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A NOTE ON THE EIGENFUNCTIONS OF THE LAPLACIAN FOR A TWISTED HOLOMORPHIC PRODUCT

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ABSTRACT. Let $Z = X \times Y$ where X and Y are complex manifolds. We suppose that projection π on the second factor is a Riemannian submersion, that TX is perpendicular to TY, and that the metrics on Z and on Y are Hermetian; we do not assume Z is a Riemannian product. We study when the pull-back of an eigenfunction of the complex Laplacian on Y is an eigenfunction of the complex Laplacian on Z.

1. Introduction

Let X and Y be smooth manifolds; we assume Y is compact and impose no further restrictions on X. Let $Z := X \times Y$ and decompose the tangent bundle

$$TZ = TX \oplus TY. \tag{1}$$

We assume the Riemannian metric on Z is a twisted product; this means that(1) is an orthogonal decomposition and that the projection π on the second factor Y is a Riemannian submersion. Equivalently, let $x = (x^i)$ and $y = (y^a)$ be local real coordinates on X and Y respectively. We adopt the Einstein convention and sum over repeated indices. The metric is a twisted product if

$$ds_Z^2 = g_{ij}(x,y)dx^i \circ dx^j + h_{ab}(y)dy^a \circ dy^b. \tag{2}$$

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Let $dvol_Y$ and $dvol_Z$ be the Riemannian volume forms on Y and on Z. Let

$$E(\lambda, \Delta_Y) := \{\phi \in C^\infty(Y) : \Delta_Y \phi = \lambda\phi\}$$
$$E(\mu, \Delta_Z) := \{\Phi \in C^\infty(Z) : \Delta_Z \Phi = \mu\Phi\}$$

be the eigenspaces of the Laplace Beltrami operators $\Delta := \delta d$ on Y and on Z respectively. Define pull-back $\pi^* : C^\infty(Y) \to C^\infty(Z)$ by $\pi^*\phi := \phi \circ \pi$. In earlier work [5], we determined necessary and sufficient conditions for π^* to intertwine Δ_Y and Δ_Z; see also [2, 4] for related work.

THEOREM 1.1. *Let $Z = X \times Y$ where X and Y are real manifolds with Y closed. We suppose that projection π on the second factor is a Riemannian submersion and that TX is perpendicular to TY or equivalently that the metric on Z has the form given by equation (2). Then the following assertions are equivalent*

(a) *For all $\lambda \in \mathbb{R}$, we have $\pi^* E(\lambda, \Delta_Y) \subset E(\lambda, \Delta_Z)$.*
(b) *For all $\lambda \in \mathbb{R}$, there exists $\mu(\lambda) \in \mathbb{R}$ so $\pi^* E(\lambda, \Delta_Y) \subset E(\mu(\lambda), \Delta_Z)$.*
(c) *There exists a volume form $d\nu_X$ on X so that $dvol_Z = d\nu_X \wedge dvol_Y$.*

In this brief note, we study the complex or Dolbeault Laplacian and establish the corresponding result for that operator. In §2, we review the material from complex geometry that we shall need. In §3, we state and prove the main result of this paper.

2. Review of complex geometry

Let $w = (w^i)$ for $w^i = u^i + \sqrt{-1}v^i$ be local coordinates on a complex manifold M. We define:

$$J(\partial/\partial u^i) := \partial/\partial v^i, \ J(\partial/\partial v^i) := -\partial/\partial u^i,$$
$$\partial_i^w := (\partial/\partial u^i - \sqrt{-1}\partial/\partial v^i)/2, \ \partial_i^{\bar w} := (\partial/\partial u^i + \sqrt{-1}\partial/\partial v^i)/2,$$
$$dw^i := du^i + \sqrt{-1}dv^i, \ d\bar w^i := du^i - \sqrt{-1}dv^i,$$
$$\Lambda^{(1,0)}(M) := \text{span}_{\mathbb{C}}\{dw^i\}, \ \Lambda^{(0,1)}(M) := \text{span}_{\mathbb{C}}\{d\bar w^i\},$$
$$\partial f := \partial_i^w(f)dw^i, \ \bar\partial f := \partial_i^{\bar w}(f)d\bar w^i.$$

