Comm. Korean Math. Soc. **12**(1997), No. 2, pp. 311-324

ON THE BEREZIN TRANSFORM ON D^n

JAESUNG LEE

ABSTRACT. We show that if $f \in L^\infty(D^n)$ satisfies $Sf = rf$ for some r in the unit circle , where S is any convex combinatiom of the iterations of Berezin operator , then f is $n-$ harmonic. And we give some remarks and a conjecture on the space

$$M_2 = \{f \in L^2(D^2, m \times m) | Bf = f\}.$$

1. Introduction

Let m be the Lebesque measure on C normalized to $m(D) = 1$ for the unit disc D, and B be the Berezin operator on the polydisc D^n defined by
For $f \in L^1(D^n, m \times \cdots \times m)$

$$(Bf)(z_1, \cdots, z_n) = \int_D \cdots \int_D f\Big(\varphi_{z_1}(x_1), \cdots, \varphi_{z_n}(x_n)\Big)\, dm(x_1) \cdots dm(x_n)$$

where

$$\varphi_a(w) = \frac{a - w}{1 - \bar{a}w}.$$

In [5], the author showed that if $f \in L^\infty(D^n)$ satisfies $Bf = f$, then f is $n-$ harmonic (Theorem 3.1). In this paper we extend that result to more generalized cases (Theorem 2.5).
Also in [5], the author showed that for $1 \le p < \infty$ there are joint eigenfunctions of invariant Laplacians with uncountably many eigenvalues which are invariant under the Berizin transform in $L^p(D^n, m \times \cdots \times m)$.

Received March, 8, 1997. Revised April 1, 1997.
1991 Mathematics Subject Classification: 47A15, 46C15.
Key words and phrases: Berezin transform, joint eigenspaces, Hilbert space, n-harmonic.
This research is partially suppoted by TGRC-KOSEF.

In this paper , we try to characterize $f \in L^2(D^2, m \times \cdots \times m)$ which satisfy $Bf = f$ by proposing a conjecture and support that conjecture in some special cases.

2. Functions fixed by Berezin transform

Here we generalize the Theorem 3.1 of [5]. Following definitions coincide with those of [5].

DEFINITION 2.1. The invariant measure μ on D is defined by $d\mu(z) = (1-|z|^2)^{-2}\, dm(z)$, which satisfies

$$\int_D u \circ \psi \, d\mu = \int_D u \, d\mu, \quad \text{for all } u \in L^1(D,\mu), \text{ and for all } \psi \in Aut(D).$$

Then we define $L^p_R = L^p_R(D^n)$ the subspace of $L^p(D^n, \mu \times \cdots \times \mu)$ consists of radial functions i.e

$$L^p_R = \{ f \in L^p(D^n, \mu \times \cdots \times \mu) \mid f(\,|z_1|, \cdots, |z_n|\,) = f(z_1, \cdots, z_n) \}.$$

For $f \in L^p(D^n, \mu \times \cdots \times \mu), g \in L^q(D^n, \mu \times \cdots \times \mu)$ we denote

$$\langle f, g \rangle = \int_D \cdots \int_D f \cdot g \, d\mu \cdots d\mu.$$

By the same methods as Lemma 3.3 of [5], we immediately get the following

LEMMA 2.2. *For $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ ($p = \infty$ means $q = 1$)*

(a) *B is a bounded linear operator on $L^p(D^n, \mu \times \cdots \times \mu)$ with $\|B\| \le 1$.*
(b) *For $f \in L^p(D^n, \mu \times \cdots \times \mu)$, $g \in L^q(D^n, \mu \times \cdots \times \mu)$ we have $\langle Bf, g\rangle = \langle f, Bg\rangle$.*

LEMMA 2.3. *For $f \in L^1_R(D^n)$*

$$\lim_{n \to \infty} \|B^n f\|_1 = 0 \quad \text{if and only if} \quad \int_D \cdots \int_D f \, d\mu \cdots d\mu = 0.$$

