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INFINITESIMALLY GENERATED STOCHASTIC TOTALLY POSITIVE MATRICES

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ABSTRACT. We show that each element in the semigroup S_n of all $n \times n$ non-singular stochastic totally positive matrices is generated by the infinitesimal elements of S_n, which form a cone consisting of all $n \times n$ Jacobi intensity matrices.

1. Introduction

Let G be a Lie group, let $L(G)$ be its Lie algebra, and let $\exp : L(G) \to G$ denote the exponential mapping. Let $gl(n, \mathbb{R})$ denote the set of all real $n \times n$ matrices and $GL(n, \mathbb{R})$ the general linear group of degree n over $\mathbb{R}$. Here $\mathbb{R}$ denotes the set of all real numbers and hereafter we shall use this notation. For $G = GL(n, \mathbb{R})$ and $L(G) = gl(n, \mathbb{R})$, it is well known that the exponential map exp: $gl(n, \mathbb{R}) \to GL(n, \mathbb{R})$ is defined by $\exp(tX) = I + tX + \frac{1}{2!}(tX)^2 + \dots$ for $X \in gl(n, \mathbb{R})$.

Let S_n be a subsemigroup of $GL(n, \mathbb{R})$ and let $X(t)$ be a differentiable matrix function of the real parameter t in an interval $0 \leq t \leq t_0$ such that $X(t) \in S_n$ for each t and $X(0) = I$. We call the matrix $(\frac{dX(t)}{dt})|_{t=0}$ an *infinitesimal element* of S_n and denote the totality of all infinitesimal elements of S_n by $\mathcal{D}(S_n)$. Let $A(t)$ be a sectionwise continuous function of t $(0 \leq t \leq t_0)$ such that $A(t) \in \mathcal{D}(S_n)$ for each t. It is standard that the differential equation

$$\frac{dX(t)}{dt} = A(t)X(t) \ ; \quad X(0) = I$$

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has a unique continuous solution and $X(t_0) \in S_n$. This $X(t_0)$ in S_n is called *generated by the infinitesimal elements* $A(t)$ $(0 \leq t \leq t_0)$.

Loewner [4] showed that each element in the semigroup of all $n \times n$ non-singular totally positive matrices is generated by the infinitesimal elements of the semigroup, which form a set of all $n \times n$ Jacobi matrices with non-negative off-diagonal elements. In general, a semigroup is not completely recreated from its infinitesimal elements, even if the semigroup is connected, and it is quite difficult to compute a semigroup generated by its infinitesimal elements.

In this paper, we show that the infinitesimal elements of the semigroup of all $n \times n$ non-singular stochastic totally positive matrices are $n \times n$ Jacobi intensity matrices and that each element in the semigroup of all $n \times n$ non-singular stochastic totally positive matrices is generated by the infinitesimal elements of the semigroup, which form a cone consisting of all $n \times n$ Jacobi intensity matrices.

2. Infinitesimally generated stochastic totally positive matrices

DEFINITION 2.1. A subset W of a real topological vector space V is called a *cone* if it satisfies the following conditions:
(1) $W + W \subseteq W$ (2) $\mathbb{R}^+ W \subseteq W$ (3) W is closed in V,
where $\mathbb{R}^+$ denotes the set of all non-negative real numbers.

It is easy to see that $\mathcal{D}(S_n)$ forms a convex cone in the matrix space $gl(n, \mathbb{R})$.

PROPOSITION 2.2. *Let S_n and T_n be subsemigroups of $GL(n, \mathbb{R})$. Then $\mathcal{D}(S_n \cap T_n) = \mathcal{D}(S_n) \cap \mathcal{D}(T_n)$.*

PROOF. Straightforward. □

DEFINITION 2.3. A matrix $A = \|a_{ij}\|$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ over $\mathbb{R}$ is called a *stochastic matrix* if $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1$ for $i = 1, 2, \ldots, m$. A matrix $B = \|b_{kl}\|$ $(k = 1, 2, \ldots, m; l = 1, 2, \ldots, n)$ over $\mathbb{R}$ such that $b_{kl} \geq 0$ for $k \neq l$ and $\sum_{l=1}^{n} b_{kl} = 0$ for $k = 1, 2, \ldots, m$ is called an *intensity matrix*. An intensity matrix C is called an *extreme intensity matrix* if C has only one nonzero off-diagonal element which is equal to 1. An extreme intensity matrix $C = \|c_{kl}\|$ is denoted by E_{pq} $(p \neq q)$ if $c_{pp} = -1$ and $c_{pq} = 1$. A rectangular matrix $A = \|a_{ik}\|$ $(i = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$ over $\mathbb{R}$ is called *totally positive* – hereafter denoted by TP – if all its minors of any order are non-negative. The square matrix $A = \|a_{ij}\|$ is called a *Jacobi matrix* if all elements outside the main diagonal and the first super-diagonal and sub-diagonal are zero.

