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CONVERGENCE OF APPROXIMATING FIXED POINTS FOR NONEXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

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Abstract. Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, $T : C \to E$ a nonexpansive mapping, and Q a sunny nonexpansive retraction of E onto C. For $u \in C$ and $t \in (0,1)$, let x_t be a unique fixed point of a contraction $R_t : C \to C$, defined by $R_t x = Q(tTx + (1-t)u)$, $x \in C$. It is proved that if $\{x_t\}$ is bounded, then the strong $\lim_{t\to 1} x_t$ exists and belongs to the fixed point set of T. Furthermore, the strong convergence of $\{x_t\}$ in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm is also given in case that the fixed point set of T is nonempty.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E, $T : C \to E$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\ y \in C$), and Q a sunny nonexpansive retraction of E onto C. Then, following Marino and Trombetta [11], given a $u \in C$ and a $t \in (0,1)$, we can define a contraction $R_t : C \to C$ by

$$R_t x = Q(tTx + (1-t)u), \quad x \in C. \tag{1}$$

By Banach's contraction principle, R_t has a unique fixed point x_t in C, that is, we have

$$x_t = Q(tTx_t + (1-t)u). \tag{2}$$

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When Q is the identity mapping, the strong convergence of $\{x_t\}$ as $t \to 1$ to a fixed point of T has been investigated by several authors; see, for example, Browder [2], Halpern [8], Jung and Kim [9], Kim and Takahashi [10], Marino and Trombetta [11], Reich [17], Singh and Waston [18], Xu and Yin [19] and others.

Recently, Xu and Yin [19, Theorem 3] proved that if C is a nonempty closed convex (not necessarily bounded) subset of Hilbert space H, if $T : C \to H$ is a nonexpansive nonself-mapping satisfying the weak inwardness condition, if Q is the nearest point projection from H onto C, and if $\{x_t\}$ is the sequence defined by (2) which is bounded, then $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T. Kim and Takahashi [10] extended Xu and Yin's result to a Banach space with a weakly sequentially continuous duality mapping.

In this paper, we establish the strong convergence of sequence $\{x_t\}$ defined by (2) which is bounded in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, thus generalizing Xu and Yin's result [19, Theorem 3] to a Banach space setting. Furthermore, we prove that if C is a nonempty closed convex subset of a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and if the fixed point set of T is nonempty, then the sequence $\{x_t\}$ defined by (2) also converges strongly as $t \to 1$ to a fixed point of T.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by (x, x^*).

For every ε with $0 \leq \varepsilon \leq 2$, the modulus $\delta(\varepsilon)$ of convexity of E is defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| \leq 1, \quad \|y\| \leq 1, \quad \|x-y\| \geq \varepsilon\}.$$

E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then E is reflexive and strictly convex.

The norm of E is said to be Gâteaux differentable (and E is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{3}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Gâteaux differentiable if each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit in (3) is attained uniformly for $(x, y) \in U \times U$. Since the dual E^* of E is uniformly convex if and only if the norm of E is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm. The converse implication is false. A discussion of these and related concepts may be found in [3].

The (normalized) duality mapping J from E into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : (x, f) = \|x\|^2 = \|f\|^2\}.$$

for each $x \in E$. It is single valued if and only if E is smooth. It is also well-known that if E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded subsets of E from the strong topology of E the weak-star topology of E^* (cf. [4, 5]).

Let μ be a mean on integers N, i.e., a continuous linear functional on ℓ^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n : n \in N\} \le \mu(a) \le \sup\{a_n : n \in N\}$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a Banach limit if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$. Using the Hahn-Banach theorem, we can prove the existence of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$. Let $\{x_n\}$ be a bounded sequence in E. Then we can define the real valued continuous convex function ϕ on E by

$$\phi(z) = \mu_n \|x_n - z\|^2$$

for each $z \in E$.

The following lemma was given in [6, 7].

LEMMA 1. *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in E. Let μ be a Banach limit and $u \in C$. Then*

$$\mu_n \|x_n - u\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if

$$\mu_n(x - u, J(x_n - u)) \leq 0$$

for all $x \in C$.

Recall that a closed convex subset C of E is said to have the fixed point propety for nonexpansive self-mappings (FPP) for short) if every nonexpansive mapping $T : C \to C$ has a fixed point, that is, there is a point $p \in C$ such that $T(p) = p$. It is well-known that every bounded closed convex subset of a uniformly convex Banach space has the FPP (cf. [5, p. 22]).

Let C be a nonempty closed convex subset of E. A mapping Q of C into C is said to be a retraction if $Q^2 = Q$. If a mapping Q of C into C is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is range of Q. Let Q be a retraction of E onto a closed subset C of E. Q is said to be sunny if each point on the ray $\{Qx + t(x - Qx) \ : t > 0\}$ is mapped by Q back onto Qx, in other words,

$$Q(Qx + t(x - Qx)) = Qx$$

for all $t \geq 0$ and $x \in E$. If there exists a retraction $Q : E \to C$ which is both sunny and nonexpansive, then C is said to be a sunny nonexpansive retract. Sunny nonexpansive retracts appear in [15, 16].