PROOF. ($\Rightarrow$) Obvious from the fact that

$$\int_D \cdots \int_D B^n f \, d\mu \cdots d\mu \;=\; \int_D \cdots \int_D f \, d\mu \cdots d\mu \quad \text{for all } n \geq 0.$$

($\Leftarrow$) The proof is very similar to that of Lemma 3.5 of [5]. We give an outline. B is the linear contraction on $L^1_R(D^n)$ with the spectrum.

$$\sigma(B) \;=\; \{\, h(\alpha_1)\cdots h(\alpha_n) \mid 0 \leq \; Re\ \alpha_i \; \leq 1,\; i = 1,\cdots,n \,\}$$

where

$$h(z) \;=\; \frac{\pi z(1-z)}{\sin \pi z}.$$

Hence by 2.7 of [5], $\sigma(B)$ intersects the unit circle only at a point $z = 1$. Thus by Theorem 1 of [4]

$$\lim_{n\to\infty} \|B^n f\|_1 \;=\; 0, \quad \text{for all } f \in (I - B)L^1_R.$$

Now define

$$X \;=\; \left\{ f \in L^1_R \mid \int_D \cdots \int_D f \, d\mu \cdots d\mu = 0 \right\}.$$

Then we immediately get $(I - B)L^1_R \subset X$. But from Theorem 3.1 of [5] we know if $g \in L^\infty_R(D^n)$ satisfies $Bg = g$ then g is a constant.
Combine this and Lemma 2.2 , then using the Hahn-Banach theorem we get that $(I - B)L^1_R$ is dense in X.
Hence

$$\lim_{n\to\infty} \|B^n f\|_1 \;=\; 0, \quad \text{for all } f \in X.$$

□

PROPOSITION 2.4. *If $f \in L^\infty(D^n)$, $f \not\equiv 0$ satisfies $B^m f = rf$ for some r with $|r| = 1$ and for some $m \in \mathbf{N}$, then f is $n-$harmonic and $r = 1$.*

PROOF. First assume that f is radial. Suppose $f \in L^\infty_R(D^n)$ satisfy $B^m f = rf$ for some $m \in \mathbf{N}$ and $|r| = 1$.
Pick any $g \in L^1_R(D^n)$ satisfying

$$\int_D \cdots \int_D g \, d\mu \cdots d\mu \;=\; 0.$$

Then by Lemma 2.3, we get

$$\lim_{k\to\infty} \| B^{mk} g \|_1 \;=\; 0.$$

Hence

$$\lim_{k\to\infty} | \langle B^{mk} g, \; f \rangle | \;\leq\; \|f\|_\infty \lim_{k\to\infty} \| B^{mk} g \|_1 \;=\; 0.$$

But for all $k \geq 0$

$$\begin{aligned} \langle B^{mk} g, f \rangle &= \langle g, \; B^{mk} f \rangle \quad \text{by 2.2 } (b) \\ &= r^{mk} \langle g, \; f \rangle \end{aligned}$$

Hence $\langle g, f \rangle = 0$. This implies that f is a constant , which implies $r = 1$ since $f \not\equiv 0$. For a general $f \in L^\infty(D^n)$, the radialization Rf satisfies

$$B(Rf) = R(Bf) = rRf.$$

Hence Rf is a constant and $r = 1$.
The remaining part of the proof is identical to the step (ii) of 3.6 in [5]. □

THEOREM 2.5. *Let $0 < \alpha_k < 1$ satisfy*

$$\sum_{k=1}^{\ell} \alpha_k \;=\; 1$$

and m_k be positive integers for $k = 1, 2, \cdots, \ell$.
If $f \in L^\infty(D^n)$ satisfies

$$\left(\sum_{k=1}^{\ell} \alpha_k B^{m_k} \right) f \;=\; rf$$

for some $|r| = 1$, then f is $n-$harmonic.

