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AN OPERATOR VALUED FUNCTION SPACE INTEGRAL OF FUNCTIONALS INVOLVING DOUBLE INTEGRALS

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ABSTRACT. The existence theorem for the operator valued function space integral has been studied, when the wave function was in $L_1(\mathbb{R})$ class and the potential energy function was represented as a double integral [4]. Johnson and Lapidus established the existence theorem for the operator valued function space integral, when the wave function was in $L_2(\mathbb{R})$ class and the potential energy function was represented as an integral involving a Borel measure [9]. In this paper, we establish the existence theorem for the operator valued function space integral as an operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$ for certain potential energy functions which involve double integrals with some Borel measures.

1. Introduction

The theory of quantum mechanics is based on the Schrödinger wave equation. In 1948, to solve the wave equation, Feynman introduced an integral, so called the Feynman integral, which had some mathematical difficulties. In 1968, Cameron and Storvick defined an integral, the operator valued function space integral or the Cameron-Storvick function space integral, which is the nearest concept to the original Feynman's suggestion [7]. In 1973, Cameron and Storvick proved the existence theorem for the operator valued function space integral, when the wave

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function was in $L_1(\mathbb{R})$ class and the potential energy function was represented as a double integral [3]. In 1986, Johnson and Lapidus established the existence theorem for the operator valued function space integral, when the wave function was in $L_2(\mathbb{R})$ class and the potential energy function was represented as an integral involving a Borel measure [9]. In this papêr, we establish the existence theorem for the operator valued function space integral as an operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$ for certain potential energy functions which involve double integrals with some Borel measures.

Now we present some necessary notations which are needed in the next section.

Let $\mathbb{C}$ and $\mathbb{C}^+$ denote the set of all complex numbers and the set of all complex numbers with positive real part, respectively. $C[a,b]$ will denote the space of all real-valued continuous functions on $[a,b]$, and the Wiener space, $C_0[a,b]$, will consist of those x in $C[a,b]$ such that $x(a)=0$, and m_w will denote Wiener measure on $C_0[a,b]$. Let $M(a,b)$ denote the space of all complex Borel measures η on the open interval (a,b) and let $M^*(a,b)$ denote the subspace of $M(a,b)$ such that if μ is the continuous part of η in $M(a,b)$ then the Radon-Nikodym derivative $d|\mu|/dm_l$ exists and is essentially bounded, and if ν is the discrete part of η then ν has a finite support, where m_l is the Lebesgue measure. We work with the space $M^*(a,b)$ throughout this paper but $M^*[a,b]$ could be treated without any essential complications. For $2 < r \le \infty$, let $L_{1r} := L_{1r}\big((a,b)^2 \times \mathbb{R}^2\big)$ be the space of Borel measurable $\mathbb{C}$-valued functions θ on $(a,b)^2 \times \mathbb{R}^2$ such that

$$(1.1) \qquad \|\theta\|_{1r} := \left\{ \int_a^b \int_a^b \|\theta(s,t,\cdot,\cdot)\|_1^r \, dsdt \right\}^{\frac{1}{r}} < \infty.$$

Note that $L_{1r} \subseteq L_{1s}$ if $1 \le s \le r \le \infty$. Let F be a real or complex functional defined on $C[a,b]$. Given $\lambda > 0, \psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$, let

$$(1.2) \qquad (I_\lambda(F)\psi)(\xi) = \int_{C_0[a,b]} F(\lambda^{-\frac{1}{2}}x + \xi)\, \psi(\lambda^{-\frac{1}{2}}x(b) + \xi)\, dm_w(x)$$

If $I_\lambda(F)\psi$ is in $L_\infty(\mathbb{R})$ as a function of ξ and if the correspondence $\psi \longrightarrow I_\lambda(F)\psi$ gives an element of $\mathcal{L} := \mathcal{L}\big(L_1(\mathbb{R}), L_\infty(\mathbb{R})\big)$, we say that the operator-valued function space integral $I_\lambda(F)$ exists for λ.

