{}^t R_{ti}$ and $S_{ji} = -S_{ij}$.

The tensor field $\overline{B}_0$ satisfies, among others, the following identities :

$$\begin{aligned}& \overline{B}_{0,kjih} = \overline{B}_{0,ihkj}, \quad \overline{B}_{0,kjih} = -\overline{B}_{0,jkih}, \quad \overline{B}_{0,kjih} = -\overline{B}_{0,kjhi}, \\ & \overline{B}_{0,kjih} + \overline{B}_{0,jikh} + \overline{B}_{0,ikjh} = 0, \\ & \overline{B}_{0,tjis}g^{ts} = \frac{2(n-2)}{n}R_{ji} - \frac{(n-2)r}{n^2}g_{ji} + \frac{(n-2)r}{n^2}\eta_j\eta_i, \\ & \overline{B}_{0,kjih}\xi^h = 0, \quad \overline{B}_{0,kjih}\varphi^{kh} = 0, \quad \overline{B}_{0,tsih}\varphi^{ts} = 0.\end{aligned}$$

A cosymplectic manifold with $\overline{B}_0 = 0$ is said to be *cosymplectic conformal flat.* If the cosymplectic conformal curvature tensor of M vanishes, that is, $\overline{B}_0 = 0$, then from (3.1), (3.4) and (3.8) we have

$$(n-2)R_{ji} = \frac{(n-2)r}{2n}g_{ji} - \frac{(n-2)r}{2n}\eta_j\eta_i.$$

Therefore we see that M is η-Einstein, provided $n \neq 2$. Using these identities, (3.1), (3.4) and (3.8), we can easily check that

$$|\overline{B}_0|^2 = |R|^2 - \frac{8}{n^2}|R_1|^2 - \frac{2(n^2-2n-2)}{n^3(n+1)}r^2, \tag{3.9}$$

$$|\overline{B}_0|^2 = |H|^2 - \frac{8}{n^2}|Q|^2. \tag{3.10}$$

On the other hand, the cosymplectic Bochner curvature tensor $\overline{B} = (\overline{B}_{kjih})$ defiend on M by (cf. [4])

$$\begin{aligned}\overline{B}_{kjih} =& R_{kjih} - \frac{1}{2(n+2)}(g_{kh}R_{ji} - g_{jh}R_{ki} + R_{kh}g_{ji} - R_{jh}g_{ki} \\ &+ \varphi_{kh}S_{ji} - \varphi_{jh}S_{ki} + S_{kh}\varphi_{ji} - S_{jh}\varphi_{ki} - 2\varphi_{kj}S_{ih} - 2S_{kj}\varphi_{ih} \\ &- \eta_k\eta_h R_{ji} + \eta_j\eta_h R_{ki} - \eta_j\eta_i R_{kh} + \eta_k\eta_i R_{jh}) \\ &+ \frac{r}{4(n+1)(n+2)}(g_{kh}g_{ji} - g_{jh}g_{ki} - g_{kh}\eta_j\eta_i + g_{jh}\eta_k\eta_i \\ &- g_{ji}\eta_k\eta_h + g_{ki}\eta_j\eta_h + \varphi_{kh}\varphi_{ji} - \varphi_{jh}\varphi_{ki} - 2\varphi_{kj}\varphi_{ih}).\end{aligned}$$

We also verified in the previous paper ([7]) that

$$|\overline{B}|^2 = |H|^2 - \frac{8}{n+2}|Q|^2. \tag{3.11}$$

From (3.10) and (3.11), we have

$$|\overline{B}_0|^2 = |\overline{B}|^2 + \frac{8(n-2)(n+1)}{n^2(n+2)}|Q|^2. \tag{3.12}$$

From (3.12), we have the following

THEOREM 3.1. *Let M be a cosymplectic manifold of dimension $\neq 5$. Then M is cosymplectic conformal flat if and only if M is η-Einstein and cosymplectic Bochner flat.*

REMARK. If M be a 5-dimensional cosymplectic manifold, then M is cosymplectic conformal flat if and only if M is cosymplectic Bochner flat.

From (3.10) and Theorem 3.1, we have the following

THEOREM 3.2. *Let M be a cosymplectic manifold of dimension > 5. Then M is of constant φ-holomorphic sectional curvature if and only if M is cosymplectic conformal flat.*

REMARK. If M be a 5-dimensional cosymplectic manifold, then M is of constant φ-holomorphic sectional curvature if and only if M is η-Einstein and cosymplectic conformal flat.