PROOF. Let

$$S = \sum_{k=1}^{\ell} \alpha_k B^{m_k} \quad \text{and} \quad X = \{ f \in L^\infty(D^n) \mid Sf = rf \}.$$

Now fix j $(1 \le j \le \ell)$ and define U on $L^\infty(D^n)$ by

$$U = \frac{1}{1-\alpha_j} \sum_{k \neq j} \alpha_k B^{m_k}.$$

Pick any $f \in X$, then

$$SB^{m_j} f = B^{m_j} Sf = rB^{m_j} f.$$

Hence $B^{m_j} f \in X$.
By the same way, $Uf \in X$. Then by Lemma 2.1, B^{m_j} and U are contractions on the Banach Space X.
And on $L^\infty(D^n)$,

$$S = \alpha_j B^{m_j} + (1-\alpha_j)U \tag{1}$$

If we show that $B^{m_j} = rI$ on X, then by the previous proposition, X consists of $n-$ harmonic functions and $r = 1$, which completes the proof.
□

Now let P be an operator on X defined by

$$P = \alpha_j B^{m_j} - \alpha_j rI \quad (\text{on } X)$$

Let X^* be the dual space of X, and $(B^{m_j})^*, U^*, P^*$ be the adjoints of B^{m_j}, U, P on X^* , respectively. For $g \in X^*$, we denote

$$(B^{m_j})^* q = q_1 \quad \text{and} \quad U^* q = q_2.$$

Since B^{m_j}, U are contractions on X, we get

$$\| q_1 \| \le \|q\| \quad \text{and} \quad \| q_2 \| \le \|q\| \quad \text{on } X^*.$$

Now let A^* be the closed unit ball of X^*. Assume that q is an extreme point of A^*. From(1),

$$rI \;=\; \alpha_j B^{m_j} \;+\; (1-\alpha_j)U \quad \text{on } X.$$

hence we get

$$rq \;= \alpha_j q_1 \;+\; (1-\alpha_j) q_2.$$

Since q is an extreme point, this forces

$$q \;=\; \frac{q_1}{r} \;=\; \frac{q_2}{r}.$$

Therefore on X^*,

$$P^* q \;=\; \alpha_j (B^{m_j})^* q \;-\; \alpha_j r q \;=\; \alpha_j q_1 - \alpha_j r q \;=\; 0.$$

But by Krein-Milman , A^* is the closed convex hull of the set of its extreme points. It follows that $P^* \;\equiv\; 0$ on A^*.
Hence $P \;\equiv\; 0$ on X. From (2), it is equivalent to saying that $B^{m_j} \;=\; rI$ on X.

This completes the proof.

3. On the space $\mathbf{M}_2 = \{\, f \in L^2(D^2, m \times m) \mid Bf = f \,\}$

In [5], the author showed that the space

$$M_p \;=\; \{\, f \in L^p(D^2, m \times m) \mid Bf = f \,\}$$

has eigenfunctions with uncountably many joint eigenvalues of invariant Laplacians $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$, when $1 \le p < \infty$.

In [1], the author showed that when $n \ge 12$ the space

$$M \;=\; \{\, f \in L^1(B_n) \mid T_0 f = f \,\}$$

is the direct sum of finitely many eigensapces of invariant Laplacian. (Here T_0 is the Berezin transform on the $n-$ dimensional unit ball B_n) Our attempt to characterize the space M_2, like [1] did in the unit ball, was not successful. Instead, we have the following.

3.1 Conjecture

"The space M_2 is generated by the point eigenfunctions of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ in M_2." (i.e the set of all finite sum of the joint eigenfunctions in M is dense in M.

We will be back to mention about the conjecture later. Here like [5], we will write T as the Berezin transform on D. i.e for $u \in L^1(D,m)$

$$\begin{aligned}(Tu)(z) &= \int_D u(\varphi_z(x))\ dm(x)\\ &= \int_D u(x)K(z,x)\ dm(x)\end{aligned}$$

where

$$K(z,x) = \frac{(1-|z|^2)^2}{|1-\bar{z}x|^4}.$$

Next proposition shows that B is bounded on L^2, which leads the boundedness of invariant Laplacian on M_2, in the proof we use similar technique to that of [3].