Finally, let C be a nonempty convex subset of E.Then for $x \in C$ we define the inward set $I_C(x)$ as follows:

$$I_C(x) = \{y \in E : y = x + a(z - x) \text{ for some } z \in C \text{ and } a \geq 0\}.$$

A mapping $T : C \to E$ is said to be inward if $Tx \in I_C(x)$ for all $x \in C$. T is also said to be weakly inward if for each $x \in C$. $Tx \in \mathrm{cl} I_C(x)$, where for $A \subset E$, $\mathrm{cl}A$ means the closure of A.

3. Main results

In this section, we study the strong convergence of $\{x_t\}$ defined by (2) in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm.

We begin with the following lemma, which is crucial in our main result.

LEMMA 2. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $\{x_n\}$ a bounded sequence of E. Then the set*

$$M = \{u \in C : \mu_n \|x_n - u\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2\}$$

consists of one point.

PROOF. Let $\phi(z) = \mu_n \|x_n - z\|^2$ for each $z \in C$ and $r = \inf\{\phi(z) : z \in C\}$. Then, since the function ϕ on C is convex and continuous, $\phi(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, it follows from [1, p.79] that there exists $u \in C$ with $\phi(u) = r$. Therefore M is nonempty. From Lemma 1, we know that $u \in M$ if and only if

$$\mu_n(x - u, J(x_n - u)) \leq 0$$

for all $x \in C$. Now we show that M consists of one point. Let $u,\ v \in M$ and suppose $u \neq v$. Then, by [12, Theorem 1], there is a positive number k such that

$$(x_n - u - (x_n - v), J(x_n - u) - J(x_n - v)) \geq k > 0$$

for every n. Therefore we get

$$\mu_n(v - u, J(x_n - u) - J(x_n - v)) \geq k > 0.$$

On the other hand, since $u,\ v \in M$, we have

$$\mu_n(v - u, J(x_n - u)) \leq 0$$

and

$$\mu_n(u - v, J(x_n - v)) \leq 0.$$

Then we have

$$\mu_n(v - u, J(x_n - u) - J(x_n - v)) \leq 0.$$

This is a contradiction. Therefore $u = v$. □

THEOREM 1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retact of E, and that for some $u \in C$ and each $t \in (0,1)$, x_t is a (unique) fixed point of the contraction R_t defined by (2), where Q is a sunny nonexpansive retraction of E onto C. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

PROOF. Let w be a fixed point of T. Then we have

$$\begin{aligned}\|w - x_t\| &= \|Qw - Q(tTx_t + (1-t)u)\| \\ &\le t\|w - Tx_t\| + (1-t)\|w - u\| \\ &\le t\|w - x_t\| + (1-t)\|w - u\|\end{aligned}$$

and hence $\|w - x_t\| \le \|w - u\|$ for all $t \in (0,1)$. So $\{x_t\}$ is bounded.

Suppose conversely that $\{x_t\}$ remains bounded as $t \to 1$. We now show that $F(T)$ is nonempty and that $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T. To this end, let $t_n \to 1$ and $x_n = x_{t_n}$. Define $\phi : C \to [0,\infty)$ by $\phi(z) = \mu_n\|x_n - z\|^2$. Since ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, ϕ attains its infimum over C. Let $z \in C$ be such that

$$\mu_n\|x_n - z\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2$$

and let

$$M = \{u \in C : \mu_n\|x_n - u\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2\}.$$

Then M is nonempty because $z \in M$. Since $\{Tx_t\}$ is bounded and

$$\begin{aligned}\|x_t - QTx_t\| &\le \|tTx_t + (1-t)u - Tx_t\| \\ &= (1-t)\|u - Tx_t\|,\end{aligned} \tag{4}$$

we have $x_t - QTx_t \to 0$. So, we have for $z \in M$,

$$\begin{aligned}&\|x_n - QTz\| \\ &\le \|x_n - QTx_n\| + \|QTx_n - QTz\| \\ &\le \|x_n - z\| + \|x_n - QTx_n\|\end{aligned}$$

and hence

$$\mu_n \|x_n - QTz\|^2 \leq \mu_n \|x_n - z\|^2.$$

Then $QTz \in M$. By Lemma 2, we know that M consists of one point. Therefore $QTz = z$. Since Q is sunny and nonexpansive retraction, from Lemma 2.4 in [13] (cf [5, p. 48]), we have

$$(5) \qquad (Tz - z, J(z - w)) \geq 0 \quad \text{for all } w \in C.$$

On the other hand, Tz belongs to $\mathrm{cl}I_C(z)$ by the weak inwardness condition. Hence for each integer $n \geq 1$, there exist $z_n \in C$ and $a_n \geq 0$ such that

$$(6) \qquad y_n := z + a_n(z_n - z) \to Tz \ \text{ strongly}.$$

Since E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*. Thus it follows from (5) and (6) that

$$\begin{aligned} 0 &\leq (Tz - z, a_n J(z - z_n)) \\ &= (Tz - z, J(a_n(z - z_n)) \\ &= (Tz - z, J(z - y_n)) \to (Tz - z, J(z - Tz)) = -\|Tz - z\|^2. \end{aligned}$$

