Comm. Korean Math. Soc. **12**(1997), No. 2, pp. 305-309

EXISTENCE OF A MULTIVORTEX SOLUTION FOR $SU(N)_g \times U(1)_l$ CHERN-SIMONS MODEL IN $\mathbb{R}^2/\mathbb{Z}^2$

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ABSTRACT. In this paper we prove the existence of a special type of multivortex solutions of $SU(N)_g \times U(1)_l$ Chern-Simons model. More specifically we prove existence of solutions of the self-duality equations for $(\Phi(x), A_\mu(x))$, $x \in \mathbb{R}^2$, $\mu = 0, 1, 2$, where each component $\Phi_j(x)$, $j = 1, \cdots, N$ has the same zeroes. In this case we find that the equation can be reduced to the single semilinear elliptic partial differential equations studied by Caffarelli and Yang.

1. Introduction

There has been a lot of attention to the abelian Higgs model with a Chern-Simons term in $(2+1)$ dimensional gauge theory both physically and mathematically [9, 10, 11]. The gauge group here is $U(1)_\ell$.

Physically this model was constructed to explain electrically charged vortices, so called anyons, in the context of some condensed matter physics [5]. Mathematical analysis of the model was started by R. Wang [11], where he proved existence of topological multivortex solutions using the variational method. In [9] and [10], respectively, Spruck-Yang proved existence of maximal topological multivortex solutions, and radially symmetric nontopological solutions both in $\mathbb{R}^2$. For the existence of solution in $\mathbb{R}^2/\mathbb{Z}^2$, Caffarelli-Yang recently constructed multivortex a solution in [3] using the super-subsolution method.

In this paper, we consider a generalization of the above model proposed in [8]. Namely, we consider a Chern-Simons model in $(2+1)$ dimension with gauge group $SU(N)_g \times U(1)_\ell$ instead of $U_\ell(1)$ for the

Received December 26, 1996. Revised March 18, 1997.
1991 Mathematics Subject Classification: 35J25.
Key words and phrases: Chern-Simons model, abelian Higgs model, multivortex solution.

spatial domain $\Omega = \mathbb{R}^2/\mathbb{Z}^2$. For this model we prove existence of a particular type of multivortex solution by reducing our system of partial differential equations to the equation considered by Caffarelli-Yang [3].

2. Main Results

We consider the pure Chern-Simons Lagrangean in $(2+1)$ dimension:

$$\mathcal{L} = \frac{1}{2}(D_\mu\Phi)^+(D^\mu\Phi) + \frac{\kappa}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda - \frac{e^4}{8\kappa^2}|\Phi|^2\left(|\Phi|^2-1\right)^2,$$

where $D^\mu = \partial^\mu + ieA^\mu$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $\mu = 0,1,2$ and $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_N) \in \mathbb{C}^N(\Omega)$. We are using the metric $(g_{\mu\nu}) = (g^{\mu\nu}) = diag(-1,1,1)$.

We have the Gauss law constraint

$$A_0 = \frac{\kappa}{2e^2}\frac{B}{|\Phi|^2},$$

where $B = \partial_1 A_2 - \partial_2 A_1$, which is the variational equation for A_0. Using this Gauss law constraint and following the procedure similar to [8], we can write the stationary energy as

$$E = \frac{1}{2}\int_\Omega d^2x\left\{|(D_1 \pm iD_2)\Phi|^2 + \left|\frac{\kappa}{e}\frac{B}{\Phi} \pm \frac{e^2}{2\kappa}\Phi^+(1-|\Phi|^2)\right|^2\right\}$$
$$\pm e\int_\Omega B\, d^2x.$$

Thus, the minimum value of energy$(= e|\int_{\mathbb{R}^2} B\, d^2x|)$ is obtained when the fields satisfy the first order self-duality equation,

$$(D_1 \pm iD_2)\Phi = 0, \quad \text{in } \Omega \tag{1}$$

$$B = \mp\frac{e^3}{2\kappa^2}|\Phi|^2\left(1-|\Phi|^2\right) \quad \text{in } \Omega. \tag{2}$$

The natural boundary condition for our case $\Omega = \mathbb{R}^2/\mathbb{Z}^2$ is the t'Hooft boundary condition (See [5] for details.)