We note that the almost complex structure J on the tangent bundle $T(M)$, that the vector bundles $\Lambda^{(1,0)}(M)$ and $\Lambda^{(0,1)}(M)$, and that the operators

$$\partial : C^\infty(M) \to C^\infty(\Lambda^{(1,0)}(M)) \text{ and } \bar\partial : C^\infty(M) \to C^\infty(\Lambda^{(0,1)}(M))$$

are all invariantly defined and are independent of the particular coordinate system chosen. Let $g_M(\cdot,\cdot)$ be a Riemannian metric on M. We say g_M is *Hermitian* if $g_M(X,Y) = g_M(JX,JY)$ for all real tangent vectors or equivalently if we can represent the metric in the form

$$(3) \qquad ds^2_M = g_{i\bar\jmath} dw^i \circ d\bar w^j \text{ where } g_{i\bar\jmath} = \bar g_{j\bar\imath}.$$

We assume g_M is Hermitian henceforth. We extend g_M to the complexified tangent bundle to be complex linear in the first factor and conjugate linear in the second factor; we extend g_M dually to the complexified cotangent bundle similarly. Then

$$\begin{aligned}
&g_M(\partial_i^w, \partial_j^w) = g_{i\bar\jmath}/2,\ g_M(\partial_i^w, \partial_j^{\bar w}) = 0,\ g_M(\partial_i^{\bar w}, \partial_j^{\bar w}) = \bar g_{i\bar\jmath}/2,\\
&g_M(dw^i, dw^j) = 2g^{i\bar\jmath},\ g_M(dw^i, d\bar w^j) = 0,\ g_M(d\bar w^i, d\bar w^j) = 2\bar g^{i\bar\jmath},\\
&dvol_M = g\,du^1 \wedge dv^1 \wedge \ldots \wedge du^m \wedge dv^m \text{ for } g = \det(g_{i\bar\jmath}).
\end{aligned}$$

Let δ' and δ'' be the adjoints of the operators ∂ and $\bar\partial$ and let Δ^c_M be the complex or Dolbeault Laplacian;

$$\begin{aligned}
&\delta' : C^\infty(\Lambda^{(1,0)}(M)) \to C^\infty(M),\\
&\delta'' : C^\infty(\Lambda^{(0,1)}(M)) \to C^\infty(M),\\
&\Delta^c_M := \delta''\bar\partial : C^\infty(M) \to C^\infty(M).
\end{aligned}$$

LEMMA 2.1.

(a) *If* $\omega = \omega_{\bar\imath} d\bar w^i \in C^\infty(\Lambda^{(0,1)}(M))$, *then* $\delta''\omega = -2g^{-1}\partial_j^w(g g^{j\bar\imath}\omega_{\bar\imath})$.

(b) *If* $f \in C^\infty(M)$, *then* $\Delta^c_M(f) = -2g^{-1}\partial_j^w(g g^{j\bar\imath}\partial_i^{\bar w} f)$.

PROOF. Let $w = (w^i)$ be complex coordinates on an open set W of M. Let $f \in C_0^\infty(W)$ be a smooth function with compact support in W. We integrate by parts in the following computation; the boundary terms vanish as f has compact support:

$$\begin{aligned}(\delta''\omega, f)_{L^2} =&\textstyle\int g_M(\omega, \bar\partial f) = \int g_M(\omega, \partial_j^{\bar w} f d\bar w^j) g du^1 ... = 2\int g g^{j\bar\imath}\omega_{\bar\imath}\overline{\partial_j^{\bar w} f} du^1 ...\\ =&\textstyle 2\int g g^{j\bar\imath}\omega_{\bar\imath}\partial_j^{w}\bar f du^1 ... = -2\int \partial_j^w (g g^{j\bar\imath}\omega_{\bar\imath})\bar f du^1 ...\\ =&\textstyle -2\int g^{-1}\partial_j^w (g g^{j\bar\imath}\omega_{\bar\imath})\bar f dvol_M = (-2g^{-1}\partial_j^w(g g^{j\bar\imath}\omega_{\bar\imath}), f)_{L^2}.\end{aligned}$$