Let β and η be in $M^*(a,b)$, say,

$$\beta = \mu + \sum_{j=1}^{l} w_j \delta_{\tau_j}, \quad \eta = \nu + \sum_{k=1}^{n} \alpha_k \delta_{\gamma_k}, \tag{1.3}$$

where δ_{τ_j} is the Dirac measure at $\tau_j \in (a,b)$ and let θ be a $\mathbb{C}$-valued function on $(a,b)^2 \times \mathbb{R}^2$ satisfying the following three conditions;

$$\theta \in L_{1r}, \quad r \in (2,\infty] \tag{1.4 a}$$

$$\theta(\tau_j, \gamma_k, v_j, u_k) = \phi_1(\tau_j, v_j)\, \phi_2(\gamma_k, u_k), \tag{1.4 b}$$

where $\phi_1(\tau_j, \cdot)$ and $\phi_2(\gamma_k, \cdot)$ are in $L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ for each $j = 1, 2, \cdots, l$, $k = 1, 2, \cdots, n$, and

$$\theta(\tau_j, t, \cdot, \cdot) = \theta(s, \gamma_k, \cdot, \cdot) = 0, \tag{1.4 c}$$

where $t \neq \gamma_k$, $s \neq \tau_j$ for $j = 1, 2, \cdots, l$, $k = 1, 2, \cdots, n$.

Let

$$F(y) = \int_{(a,b)} \int_{(a,b)} \theta(s, t, y(s), y(t))\, d\beta(s)\, d\eta(t) \tag{1.5}$$

for any $y \in C[a,b]$ for which the integral exists. Then for every $\lambda > 0$ and every $\xi \in \mathbb{R}$, $F(\lambda^{-\frac{1}{2}} x + \xi)$ is defined for $m_w \times m_l$–a.e. $(x, \xi) \in C_0[a,b] \times \mathbb{R}$ [1].

2. The existence theorem for the analytic operator valued function space integral

In this section, we will investigate the existence theorem for the analytic operator valued function space integral of the potential function which is represented as a double intergal.

DEFINITION 2.1. Let Ω be a simply connected domain of the complex λ-plane whose intersection with the positive real axis is a single non-empty open interval (α, β). Let F be a functional on $C[a, b]$ such that $I_\lambda(F)$ exists for $\lambda \in (\alpha, \beta)$. For each $\psi \in L_1(\mathbb{R})$ let a function $A(\lambda : \psi)$ exist as a weakly analytic vector-valued function of λ for $\lambda \in \Omega$ (i.e. for each $\phi \in L_1(\mathbb{R})$, $\int_{-\infty}^{\infty} A(\xi, \lambda)\phi(\xi)d\xi$ is an analytic function of λ in Ω), $A(\lambda : \psi) \in L_\infty(\mathbb{R})$ and let $A(\lambda : \psi) = I_\lambda(F)\psi$ for $\lambda \in (\alpha, \beta)$ and $\psi \in L_1(\mathbb{R})$. We define

$$I_\lambda^{an}(F)\psi = A(\lambda : \psi)$$

for $\lambda \in \Omega$ and $\psi \in L_1(\mathbb{R})$. $I_\lambda^{an}(F)$ is called the analytic operator valued function space integral.

We note that, if $I_\lambda^{an}(F)$ exists, it is uniquely defined and it is a linear operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$.

REMARK. Let θ, β and η be given as in (1.3) and (1.4) and let

$$\begin{aligned} &g_m(\beta, \eta : q_{1,1}, \ldots, q_{l,n} : v_1', \cdots, v_{q-\tilde{q}}') \\ &:= \prod_{j=1}^{l}\prod_{k=1}^{n} \theta(\tau_j, \gamma_k, v_j, v_k)^{q_{j,k}}, \end{aligned}$$

where $q_{1,1} + q_{1,2} + \cdots + q_{l,n} \le m - q_0$ and $0 \le \hat{q} \le q$. Then $g_m(\beta, \eta : q_{1,1}, \ldots, q_{l,n} : \cdot, \overset{(q-\tilde{q})}{\cdots}, \cdot) \in L_1(\mathbb{R}^{q-\tilde{q}})$.

NOTATION. *Throughout this paper, we let*

(a) $|||g_m||| := \sup_{q_{1,1}+q_{1,2}+\cdots+q_{l,n}=m-q_0} \|g_m(\beta, \eta : q_{1,1}, \ldots, q_{l,n} : \cdot \cdots \cdot)\|_1.$

(b) *For any nonnegative integer* q_0, $r > 2$ *and for* r' *with* $\frac{1}{r} + \frac{1}{r'} = 1$, *let*