PROPOSITION 3.2. *B is a bounded operator on $L^p(D^2, m\times m)$ when $p>1$, but not bounded on $L^1(D^2, m\times m)$.*

PROOF. Step (i) : First we will prove that the operator T is bounded on $L^p(D,m)$ when $p>1$. For $p>1$, let $q=p/(p-1)$ so that $1/p+1/q=1$.

By 1.4.10 of [6] and simple calculation , there exist $c_1, c_2 > 0$ such that

$$\int_D K(z,x)\ (1-|x|^2)^{-\frac{1}{p}}\ dm(x) \le c_1\ (1-|z|^2)^{-\frac{1}{p}} \tag{2}$$

and

$$\int_D K(z,x)\ (1-|z|^2)^{-\frac{1}{q}}\ dm(z) \le c_2\ (1-|x|^2)^{-\frac{1}{q}}. \tag{3}$$

Now for $u \in L^1(D,m)$, we have

$$
\begin{aligned}
|Tu(z)| &\le \int_D K(z,x)|u(x)|\ dm(x)\\
&= \int_D K(z,x)^{\frac{1}{q}}(1-|x|^2)^{-\frac{1}{pq}}K(z,x)^{\frac{1}{p}}(1-|x|^2)^{\frac{1}{pq}}|u(x)|\ dm(x)\\
&\le \left\{ \int_D K(z,x)(1-|x|^2)^{-\frac{1}{p}}\ dm(x) \right\}^{\frac{1}{q}}\cdot\\
&\qquad \left\{ \int_D K(z,x)(1-|x|^2)^{\frac{1}{q}}|u(x)|^p\ dm(x)\right\}^{\frac{1}{p}}\\
&\le c_1^{\frac{1}{q}}(1-|z|^2)^{-\frac{1}{pq}}\\
&\qquad \left\{ \int_D K(z,x)(1-|x|^2)^{\frac{1}{q}}|u(x)|^p dm(x)\right\}^{\frac{1}{p}} \quad \text{by } (2)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_D |Tu(z)|^p\ dm(z)\\
&\le \int_D c_1^{\frac{p}{q}}(1-|z|^2)^{-\frac{1}{q}}\int_D K(z,x)(1-|x|^2)^{\frac{1}{q}}|u(x)|^p\ dm(x)\ dm(z)\\
&= c_1^{\frac{p}{q}}\int_D (1-|x|^2)^{\frac{p}{q}}|u(x)|^p\\
&\qquad \int_D K(z,x)(1-|z|^2)^{-\frac{1}{q}}\ dm(z)\ dm(x) \quad \text{by Fubini}\\
&\le c_1^{\frac{p}{q}}\int_D (1-|x|^2)^{\frac{1}{q}}|u(x)|^p\ c_2(1-|x|^2)^{-\frac{1}{q}}\ dm(x) \quad \text{by } (3)\\
&= c_1^{\frac{p}{q}}\ c_2\int_D |u(x)|^p\ dm(x)
\end{aligned}
$$

Hence if we let $c = c_1^{\frac{1}{q}}\ c_2^{\frac{1}{p}}$, then we have

(4) $$\|Tu\|_p \le c\|u\|_p.$$

Step (ii) : Let $f \in L^1(D^2, m\times m)$, then

$$(Bf)(z,w) \ = \ \int\int_{D^2} f(x,y)\ K(z,x)\ K(w,y)\ dm(x)\ dm(y).$$

Thus

$$\begin{aligned}
&\int\int_{D^2} |Bf(z,w)|^p \, dm(z)\, dm(w) \\
&\leq \int\int_{D^2} \left\{ \int\int_{D^2} |f(x,y)|\; K(z,x)\; K(w,y)\; dm(x)\; dm(y) \right\}^p dm(z)\; dm(w) \\
&= \int\int_{D^2} \left\{ \int_D K(w,y) \left(\int_D |f(x,y)|\; K(z,x)\; dm(x) \right) dm(y) \right\}^p dm(z) dm(w) \\
&\leq \int\int_{D^2} c^p \left(\int_D |f(x,y)|\; K(z,x)\; dm(x) \right)^p dm(z)\; dm(w) \quad \text{by } (4) \\
&\leq \int\int_{D^2} c^{2p}\; |f(z,w)|^p\; dm(z)\; dm(w)
\end{aligned}$$