This identity for all $f \in C_0^\infty(M)$ establishes the first asssertion; the second assertion now follows. □

We shall need the following technical fact. Let $Z = X \times Y$ where X and Y are complex manifolds. Let π be projection on the second factor; $\pi^*\bar\partial_Y = \bar\partial_Z \pi^*$. Let $x = (x^i)$ and $y = (y^a)$ be local holomorphic coordinates on X and Y. We suppose that π is a Riemannian submersion, that TX is perpendicular to TY, and that the metrics on Z and on Y are Hermitian; this is equivalent to supposing that

$$ds_Z^2 = g_{i\bar j}(x,y)dx^i \circ d\bar x^j + h_{a\bar b}(y)dy^a \circ d\bar y^b. \tag{4}$$

LEMMA 2.2. *Expand $dvol_Z = e^{\theta(x,y)} d\tilde\nu_X dvol_Y$ where $d\tilde\nu_X$ is any volume form on X. If $\phi \in C^\infty(Y)$, then*

$$\begin{aligned}\Delta_Z \pi^*\phi &= \pi^*\Delta_Y\phi - g_Z(d\pi^*\phi, d\theta)\\ \Delta_Z^c \pi^*\phi &= \pi^*\Delta_Y^c\phi - g_Z(\bar\partial\pi^*\phi, \bar\partial\theta).\end{aligned}$$

PROOF. We refer to [5] for the proof of the first identity. We use equation (4) and Lemma 2.1 to prove the second identity. Let $g := \det(g_{i\bar j})$ and $h := \det(h_{a\bar b})$. Expand $x_i = s_i + \sqrt{-1}t_i$ and $y_a = u_a + \sqrt{-1}v_a$. Choose $\psi(x)$ so $d\tilde\nu_X = e^{\psi(x)} ds_1 \wedge dt_1 \wedge ....$ Then

$$\begin{aligned}dvol_Z =& g h ds_1 \wedge dt_1 \wedge ... \wedge du_1 \wedge dv_1 \wedge ... = e^{\theta(x,y)} d\tilde\nu_X \wedge dvol_Y\\ =&\{e^{\theta(x,y)+\psi(x)} ds_1 \wedge dt_1 \wedge ...\} \wedge \{h(y) du_1 \wedge dv_1 \wedge ...\}.\end{aligned}$$

Consequently $g = e^{\theta(x,y)+\psi(x)}$. Let $\Phi := \pi^*\phi$; $\partial_j^x\Phi = 0$ so by Lemma 2.1

$$\begin{aligned}\Delta_Z^c\Phi - \pi^*\Delta_Y^c\phi =& -2g^{-1}h^{a\bar b}(\partial_a^Y g)(\partial_b^{\bar Y}\Phi) = -2h^{a\bar b}(\partial_a^Y\theta)(\partial_b^{\bar Y}\Phi)\\ =& -g_Z(\bar\partial\Phi, \bar\partial\theta)(x,y).\end{aligned}$$

□

3. The main results of the paper

We first generalize Theorem 1.1 to the complex setting where the base Y is closed. Then we consider the case in which the base is compact with smooth boundary and impose Dirichlet boundary conditions. We begin with

THEOREM 3.1. *Let $Z = X \times Y$ where X and Y are complex manifolds with Y closed. We suppose that projection π on the second factor is a Riemannian submersion, that TX is perpendicular to TY, and that the metrics on Z and on Y are Hermitian. This is equivalent to assuming that the metric on Z has the form given by equation (4). Then the following assertions are equivalent*

(a) *For all $\lambda \in \mathbb{R}$, we have $\pi^* E(\lambda, \Delta_Y) \subset E(\lambda, \Delta_Z)$.*

(b) *For all $\lambda \in \mathbb{R}$, there exists $\mu(\lambda) \in \mathbb{R}$ so $\pi^* E(\lambda, \Delta_Y) \subset E(\mu(\lambda), \Delta_Z)$.*