(2.1)

$$\begin{aligned} &A(2q_0 : a_1, \ldots, a_q : r') \\ &:= \Bigg\{ \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\ldots,a_q}_{2q_0:j_1,\ldots,j_{q+1}}} [(r_1 - a) \cdots (a_1 - r_{j_1})(r_{j_1+1} - a_1) \cdot \\ &\qquad \cdots (b - r_{j_1+\cdots+j_{q+1}})]^{-\frac{r'}{2}} \, d \overset{2q_0}{\underset{i=1}{\times}} r_i \Bigg\}^{\frac{1}{r'}} \end{aligned}$$

where $a = a_0 < a_1 < a_2 < \cdots < a_q < a_{q+1} = b$, and $j_1, \ldots, j_{q+1}, q$ are nonnegative integers, and

(2.2)

$$\Delta^{a_1,\ldots,a_q}_{2q_0:j_1,\ldots,j_{q+1}} := \Big\{(r_1, \ldots, r_{2q_0}) \in (a,b)^{2q_0} \;:\; a < r_1 < \cdots < r_{j_1} < a_1 < r_{j_1+1} < \cdots < r_{j_1+\cdots+j_q} < a_q < r_{j_1+\cdots+j_q+1} < \cdots < r_{2q_0} < b\Big\}.$$

From [1], we have the following lemma which will be applied in the proof of Theorem 2.3 and it will also serve to illustrate the notations in the theorem.

LEMMA 2.2. *Consider a set $\Delta^{a_1,\ldots,a_q}_{2q_0:j_1,\ldots,j_{q+1}}$, then for $0 \le \tilde{q} \le q$,*

$$\begin{aligned} &\tilde{A}(2q_0 : a'_1, \ldots, a'_{q-\tilde{q}} : r') \\ &:= \Big\{ \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\ldots,a_q}_{2q_0:j_1,\ldots,j_{q+1}}} [(r_1 - a) \cdots (r_{\sigma(1)+1} - r_{\sigma(1)}) \cdots \\ &\quad \cdot (r_{\sigma(\tilde{q})+1} - r_{\sigma(\tilde{q})}) \cdots (b - r_{j_1+\cdots+j_{q+1}})]^{-\frac{r'}{2}} \, d \mathop{\times}_{i=1}^{2q_0} r_i \Big\}^{\frac{1}{r'}} \qquad (2.3) \\ &\le \left(\frac{{}_{(2q_0+q)}P_{\tilde{q}}}{{}_{q}P_{\tilde{q}}} \right)^{\frac{1}{r'}} A(2q_0 : a'_1, \ldots, a'_{q-\tilde{q}} : r'), \end{aligned}$$

where $\{a_1, \ldots, a_q\} = \{a_{i,\tilde{q}} : i = 0, 1, \ldots, \tilde{q}\} \cup \{a'_j : j = 1, \ldots, q - \tilde{q}\}$ such that $a < a_1 < a_2 < \cdots < a_q < b$, $a_{i,\tilde{q}} < a_{j,\tilde{q}}$ for $i < j$ and $a'_i < a'_j$ for $i < j$. And σ is a function from $\{a_{i,\tilde{q}}, \ldots, a_{\tilde{q},\tilde{q}}\}$ to $\left\{\sum_{k=1}^{l} j_k : l = 1, \ldots, q\right\}$ defined by

$$\sigma(i) := \sigma(a_{i,\tilde{q}}) = \sum_{k=1}^{t} j_k,$$

where $a_{i,\tilde{q}} = a_t$. And $j'_i = \sum_{k=c+1}^{d} j_k$ where $a'_{i-1} = a_c$, $a'_i = a_d$ and $j'_{q-\tilde{q}+1} = \sum_{k=1}^{q+1} j_q - \sum_{i=1}^{q-\tilde{q}} j'_i$, for $i = 1, \ldots, q - \tilde{q}$.

In order to demonstrate the notations in the above lemma, we give the following specific example.

$$\begin{aligned}
&\Delta^{a_1,a_2,a_3,a_4,a_5}_{10:3,2,2,1,0,2}\\
&=\{(r_1,r_2,\dots,r_{10})\in(a,b)^{10}\ :\ a<r_1<r_2<r_3<a_1=a_1'\\
&\qquad <r_4<r_5<a_2=a_{1,3}<r_6<r_7<a_3=a_2'\\
&\qquad <r_8<a_4=a_{2,3}<a_5=a_{3,3}<r_9<r_{10}<b\},
\end{aligned}$$