This proves that B is a bounded operator on $L^p(D^2, m\times m)$, for $p > 1$.
When $p = 1$.
From its definition, the norm of B on $L^1(D^2, m \times m)$ is

$$\begin{aligned}
\|B\|_1 &= \sup_{(x,y)\in D\times D} \int\int_{D^2} K(z,x)\; K(w,y)\; dm(z) dm(w) \\
&= \sup_{(x,y)\in D\times D} \int_D \frac{(1-|z|^2)^2}{|1-x\overline{z}|^4}\; dm(z) \int_D \frac{(1-|w|^2)^2}{|1-y\overline{w}|^4}\; dm(y)
\end{aligned}$$

But by 1.4.10 of [6], we get

$$\int_D \frac{(1-|z|^2)^2}{|1-z\overline{x}|^4}\; dm(z) \;\approx\; \log\frac{1}{1-|x|^2}$$

which is unbounded on D. Hence B is not bounded on $L^1(D^2, m \times m)$ and this completes the proof of proposition. □

DEFINITION 3.3. For $f \in L^1(D^2, m \times m)$ and $k, \ell = 0, 1, 2, \cdots$, we define the operator $T_{k,\ell}$ on $L^1(D^2, m \times m)$ by

$$\begin{aligned}
(\, T_{k,\ell} f\,)(z,w) = \;& (k+1)(\ell+1)\cdot \\
& \int\int_{D^2} (1-|x|^2)^k (1-|y|^2)^\ell f(\varphi_z(x), \varphi_w(y))\; dm(x)\; dm(y)
\end{aligned}$$

and by replacing x, y by $\varphi_z(x)$ and $\varphi_w(y)$ we get

$$\big(T_{k,\ell}f\big)(z,w) = (k+1)\,(\ell+1)\cdot \int\!\!\int_{D^2}\left(\frac{(1-|x|^2)^k(1-|z|^2)^{k+2}}{|1-z\overline{x}|^{2k+4}}\cdot\frac{(1-|y|^2)^\ell(1-|w|^2)^{\ell+2}}{|1-w\overline{y}|^{2\ell+4}}\right)\cdot f(x,y)\;dm(x)\;dm(y)$$

In our definition we can see $T_{0,0} = B$.
Using the same method as Proposition 3.2 we get the following corollary.

COROLLARY 3.4. *For $k, \ell \geq 0, T_{k,\ell}$ is a bounded operator on $L^p(D^2, m\times m)$ when $p > 1$.*

The following properties of $T_{k,\ell}$ can be obtained using methods of [1], [5] and some straightforward calculations.

3.5 Properties of $T_{k,\ell}$.

(a) For $k, \ell \geq 0, \psi \in Aut(D^2), f \in L^1(D^2, m\times m)$
$\big(T_{k,\ell}f\big)\circ\psi = T_{k,\ell}\big(f\circ\psi\big)$.
(b) For $k, \ell > 0, T_{k,\ell}$ is a bounded linear operator on $L^1(D^2, m\times m)$.
(c) For $f \in L^1(D^2, m\times m)$

$$\tilde{\Delta}_1 T_{k,\ell}f = 4(k+1)(k+2)\big(\,T_{k,\ell}f - T_{k+1,\ell}f\,\big)$$

$$\tilde{\Delta}_2 T_{k,\ell}f = 4(\ell+1)(\ell+2)\big(\,T_{k,\ell}f - T_{k,\ell+1}f\,\big)$$