4. $Spec^0 M$ and the geometry of M

Assume that M is a compact cosymplectic manifold of dimension $2n+1$ and consider $Spec^0 M$. With the help of (3.7) and (3.9), the coefficient $a_{2,0}$ given by (2.3) may be written as follows :

$$(4.1) \qquad a_{2,0} = \frac{1}{180}\int_M [|\overline{B}_0|^2 + \frac{8-n^2}{n^2}|Q|^2] dM + \frac{C_0(n)}{180}\int_M r^2 dM,$$

where $C_0(n)$ is constant depending only on n and $C_0(n) > 0$.

We shall often use the following Lemma 4.1.

LEMMA 4.1 ([9]). *Let (M, g) and (M', g') be compact orientable Riemannian manifolds with $Vol(M, g) = Vol(M', g')$ and $\int_M r dM = \int_{M'} r' dM'$. If $r' =$ constant, then $\int_M r^2 dM \geq \int_{M'} r'^2 dM'$ with equality if and only if $r =$ constant $= r'$.*

THEOREM 4.2. *Let M and M' be compact cosymplectic manifolds. Assume that $Spec^0 M = Spec^0 M'$. Then* $\dim M = \dim M' = 2n+1 = m$ *and*

(a) *for $m = 3$, M is cosymplectic conformal flat with constant scalar curvature r if and only if M' is cosymplectic conformal flat with constant scalar curvature r', moreover $r' = r$,*

(b) *when M and M' are η-Einstein and η'-Einstein, respectively and $m \geq 5$, M is cosymplectic conformal flat if and only if M' is cosymplectic conformal flat, moreover $r' =$ constant$= r$.*

PROOF. Because of (2.1) and (2.2), $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$ imply $Vol(M) = Vol(M')$ and $\int_M r dM = \int_{M'} r' dM'$. Moreover, by virtue of (4.1), $a_{2,0} = a'_{2,0}$ yields

$$(4.2) \qquad \begin{aligned} &\int_M [|\overline{B}_0|^2 + \frac{8-n^2}{n^2}|Q|^2] dM + C_0(n)\int_M r^2 dM \\ &= \int_{M'} [|\overline{B}_0{}'|^2 + \frac{8-n^2}{n^2}|Q'|^2] dM' + C_0(n)\int_{M'} r'^2 dM'. \end{aligned}$$

(a) If $n = 1$ and $\overline{B}_o{}' = 0$, then $Q' = 0$ and it follows from (4.2) that

$$\int_M [|\overline{B}_0|^2 + 7|Q|^2] dM + C_0(1)(\int_M r^2 dM - \int_{M'} r'^2 dM') = 0,$$

which, by $r' =$ constant and Lemma 4.1, gives our assertion.

(b) Let $Q = 0$ and $Q' = 0$. Then r and r' are constant for $n \geq 2$. Thus by Lemma 4.1, we obtain $r = r'$ and it follows from (4.2) that

$$\int_M |\overline{B}_0|^2 dM = \int_{M'} |\overline{B}_0{}'|^2 dM'.$$

If M' is cosymplectic conformal flat, that is, $\overline{B}_0{}' = 0$, then from the above equation, we obtain $\overline{B}_0 = 0$. This completes the proof of our Theorem 4.2. □

COROLLARY 4.3. *Under the same assumptions as in Theorem 4.2, M is cosymplectic Bochner flat if and only if M' is cosymplectic Bochner flat.*

5. $Spec^1 M$ and the geometry of M

Assume that M is a compact cosymplectic manifold of dimension $2n+1$ and consider $Spec^1 M$. With the help of (3.7) and (3.9), the coefficient $a_{2,1}$ given by (2.6) reduces to

$$a_{2,1} = \frac{1}{180}\int_M [2(n-7)|\overline{B}_0|^2 - A_1(n)|Q|^2]dM + \frac{C_1(n)}{180}\int_M r^2 dM, \tag{5.1}$$

where $A_1(n) = \frac{(2n^3-89n^2-16n+112)}{n^2}$ and $C_1(n) = \frac{(n-3)(5n-11)(2n+1)}{2n(n+1)}$.

THEOREM 5.1. *Let M and M' be compact cosymplectic manifolds. Assume that $Spec^1 M = Spec^1 M'$. Then $\dim M = \dim M' = 2n+1 = m$ and*

(a) for $17 \leq m \leq 89$, M is cosymplectic conformal flat if and only if M' is cosymplectic conformal flat, moreover $r' =$ constant $= r$,

(b) when M and M' are η-Einstein and η'-Einstein, respectively and $m \geq 5$ and $m \neq 15$, M is cosymplectic conformal flat if and only if M' is cosymplectic conformal flat, moreover $r' =$ constant $= r$.