that is, $q=5$, $\tilde{q}=3$. Then

$$\begin{aligned}
\sigma(1)&:=\sigma(a_{1,3})=3+2=5\\
\sigma(2)&:=\sigma(a_{2,3})=3+2+2+1=8\\
\sigma(3)&:=\sigma(a_{3,3})=3+2+2+1+0=8
\end{aligned}$$

and $j_1'=3$, $j_2'=2+2=4$, $j_3'=10-7=3$.
Thus we obtain a set

$$\begin{aligned}
&\Delta^{a_1',\,a_2'}_{10:3,4,3}\\
&=\{(r_1,r_2,\dots,r_{10})\in(a,b)^{10}\,|\,a<r_1<r_2<r_3<a_1=a_1'<r_4<\\
&\qquad \cdots<r_7<a_3=a_2'<r_8<r_9<r_{10}<b\}
\end{aligned}$$

and

$$\Delta^{a_1,a_2,a_3,a_4,a_5}_{10:3,2,2,1,0,2}\subset\Delta^{a_1',\,a_2'}_{10:3,4,3}.$$

In order to prove our main theorem, Theorem 2.4, We need the following theorem. We state it without proof, for the proof of the theorem, see [1].

THEOREM 2.3. *(β, η : finitely supported measures) Let θ, β and η be given as in (1.3) and (1.4). Let*

$$F_m(y)=\left[\int_{(a,b)}\int_{(a,b)}\theta(s,t,y(s),y(t))\,d\beta(s)\,d\eta(t)\right]^m$$

for any $y \in C[a,b]$. Then the operator $I_\lambda^{an}(F_m)$ exists for all $\lambda \in \mathbb{C}^+$. Further for $\lambda \in \mathbb{C}^+$, $\psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$,

(2.4)

$$(I_\lambda^{an}(F_m)\psi)(\xi) = \sum_{q_0+q_{1,1}+\cdots+q_{l,n}=m} \frac{m!(w_1\alpha_1)^{q_{1,1}}\cdots(w_l\alpha_n)^{q_{l,n}}}{q_0!q_{1,1}!\cdots q_{l,n}!} \sum_{(m_1,\ldots,m_{q_0},k_1,\ldots,k_{q_0})\in P} \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\ldots,a_q}_{2q_0:j_1,\ldots,j_{q+1}}} Y\, d \mathop{\times}_{n=1}^{2q_0} \tilde{\mu}_{p,n}(r_n),$$

where $\{r_1,\ldots,r_{2q_0}\}$ is the set of numbers $s_1,\ldots,s_{q_0},t_1,\ldots,t_{q_0}$ in some rearrangement, P is the set of all permutations of $\{1,2,\ldots,2q_0\}$, $s_j := r_{m_j}$, $t_j := r_{k_j}$ and $\tau_j := a_{1,j}$, $\gamma_k := a_{2,k}$ and $\{a_1,\ldots,a_q\} = \{\tau_j,\gamma_k : j = 1,2,\ldots,l,\ k = 1,2,\ldots,n\}$ such that $a < a_1 < \cdots < a_q < b$ and $q_{j,k}$'s are nonnegative integers. And $\int f\, d\tilde{\mu}_{p,i}(r_i)$ means that $\int f\, d\mu(r_i)$ when $r_i = r_{m_j}$ for some r_{m_j} and $\int f\, d\tilde{\mu}_{p,i}(r_i)$ means that $\int f\, d\nu(r_i)$ when $r_i = r_{k_j}$ for some r_{k_j}.
And

$$Y = \left(\frac{\lambda}{2\pi}\right)^{\frac{2q_0+q+1}{2}} [(r_1-a)\cdots(a_1-r_{j_1})(r_{j_1+1}-a_1)\cdots \cdot (b-r_{j_1+\cdots+j_{q+1}})]^{-\frac{1}{2}} \int_{-\infty}^{\infty} \overset{2q_0+q+1}{\cdots} \int_{-\infty}^{\infty} \prod_{j=1}^{q_0} \theta(r_{m_j}, r_{k_j}, v_{m_j}, v_{k_j}) \tag{2.5}$$

$$\cdot \prod_{j=1}^{l}\prod_{k=1}^{n}\left[\theta(a_{1,j},a_{2,k},v'_{1,j},v'_{2,k})\right]^{q_{j,k}} \psi(v'_{q+1}) \exp\left\{-\frac{\lambda(v_1-v_0)^2}{2(r_1-a)} - \cdots \cdot -\frac{\lambda(v'_1-v_{j_1})^2}{2(a_1-r_{j_1})} - \frac{\lambda(v_{j_1+1}-v'_1)^2}{2(r_{j_1+1}-a_1)} - \cdots - \frac{\lambda(v'_{q+1}-v_{j_1+\cdots+j_{q+1}})^2}{2(b-r_{j_1}+\cdots+j_{q+1})}\right\} \mathop{\times}_{i=1}^{2q_0} dv_i \mathop{\times}_{j=1}^{q+1} dv'_j.$$