(c) *There exists a volume form $d\nu_X$ on X so that $dvol_Z = d\nu_X \wedge dvol_Y$.*

(d) *For all $\lambda \in \mathbb{R}$, we have $\pi^* E(\lambda, \Delta_Y^c) \subset E(\lambda, \Delta_Z^c)$.*

(e) *For all $\lambda \in \mathbb{R}$, there exists $\mu(\lambda) \in \mathbb{R}$ so $\pi^* E(\lambda, \Delta_Y^c) \subset E(\mu(\lambda), \Delta_Z^c)$.*

PROOF. The equivalence of (a), (b), and (c) was previously established in [5] in the real category. We must now establish the equivalence of (c), (d), and (e). If (c) holds, we may take $\theta = 1$ and use Lemma 2.2 to see $\Delta_Z^c \pi^* = \pi^* \Delta_Y^c$ which implies (d). It is immediate that (d) implies (e). Finally, suppose that (e) holds. Let $\phi \in E(\lambda, \Delta_Y^c)$ for $\lambda \neq 0$. Let $\Phi := \pi^* \phi \in E(\mu, \Delta_Z^c)$. Since Y is compact and $\lambda \neq 0$, ϕ is orthogonal to the constant functions in $L^2(Y)$. We fix $x \in X$ and use Lemma 2.2 to compute

$$\begin{aligned} 0 &= (\lambda - \mu) \textstyle\int_Y \phi(y) dvol_Y = \int_Y g_Y(\bar\partial_Y \phi(y), \bar\partial_Y \theta(x,y)) dvol_Y \\ &= \textstyle\int_Y (\delta_Y'' \bar\partial_Y \phi(y)) \theta(x,y) dvol_Y = \lambda \int_Y \phi(y) \theta(x,y) dvol_Y. \end{aligned}$$

This implies that as a function of y with x fixed, $\theta(x,y)$ is perpendicular to ϕ in $L^2(Y)$ to every non-constant eigenfunction of Δ_Y^c. Since Y is compact and since Δ_Y^c is a self-adjoint, elliptic second order partial differential operator, we may take a spectral resolution of Δ_Y^c to construct an orthogonal direct sum decomposition $L^2(Y) = \oplus_\lambda E(\lambda, \Delta_Y^c)$.

Consequently if we fix x, $\theta(x,y) \in E(0,\Delta_Y^c)$. This implies $\theta(x,y)$ is holomorphic in y and hence constant in y. Therefore $\theta(x,y)=\theta(x)$. We now set $d\nu_X = e^{\theta(x)} d\tilde{\nu}_X$ to construct a measure on X so that (c) holds. □

REMARK 3.2. If M is Kaehler, then $\Delta_M = 2\Delta_M^c$ and consequently the equivalence of (c) with (d) and (e) is a trivial consequence of Theorem 1.1. However, the metrics in (4) need not be Kaehler and thus Theorem 3.1 is not a direct consequence of Theorem 1.1.

In [4] we considered warped product metrics. We proved

THEOREM 3.3. *If $Z = X \times Y$ with $ds_Z^2 = e^{2h(x,y)} ds_X^2 + ds_Y^2$ is a warped product, if $\Delta_Z \pi^* = \pi^* \Delta_Y$, and if Y is compact, then $h = h(x)$ so Z is a Riemannian product.*

REMARK 3.4. If $\Delta_Z^c \pi^* = \pi^* \Delta_Y^c$, then Theorem 3.1 implies $\Delta_Z \pi^* = \pi^* \Delta_Y$ and we may conclude again Z is a Riemannian product. Thus Theorem 3.3 generalizes to the complex category. However, Theorem 3.3 is false if we replace the metric $ds_Z^2 = e^{2h(x,y)} ds_X^2 + ds_Y^2$ by a more general metric as given in equation (4). Let T be the complex torus $S^1 \times S^1$. Let $X = T \times T$ and $Y = T$ so $Z = T \times T \times T$ with the complex parameters (x_1, x_2, y) for $(x_1, x_2) \in X$ and $y \in Y$. Let

$$ds_Z^2 = e^{2h(y)} dx^1 \circ d\bar{x}^1 + e^{-2h(y)} dx^2 \circ d\bar{x}^2 + dy \circ d\bar{y};$$

this is not a Riemannian product. However, since condition (c) of Theorem 3.1 holds for this metric, π^* intertwines both the real and complex Laplacians. This metric is an example of a Riemannian submersion with minimal and not totally geodesic fibers and is is closely related to [3, Example 4.1].