And

$$T_{k,\ell}f = G_k(\tilde{\Delta}_1)G_\ell(\tilde{\Delta}_2)Bf$$

Where

$$G_k(z) = \prod_{i=1}^{k}\left(1 - \frac{z}{4i(i+1)}\right).$$

(d) On $L^1(D^2, m\times m)$, the operators B and $T_{k,\ell}$ commute for all $k, \ell \geq 0$.
(e) For all $f \in L^1(D^2, m\times m)$,

$$\lim_{n\to} \|\, f - T_{n,n}f \,\|_1 = 0.$$

3.6 On the space $\mathbf{M_2} = \{f \in L^2(D^2, m \times m) | Bf = f\}$

In an attempt to characterize M_2 as [1] did in the unit ball B_n , we use the following approach.
The space M_2 is a closed Hilbert space which consists of real analytic functions. For convenience , we denote Δ_1 (Δ_2) as the restriction of $\tilde{\Delta}_1$ ($\tilde{\Delta}_2$) to M_2. Then by 3.5 (c), for $f \in M_2$ we get

$$\Delta_1 f = \Delta_1 Bf = 8\big(f - T_{1,0}f\big)$$

and $T_{1,0}$ is bounded on $L^2(D^2)$ (corollary 3.4) , and by 3.5(d) ,

$$\begin{aligned} B\big(\Delta_1 f\big) &= 8\big(\ Bf \ - \ BT_{1,0}f\ \big) \\ &= 8\big(f - T_{1,0}Bf\big) \\ &= 8\big(f - T_{1,0}f\big) = \Delta_1 f \end{aligned}$$

Hence

Δ_1 is a bounded operator on M_2.

Furthermore, for $f \in M_2$

$$T_{n,n}f = G_n(\Delta_1)G_n(\Delta_2)f \quad \text{by 3.5 (c)}$$

If we define an entire function

$$G(z) = \prod_{n=1}^{\infty}\big(1 - \frac{z}{4n(n+1)}\big)$$

then $G_n(\Delta_1) \ \rightarrow \ G(\Delta_1)$ in the operator norm since $G_n \ \rightarrow \ G$ uniformly on compact set of $\mathbf{C}$.
Now take $n \to \infty$, by 3.5 (e) we get

$$f \ = \ G(\Delta_1)G(\Delta_2)f.$$

Therefore,

$G(\Delta_1)G(\Delta_2)$is the identity operator onM_2.

On the other hand, from 3.5 (c) and 3.6 (3) of [5], we get

$$G(\lambda) = \frac{\sin(\pi\alpha)}{\pi\alpha(1-\alpha)}$$

where $\lambda = -4\alpha(1-\alpha)$.
Hence, if we define

$$\Omega_2 = \{ \lambda = -4\alpha(1-\alpha) \mid -\frac{1}{2} < Re\ \alpha < \frac{3}{2} \}.$$

Then by 2.4 of [5] we can see
The set

$$E = \{(\lambda,\mu) \in \Omega_2 \times \Omega_2 | G(\lambda)G(\mu) = 1\}$$

is the set of all joint eigenvalues of Δ_1 *and*Δ_2.

Since

$$G(\Delta_1)G(\Delta_2) = I \quad \text{on } M_2$$

by the holomorphic functional calculus (3.11 of [2]) ,

$$1 = \sigma\big(G(\Delta_1)G(\Delta_2)\big) = \{ G(\lambda)G(\mu) \mid (\lambda,\mu) \in \sigma(\Delta_1,\Delta_2) \}.$$

Hence, the joint spectrum of Δ_1 and Δ_2 is

$$\sigma(\Delta_1,\Delta_2) = \{ (\lambda,\mu) \in \bar{\Omega}_2 \times \bar{\Omega}_2 \mid G(\lambda)G(\mu) = 1 \}.$$

But since the operators Δ_1, Δ_2 are not normal (they have uncountably many eigenvalues) , no type of spectral decomposition of M_2 with respect to Δ_1 and Δ_2 is available.
Another way to state the conjecture 3.1 is that

If $f \in M$ is orthogonal to all the joint eigenfunctions in M , then $f = \equiv 0$.

If the conjecture is right , then any $g \in M_2$ can be written as

$$g = \int_E g_v \, d\tau(v)$$

for some finite measure τ on E and g_v the corresponding jointeigenfunction.
The author hope to return to this problem in the future work.