Hence we have $Tz = z$. For any $v \in F(T)$, we have

$$t(v - u) + u = tv + (1 - t)u = Q(tv + (1 - t)u)$$

and hence

$$\begin{aligned} &\|(x_t - u) + t(v - u)\|^2 \\ &= \|Q(tTx_t + (1 - t)u) - u - t(v - u)\|^2 \\ &= \|Q(tTx_t - u) + u) - Q(t(v - u) + u)\|^2 \\ &\leq \|t(Tx_t - u) - t(v - u)\|^2 \\ &\leq t^2 \|x_t - v\|^2 \\ &= t^2 \|(x_t - u) - (v - u)\|^2. \end{aligned}$$

So, we have

$$\begin{aligned} 0 &\geq \|(x_t - u) - t(v - u)\|^2 - \|t(x_t - u) - t(v - u)\|^2 \\ &\geq 2((1-t)(x_t - u), J(t(x_t - v))) \\ &= 2(1-t)t(x_t - u, J(x_t - v)) \end{aligned}$$

and hence

$$(x_t - u, J(x_t - v)) \leq 0 \tag{7}$$

for $v \in F(T)$. From Lemma 1, it follows that

$$\mu_n(x - z, J(x_n - z)) \leq 0$$

for all $x \in C$. In particular, we have

$$\mu_n(u - z, J(x_n - z)) \leq 0. \tag{8}$$

Combining (7) with $v = z$ and (8), we get

$$\mu_n(x_n - z, J(x_n - z)) = \mu_n\|x_n - z\|^2 \leq 0$$

Therefore, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges strongly to z. To complete the proof, suppose there is another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to (say) y. Then y is a fixed point of QT by (4) and also of T. It then follows from (7) that

$$(z - u, J(z - y)) \leq 0$$

and

$$(y - u, J(y - z)) \leq 0.$$

Adding these two inequalities yields

$$(z - y, J(z - y)) = \|z - y\|^2 \leq 0$$

and thus $z = y$. This prove the strong convergence of $\{x_t\}$ to z. $\square$

COROLLARY 1 [19]. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $T : C \to H$ a nonexpansive nonself-mapping satisfying the weak inwardness condition, $P : H \to C$ the nearest point projection, and $\{x_t\}$ the sequence defined by (2) with P instead of Q. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

PROOF. Note that the nearest point projection P of a Hilbert space H onto a closed convex subset C is a sunny and nonexpansive retraction. Thus the result follows from Theorem 1. □

By slightly modifying the proof of Theorem 1, we can also obtain the following result in case that the fixed point set of T is nonempty.

THEOREM 2. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the weak inwardness condition. Suppose that C is a sunny nonexpansive retact of E, and that for some $u \in C$ and each $t \in (0,1)$, x_t is a (unique) fixed point of the contraction R_t defined by (2), where Q is a sunny nonexpansive retraction of E onto C. If the fixed point set $F(T)$ of T is nonempty, then $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of T.*

PROOF. Let $w \in F(T)$. Then we have $\|w - x_t\| \le \|w - u\|$ for all $t \in (0,1)$ and hence $\{x_t\}$ is bounded. Let $t_n \to 1$ and $x_n = x_{t_n}$. As in the proof of Theorem 1, define $\phi : C \to [0,\infty)$ by $\phi(z) = \mu_n \|x_n - z\|^2$ and let

$$M = \{u \in C : \mu_n \|x_n - u\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2\}.$$

Then, by the proof of Theorem 1, M is nonempty and invariant under QT. It is also bounded, closed and convex. Further, M contains a fixed point of QT. In fact, define

$$M^o = \{v \in M : \|v - w\| = \min_{y \in M} \|w - y\|\}.$$

Then, since E is reflexive and strictly convex, M^o is a singleton (cf. [1. p.79]). Denote such a singleton by v. Then we have

$$\|QTv - w\| = \|QTv - QTw\| \le \|v - w\|$$

and hence $QTv = v$. This v is also a fixed point of T. Let $v \in M \cap F(T)$. Then, as in the proof of Theorem 1, we have $x_t \to v$. □

It is well-known (cf. [5, 14]) that if C, a bounded closed convex subset of a Banach space E, has the FPP and if a nonexpansive nonself-mapping $T : C \to E$ is weakly inward, then T has a fixed point. Thus the following result is a direct consequence of Theorem 1 (resp. Theorem 2).

COROLLARY 2. *Let E, C, T, Q be as in Theorem 1 (resp. Theorem 2). Suppose in addition that C is bounded (resp. bounded and has the FPP). Then the sequence $\{x_t\}$ defined by (2) converges strongly as $t \to 1$ to a fixed point of T.*

REMARK. Theorem 1 and Theorem 2 apply to all uniformly convex and uniformly smooth Banach spaces and in particular, to all L^p spaces, $1 < p < \infty$.

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