In addition we have for $\lambda \in \mathbb{C}^+$,

$$\|I_\lambda^{an}(F_m)\| \le b_m(|\lambda|), \tag{2.6}$$

where

$$b_m(|\lambda|) := m! \sum_{q_0+q_{1,1}+\cdots+q_{l,n}=m} \frac{\prod_{j=1}^{l}\prod_{k=1}^{n} |w_j\alpha_k|^{q_{j,k}}}{q_0!q_{1,1}!\cdots q_{l,n}!}((2q_0)!)$$
$$\left(\frac{(2q_0+q)!}{(2q_0)!q!}\right)^{\frac{r+1}{2r}} |||g_m||| \left(\frac{|\lambda|}{2\pi}\right)^{\frac{2q_0+q-\tilde{q}+1}{2}} \left(\frac{|\lambda|}{Re\lambda}\right)^{\frac{\tilde{q}}{2}}$$
$$\left(\|\frac{d|\mu|}{dm_l}\|_\infty \|\frac{d|\nu|}{dm_l}\|_\infty \|\theta\|_{1r}\right)^{q_0} \tilde{A}(2q_0 : a'_1, \cdots, a'_{q-\tilde{q}} : r'),$$

$\tilde{A}(2q_0 : a'_1, \ldots, a'_{q-\tilde{q}} : r')$ *is given by* (2.3) *and*

$$g_m(\beta, \eta : q_{1,1}, \ldots, q_{l,n} : v'_1, \ldots, v'_{q-\tilde{q}}) := \prod_{j=1}^{l}\prod_{k=1}^{n} \theta(\tau_j, \gamma_k, v_{1,j}, v_{2,k})^{q_{j,k}}.$$

Now let $\lambda_0, \lambda_1 \in (0,\infty]$ with $\lambda_0 < \lambda_1$, $0 \le \alpha < \frac{\pi}{2}$ and $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be an analytic function satisfying

$$\sum_{m=0}^{\infty} |a_m|\, \tilde{b}_m(|\lambda|) < \infty \tag{2.7}$$

for every $\lambda \in \mathbb{C}^+_{\lambda_0,\lambda_1:\alpha} := \{z \in \mathbb{C}^+ : \lambda_0 < |z| < \lambda_1, |Arg\, z| < \alpha\}$ where $\tilde{b}_m(|\lambda|)$ is defined as in (2.6) with $\left(\frac{|\lambda|}{\mathrm{Re}\lambda}\right)^{\frac{\tilde{q}}{2}}$ replaced by $\left(\frac{|\lambda|}{\lambda_0 \cos\alpha}\right)^{\frac{\tilde{q}}{2}}$. Note that $b_m(|\lambda|) \le \tilde{b}_m(|\lambda|)$ in $\mathbb{C}^+_{\lambda_0,\lambda_1:\alpha}$ and $\tilde{b}_m(|\lambda|)$ is an increasing function of $|\lambda|$. Consider the functional

$$F(y) := f\left[\int_{(a,b)}\int_{(a,b)} \theta(s,t,y(s),y(t))\, d\beta(s)\, d\eta(t)\right] \tag{2.8}$$

for $y \in C[a,b]$; that is,

$$F(y) = \sum_{m=0}^{\infty} a_m F_m(y) \tag{2.9}$$

where F_m is given as in Theorem 2.3.

THEOREM 2.4. *(β, η : finitely supported measures)*
Let θ, β and η be given as in (1.3) and (1.4) and let F be given by (2.9) with the assumptions discussed above (in particular, F satisfies (2.7)). Then the operator $I_\lambda^{an}(F)$ exists, for all $\lambda \in \mathbb{C}^+_{\lambda_0,\lambda_1:\alpha}$ and is given by

$$I_\lambda^{an}(F) = \sum_{m=0}^{\infty} a_m I_\lambda^{an}(F_m) \tag{2.10}$$

and the series in (2.10) *satisfies*

$$\|I_\lambda^{an}(F)\| \le \sum_{m=0}^{\infty} |a_m|\, \tilde{b}_m(|\lambda|) \tag{2.11}$$

and so it converges in the operator norm.