3.7

Herewe will show that if $f \in L^1(D^2, m \times m)$ is of the form $f(z,w) = u(z)v(w)$, then f can be written as a finite sum of joint eigenfunctions. Now let $f(z,w) = u(z)v(w)$ for some $u, v \in L^1(D, m)$. Then

$$\begin{aligned}(Bf)(z,w) &= \int\int_{D^2} u(\varphi_z(x))\ v(\varphi_w(y))\ dm(x)\ dm(y)\\ &= \int_D u(\varphi_z(x))\ dm(x) \int_D v(\varphi_w(y))\ dm(y)\\ &= f(z,w) \quad = \quad u(z)\ v(w)\end{aligned}$$

Hence there exists $\alpha \in \mathbf{C}$, such that

$$\int_D u(\varphi_z(x))\ dm(x) \ = \ \alpha\ u(z)$$

and

$$\int_D v(\varphi_w(y))\ dm(y) \ = \ \frac{1}{\alpha}\ v(w).$$

Now let's define the space M_α $(\subset L^1(D, dm))$ by

$$M_\alpha = \{u \in L^1(D, dm) \mid Tu = \alpha u.\}$$

and denote Δ as the operator $\tilde{\Delta}$ restricted to M_α .

Define

$$(T_k u)(z) = \int_D (k+1)(1-|x|^2)^k u(\varphi_z(x))\ dm(x)$$

then for $k > 0$, T_k is bounded on $L^1(D, dm)$.
Hence for $u \in M_\alpha$, by the same way as 3.5 , $\Delta u = \frac{1}{\alpha}\ 8\ (\alpha u - T_1 u)$ and T_1 is bounded on $L^1(D, dm)$.

Thus we get, just as 3.6
(i) Δ is a bounded operator on M_α.
(ii) $\alpha G(\Delta)$ is the identity operator on M_α.
(iii) The set $E_\alpha = \{\lambda \in \Omega_1 | G(\lambda) = \frac{1}{\alpha}\}$ is the set of all eigenvalues of Δ on M_α.

(iv) If $E_\alpha = \{\lambda_1, \cdots, \lambda_N\}$ and

$$Q(z) = \prod_{i=1}^{N} (z - \lambda_i)$$

then $Q(\Delta) = 0$ on M_α.
Hence by lemma 4.1 of [1] we can write

$$u = u_{\lambda_1} + \cdots + u_{\lambda_N}$$

for $u_{\lambda_i} \in L^1(D, dm)$, where $\Delta u_{\lambda_i} = \lambda_i u$.
By the same way we can write

$$v = v_{\mu_1} + \cdots + v_{\mu_m} \ , \ \Delta v_{\mu_j} = \mu_j v \ , \ 1 \le j \le m$$

where

$$\{ \mu_1 \ , \cdots , \ \mu_m \} = \{ \mu \in \Omega_1 \mid G(\mu) = \alpha \}.$$

Hence we can write f as

$$f(z,w) = (u_{\lambda_1}(z) + \cdots + u_{\lambda_N}(z)) (v_{\mu_1}(w) + \cdots + v_{\mu_m}(w))$$

which is a finite sum of joint eigenfunctions.

References

[1] P. Ahern, M. Flores and W. Rudin, *An invariant volume mean value property*, J. Funct. Anal **111** (1993), 380-397.
[2] J. P Ferrier, *Spectral Theory and Complex Analysis*, North-Holland, 1973.
[3] F. Forelli and W. Rudin, *Projections on spaces of holomorphic functions in balls*, Indiana U. Math. Journal **24** (1974), 593-602.
[4] Y. Katznelson and L. Tzafriri, *On Power bounded operators*, J. Funct. Anal. **68** (1986), 313-328.
[5] J. Lee, *An invariant mean value property in the polydisc*, To appear.
[6] W. Rudin, *Function Theory in the unit Ball of* $\mathbf{C}^n$ (1980), Springer-Verlag.

Topology and Geometry Research Center
Kyungpook National University
Taegu 702-701, Korea