PROOF. Because of (2.4) and (2.5), $a_{0,1} = a'_{0,1}$ and $a_{1,1} = a'_{1,1}$ imply $Vol(M) = Vol(M')$ and $\int_M rdM = \int_{M'} r'dM'$. Moreover, by virtue of (5.1), $a_{2,1} = a'_{2,1}$ yields

$$\begin{aligned} &\int_M [2(n-7)|\overline{B}_0|^2 - A_1(n)|Q|^2]dM + C_1(n)\int_M r^2 dM \\ (5.2)\qquad &= \int_{M'} [2(n-7)|\overline{B}_0{}'|^2 - A_1(n)|Q'|^2]dM' + C_1(n)\int_{M'} r'^2 dM'. \end{aligned}$$

Using (5.2) and Lemma 4.1, we easily obtain our assertions. □

THEOREM 5.2. *Let M and M' be compact cosymplectic manifolds. Assume that $Spec^0 M = Spec^0 M'$ and $Spec^1 M = Spec^1 M'$. Then $\dim M = \dim M' = 2n+1 = m$ and*

(a) *for $m \geq 5$, M is η-Einstein if and only if M' is η'-Einstein, moreover $r' = constant = r$,*

(b) *for $m \geq 7$, M is cosymplectic conformal flat if and only if M' is cosymplectic conformal flat, moreover $r' = constant = r$.*

PROOF. Because of (2.1) and (2.2), $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$ imply $Vol(M) = Vol(M')$ and $\int_M rdM = \int_{M'} r'dM'$. Moreover, by virtue of (2.3) and (2.6), $a_{2,0} = a'_{2,0}$ and $a_{2,1} = a'_{2,1}$ yield

$$(5.3)\qquad \int_M [5|R|^2 + 13r^2]dM = \int_{M'} [5|R'|^2 + 13r'^2]dM',$$

$$(5.4)\qquad \int_M [10|R_1|^2 + r^2]dM = \int_{M'} [10|R_1{}'|^2 + r'^2]dM'.$$

(a) By (3.7), the equation (5.4) may be written as

$$\int_M |Q|^2 dM - \int_{M'} |Q'|^2 dM' + \frac{n+5}{10n}\left(\int_M r^2 dM - \int_{M'} r'^2 dM'\right) = 0.$$

If $Q' = 0$, then r' is constant for $n \geq 2$. Thus, by Lemma 4.1, the last equality leads to $Q = 0$ and $r = constant = r'$.

(b) Using (3.9), we rewrite (5.3) in the form

$$\int_M [5|\overline{B}_0|^2 + \frac{40}{n^2}|R_1|^2 + C_2(n)r^2]dM$$
$$= \int_{M'} [5|\overline{B}_0{}'|^2 + \frac{40}{n^2}|R_1{}'|^2 + C_2(n)r'^2]dM',$$

where $C_2(n) = \frac{1}{n^3(n+1)}(13n^4 + 13n^3 + 10n^2 - 20n - 20)$. This equality together with (5.4) gives

$$\int_M |\overline{B}_0|^2 dM - \int_{M'} |\overline{B}_0{}'|^2 dM'$$
$$+ \frac{13n^4 + 13n^3 + 6n^2 - 24n - 20}{5n^3(n+1)} (\int_M r^2 dM - \int_{M'} r'^2 dM') = 0.$$

Assume that $\overline{B}_0{}' = 0$. Then r' is constant for $n \geq 3$. In view of Lemma 4.1, the last equation yields now $\overline{B}_0 = 0$ and r = constant $= r'$. Hence we complete the proof. □

6. *Spec*$^2 M$ and the geometry of M

Assume that M is a compact cosymplectic manifold of dimension $2n+1$ and consider $Spec^2 M$. With the help of (3.7) and (3.9), the coefficient $a_{2,2}$ given by (2.9) may be written as follows :

$$\begin{aligned} a_{2,2} = \frac{1}{180} \int_M & [(n-7)(2n-15)|\overline{B}_0|^2 \\ & - \frac{2n^4 - 179n^3 + 434n^2 + 232n - 840}{n^2}|Q|^2]dM \\ & + \frac{10n^4 - 107n^3 + 310n^2 - 147n - 30}{360n(n+1)} \int_M r^2 dM. \end{aligned} \tag{6.1}$$