PROOF. For $\lambda \in \mathbb{C}^+_{\lambda_0,\lambda_1:\alpha}$, we have by (2.6) and (2.7),

$$\sum_{m=0}^{\infty} \|a_m I_\lambda^{an}(F_m)\| \le \sum_{m=0}^{\infty} |a_m| \tilde{b}_m(|\lambda|) < \infty. \tag{2.12}$$

Hence the right-hand side of (2.10) defines an element of $\mathcal{L}$ for all $\lambda \in \mathbb{C}^+_{\lambda_0,\lambda_1:\alpha}$. Since $\tilde{b}_m(|\lambda|)$ is an increasing function of $|\lambda|$, the series (2.10) converges uniformly in any compact subset of $\mathbb{C}^+_{\lambda_0,\lambda_1:\alpha}$.

Now we claim that for $0 < \lambda_0 < \lambda$,

$$\big(I_\lambda(F)\psi\big)(\xi) = \sum_{m=0}^{\infty} a_m \big(I_\lambda(F_m)\psi\big)(\xi). \tag{2.13}$$

Formally this follows from the following equations.

$$\begin{aligned}
&\big(I_\lambda(F)\psi\big)(\xi)\\
&= \int_{C_0[a,b]} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(b) + \xi)\, dm_w(x)\\
&= \int_{C_0[a,b]} \sum_{m=0}^{\infty} a_m F_m(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(b) + \xi)\, dm_w(x)\\
&= \sum_{m=0}^{\infty} a_m \int_{C_0[a,b]} F_m(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(b) + \xi)\, dm_w(x)\\
&= \sum_{m=0}^{\infty} a_m \big(I_\lambda(F_m)\psi\big)(\xi).
\end{aligned} \tag{2.14}$$

The interchange of integral and sum in (2.14) follows from the Fubini-theorem and the fact that

$$
\begin{aligned}
&\int_{C_0[a,b]} \sum_{m=0}^{\infty} |a_m|\,|F_m(\lambda^{-\frac{1}{2}}x+\xi)||\psi(\lambda^{-\frac{1}{2}}x(b)+\xi)|\,dm_w(x) \\
(2.15)\quad &= \sum_{m=0}^{\infty} |a_m| \int_{C_0[a,b]} |F_m(\lambda^{-\frac{1}{2}}x+\xi)|\,|\psi(\lambda^{-\frac{1}{2}}x(b)+\xi)|\,dm_w(x) \\
&\le \sum_{m=0}^{\infty} |a_m|\tilde{b}_m(|\lambda|)\|\psi\|_1 < \infty,
\end{aligned}
$$

where the last inequality comes from the same argument that yields the norm inequality (2.6) and the fact that $b_m(|\lambda|) \le \tilde{b}_m(|\lambda|)$ for $\lambda \in \mathbb{C}^+_{\lambda_0,\lambda_1:\alpha}$. Choosing $\psi \in L_1(\mathbb{R})$, we see that for $\lambda_0 < \lambda < \lambda_1$ and $\xi \in \mathbb{R}$, $\sum_{m=0}^{\infty} a_m F_m(\lambda^{-\frac{1}{2}}x+\xi)$ converges absolutely for *a.e.* $x \in C_0[a,b]$.

Now by Theorem 2.3, for each m, $I^{an}_\lambda(F_m)$ is weakly analytic ; i.e., for each $\phi \in L_1(\mathbb{R})$, $g(\lambda) = \int_{-\infty}^{\infty} \big(I^{an}_\lambda(F_m)\psi\big)(\xi)\phi(\xi)\,d\xi$ is an analytic function of λ. Thus uniform convergence of $\sum_{k=0}^{n} a_k I^{an}_\lambda(F_m)$ with respect to λ in $\mathbb{C}^+_{\lambda_0,\lambda_1:\alpha}$ noted earlier imply that the sum in (2.10) is an $\mathcal{L}$-valued weakly analytic function of λ in $\mathbb{C}^+_{\lambda_0,\lambda_1:\alpha}$ [8,Theorem 3.11.6]. By (2.13) and the fact that $I_\lambda(F_m) = I^{an}_\lambda(F_m)$ for $0 < \lambda_0 < \lambda$ and $m = 0, 1, 2, \ldots$, we see that $I^{an}_\lambda(F)$ exists and the equality in (2.10) holds. □

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