We have assumed previously that Y was closed. We now relax this condition.

LEMMA 3.5. *Let $Z = X \times Y$ where X and Y are real manifolds. We suppose that projection π on the second factor is a Riemannian submersion and that TX is perpendicular to TY or equivalently that the metric on Z has the form given by equation (2). Assume Y is compact with* *smoooth boundary and impose Dirichlet boundary conditions to define the eigenspaces $E(\lambda, \Delta_Y)$. If $\phi \in E(\lambda, \Delta_Y)$ and if $\pi^*\phi \in E(\mu, \Delta_Z)$, then $\lambda = \mu$.*

PROOF. Suppose the contrary and choose λ so $\mu(\lambda) \neq \lambda$ and $E(\lambda, \Delta_Y) \neq 0$. Then $(\lambda - \mu)\phi(y) = g_Y(d_Y\theta(x, y), d_Y\phi(y))$. Since Δ_Y is a real operator, we may assume without loss of generality that ϕ is real and non-trivial. Choose y so $\phi(y)$ is minimal or maximal. Since ϕ satisfies Dirichlet boundary conditions, y is in the interior of M. Then $d_Y\phi(y) = 0$ and thus $\phi(y) = 0$. This shows $\phi = 0$ which contradicts our basic assumption. □

REMARK 3.6. We note that the argument given to prove Lemma 3.5 fails for Δ_Y^c because this operator is not real; we do not know whether Lemma 3.5 holds for the complex Laplacian; i.e. we do not know if the pull back of an eigenfunction of the complex Laplacian on the base can be an eigenfunction of the complex Laplacian on the total space with a different eigenvalue.

THEOREM 3.7. *Let $Z = X \times Y$ where X and Y are complex manifolds. We suppose that projection π on the second factor is a Riemannian submersion, that TX is perpendicular to TY, and that the metrics on Z and on Y are Hermitian. This is equivalent to assuming that the metric on Z has the form given by equation (4). Assume Y is compact with smoooth boundary and impose Dirichlet boundary conditions to define the eigenspaces $E(\lambda, \Delta_Y)$ and $E(\lambda, \Delta_Y^c)$. Then the following assertions are equivalent*

(a) *For all $\lambda \in \mathbb{R}$, we have $\pi^*E(\lambda, \Delta_Y) \subset E(\lambda, \Delta_Z)$.*
(b) *For all $\lambda \in \mathbb{R}$, there exists $\mu(\lambda) \in \mathbb{R}$ so $\pi^*E(\lambda, \Delta_Y) \subset E(\mu(\lambda), \Delta_Z)$.*
(c) *There exists a volume form $d\nu_X$ on X so that $dvol_Z = d\nu_X \wedge dvol_Y$.*
(d) *For all $\lambda \in \mathbb{R}$, we have $\pi^*E(\lambda, \Delta_Y^c) \subset E(\lambda, \Delta_Z^c)$.*

PROOF. If (c) holds, we can use Lemma 2.2 to see that (a) and (d) hold. Conversely suppose (a) holds. Fix $x \in X$. Then $g_Y(d_Y\theta(x, y), d_Y\phi(y)) = 0$ for all $y \in Y$. Since $\text{span}\{\phi : \phi \in E(\lambda, \Delta_Y)\}$ is dense in $C_0^\infty(Y)$, we conclude $d_Y\theta(x, y) = 0$ for y in the interior of Y and hence

$\theta(x, y)$ is independent of y. Thus (a) implies (c); similarly (d) implies (c). It is immediate that (a) implies (b). By Lemma 3.5, (b) implies (a). □

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