THEOREM 6.1. *Let M and M' be compact cosymplectic manifolds. Assume that $Spec^2 M = Spec^2 M'$. Then $\dim M = \dim M' = 2n+1 = m$ and*

(a) *for* $m = 7, 9, 13$ *or* $17 \leq m \leq 173$, M *is coymplectic conformal flat if and only if* M' *is cosymplectic conformal flat, moreover* $r' =$ *constant* $= r$,

(b) *for* $m = 15$, M *is* η*-Einstein if and only if* M' *is* η'*-Einstein, moreover* $r' =$ *constant* $= r$,

(c) *when* M *and* M' *are* η*-Einstein and* η'*-Einstein, respectively and* $m \geq 5$ *and* $m \neq 15$, M *is cosymplectic conformal flat if and only if* M' *is cosymplectic conformal flat, moreover* $r' =$ *constant* $= r$.

PROOF. The proof is based on the equalities $a_{0,2} = a'_{0,2}$, $a_{1,2} = a'_{1,2}$ and $a_{2,2} = a'_{2,2}$, where the coefficients are given by (2.7), (2.8) and (6.1). The idea of the proof is similar to that of Theorem 4.2. Therefore, we shall omit the details. □

THEOREM 6.2. *Let* M *and* M' *be compact cosymplectic manifolds. Assume that* $Spec^0 M = Spec^0 M'$ *and* $Spec^2 M = Spec^2 M'$. *Then* $\dim M = \dim M' = 2n+1 = m$ *and*

(a) *for* $m = 5$ *or* $m \geq 15$, M *is* η*-Einstein if and only if* M' *is* η'*-Einstein, moreover* $r' =$ *constant* $= r$,

(b) *when for* $m \geq 7$, M *is cosymplectic conformal flat if and only if* M' *is cosymplectic conformal flat, moreover* $r' =$ *constant* $= r$.

PROOF. Because of (2.1) and (2.2), $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$ imply $Vol(M) = Vol(M')$ and $\int_M r dM = \int_{M'} r' dM'$. Moreover, by virtue of (2.3) and (2.9), $a_{2,0} = a'_{2,0}$ and $a_{2,2} = a'_{2,2}$ yield

$$\begin{aligned}(6.2)\qquad &\int_M [(10n-23)|R|^2 + (26n-67)r^2]dM \\ &= \int_{M'} [(10n-23)|R'|^2 + (26n-67)r'^2]dM',\end{aligned}$$

$$\begin{aligned}(6.3)\qquad &\int_M [2(10n-23)|R_1|^2 + (2n-19)r^2]dM \\ &= \int_{M'} [2(10n-23)|R_1{}'|^2 + (2n-19)r'^2]dM'.\end{aligned}$$

(a) By (3.7), the equation (6.3) may be written as

$$\int_M (10n-23)|Q|^2 dM - \int_{M'} (10n-23)|Q'|^2 dM' + \frac{2n^2-9n-23}{2n}\left(\int_M r^2 dM - \int_{M'} r'^2 dM'\right) = 0.$$

Let $Q' = 0$, then r' is constant for $n \geq 2$. Thus, by Lemma 4.1, our last equality leads to $Q = 0$ and $r = \text{constant} = r'$ for $n = 2$ or $n \geq 7$.

(b) Using (3.9), we rewrite (6.2) in the form

$$\int_M [(10n-23)|\overline{B}_0|^2 + \frac{8(10n-23)}{n^2}|R_1|^2 + C_3(n)r^2]dM = \int_{M'} [(10n-23)|\overline{B}_0{}'|^2 + \frac{8(10n-23)}{n^2}|R_1{}'|^2 + C_3(n)r'^2]dM',$$

where $C_3(n) = \frac{2(10n-23)(n^2-2n-2)}{n^3(n+1)}$. This equality together with (6.3) gives

$$\int_M (10n-23)|\overline{B}_0|^2 dM - \int_{M'} (10n-23)|\overline{B}_0{}'|^2 dM' + \frac{2(6n^3-9n^2+64n+46)}{n^3(n+1)}\left(\int_M r^2 dM - \int_{M'} r'^2 dM'\right) = 0.$$

If $B_0{}' = 0$, then r' is constant for $n \geq 3$. Thus, by Lemma 4.1, the last equation yields $B_0 = 0$ and $r = \text{constant} = r'$. hence we complete the proof. □

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