Introducing

$$z = x_1 + ix_2, \quad \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2),$$

and

$$\Phi = \Phi^1 + i\Phi^2, \quad A = \alpha\, dz + \bar\alpha\, d\bar z,$$

where $\alpha = \frac{1}{2}(A_1 - iA_2)$ and $\bar\alpha = \frac{1}{2}(A_1 + iA_2)$, we can rewrite (1) in the form

$$\partial_{\bar z}\Phi_j = i\bar\alpha\Phi_j, \qquad j = 1, 2, \ldots, N.$$

Using the $\partial_{\bar z}$-Poincaré lemma as in [7], we can easily see that the zeros of each Φ_j $(j = 1, 2, \ldots, N)$ is finite. Thus following Jaffe-Taubes, we can introduce $(u_1, u_2, \ldots, u_N)$ and $(\theta_1, \theta_2, \ldots, \theta_N)$ such that

(3) $$\Phi_j = e^{\frac{1}{2}(u_j + i\theta_j)},$$

(4) $$\theta_j = 2\sum_{k=1}^{N_j} \arg(z - z_{jk}), \qquad j = 1, 2, \ldots, N,$$

where $\{z_{j1}, z_{j2}, \ldots, z_{jk}\} \equiv Z(\Phi_j)$ is the zeros of Φ_j.

We now state our main theorem.

THEOREM. *Consider the self-duality equation (1), (2) in $\mathbb{R}^2/\mathbb{Z}^2$ with the t' Hooft boundary condition. Then, to have the existence of solution to (1)-(2) it is necessary that*

(5) $$\frac{e^3}{8\kappa^2} > 4\pi m.$$

Furthermore, for sufficiently large κ there exists a multivortex solution of the equation having the property

(6) $$Z(\Phi_j) = Z(\Phi_k)$$

for all $k, j = 1, 2, \ldots, N$.

PROOF. Using the substitutions of (3)-(4), we can rewrite (1)-(2) as

$$\Delta u_j = \frac{e^3}{2\kappa^2}\left(e^{u_1}+e^{u_2}+\cdots+e^{u_N}\right)\left(1-e^{u_1}-e^{u_2}-\cdots-e^{u_N}\right)$$

$$+4\pi\sum_{k=1}^{N_j}\delta(z-z_{jk}), \tag{7}$$

in Ω. Since $Z(\Phi_j)=Z(\Phi_k)$ for all $j,k=1,2,\ldots,N$, we set $\{z_{jk}\}_{k=1}^{N_j}=\{z_1,z_2,\ldots,z_m\}$. From (6)-(7), we obtain

$$\Delta(u_j-u_k)=0 \qquad \text{in } \Omega \qquad \forall j,k=1,2,\ldots,N.$$

By the maximum principle in Ω, we have

$$u_j=u_1+C_j, \qquad \forall j=1,2,\ldots,N,$$

where C_j is a constant for each $j=1,2,\ldots,N$, and $C_1=0$. We thus have from (7)

$$\Delta u_1 = -\frac{e^3}{2\kappa^2}e^{u_1+\alpha}\left(1-e^{u_1+\alpha}\right)+4\pi\sum_{k=1}^{m}\delta(z-z_k),$$

where we set $e^\alpha = 1+e^{C_2}+e^{C_3}+\cdots+e^{C_N}$.

We set $v=u_1+\alpha+u_0$, where u_0 is the solution (see [1]) of

$$\Delta u_0 = -4\pi m + 4\pi\sum_{k=1}^{m}\delta(z-z_k).$$

Then, we have

$$\Delta v = \frac{e^3}{2\kappa^2}e^{v+u_0}\left(e^{v+u_0}-1\right)+4\pi m. \tag{8}$$

Using $e^{v+u_0}(e^{v+u_0}-1)=(e^{v+u_0}-\frac{1}{2})^2-\frac{1}{4}$, and integrating (8) over Ω, we obtain the necessary condition for existence of our solution

$$\frac{e^3}{8\kappa^2} > 4\pi m.$$

Now, we observe that the equation (8) is the exactly same form studied in [3], where they showed the existence of solution for sufficiently large κ. Applying the result [3] for the equation (8), we conclude our proof.
□

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