It is easy to see that the set of all non-singular $n \times n$ stochastic matrices forms a subsemigroup of $GL(n, \mathbb{R})$ and that the set of all non-singular $n \times n$ totally positive matrices forms a subsemigroup of $GL(n, \mathbb{R})$ from the Binet-Cauchy formula ([**3**]). Thus the set of all non-singular $n \times n$ stochastic totally positive matrices forms a semigroup.

LEMMA 2.4. *Let S_n be the semigroup of all real $n \times n$ non-singular matrices with non-negative entries. Then $\mathcal{D}(S_n)$ coincides with the set of all real $n \times n$ matrices which are non-negative off the diagonal.*

PROOF. Let $A = \|a_{ij}\| \in \mathcal{D}(S_n)$. Then $A = (\frac{dX(t)}{dt})|_{t=0}$ with $X(t) \in S_n$ for each t and $X(0) = I$. Since $X(t) \in S_n$, $x_{ij}(t) \geq 0$ for $i, j = 1, 2, \ldots n$. From $X(0) = I$, $x_{ij}(0) = 0$ for $i \neq j$. Thus $a_{ij} = (\frac{dx_{ij}(t)}{dt})|_{t=0}$ ≥ 0 for $i \neq j$.

Conversely let $E_{ij} (i \neq j)$ be an extreme intensity matrix as denoted in the above definition. Since ${E_{ij}}^2 = -E_{ij}$, $\exp(tE_{ij}) = I + tE_{ij} - \frac{t^2}{2!}E_{ij} + \frac{t^3}{3!}E_{ij} + \cdots = I + (1 - e^{-t})E_{ij}$, and hence $\exp(tE_{ij}) \in S_n$ for $t \geq 0$. Since $E_{ij} = \frac{d}{dt}(\exp(tE_{ij}))|_{t=0}$, $E_{ij} \in \mathcal{D}(S_n)$. Let E_k be the matrix whose elements are 0 except that the k-th diagonal element is equal to 1. Since ${E_k}^2 = E_k$, $\exp(tE_k) = I + tE_k + \frac{t^2}{2!}E_k + \frac{t^3}{3!}E_k + \cdots = I + (e^t - 1)E_k$, and hence $\exp(tE_k) \in S_n$ for $t \geq 0$. Thus $E_k \in \mathcal{D}(S_n)$. Similarly we may show $-E_k \in \mathcal{D}(S_n)$. Since $\mathcal{D}(S_n)$ forms a convex cone in the matrix space $gl(n, \mathbb{R})$, $\sum_{1 \leq i \neq j \leq n} \alpha_{ij} E_{ij} + \sum_{k=1}^{n} \beta_k E_k - \sum_{k=1}^{n} \gamma_k E_k \in \mathcal{D}(S_n)$ for all α_{ij}, β_k, $\gamma_k \geq 0$. Thus every real $n \times n$ matrix which is non-negative off the diagonal is contained in $\mathcal{D}(S_n)$. □

LEMMA 2.5. *Let T_n be the semigroup of all real non-singular $n \times n$ matrices with each row sum equal to 1. Then*

$$\mathcal{D}(T_n) = \{\|c_{ij}\| \in gl(n, \mathbb{R}) : \sum_{j=1}^{n} c_{ij} = 0 \text{ for } i = 1, 2, \ldots, n\}.$$

PROOF. Let $\Omega = \|\omega_{ij}\| \in \mathcal{D}(T_n)$. Then there exists $U(t) \in T_n$ such that $\Omega = (\frac{dU(t)}{dt})|_{t=0}$, $\sum_{j=1}^{n} u_{ij}(t) = 1$ for $i = 1, 2, \ldots, n$, and $U(0) = I$. Hence

$$\sum_{j=1}^{n} \omega_{ij} = \sum_{j=1}^{n} \frac{d}{dt}(u_{ij}(t))|_{t=0}$$
$$= \frac{d}{dt}(\sum_{j=1}^{n} u_{ij}(t))|_{t=0} = \frac{d}{dt}(1)|_{t=0} = 0 \text{ for } i = 1, 2, \ldots, n.$$

Conversely suppose that $C = \|c_{ij}\|$ with $\sum_{j=1}^{n} c_{ij} = 0$ for $i = 1, 2, \ldots, n$. Let

$$W = \{\|b_{ij}\| \in gl(n, \mathbb{R}) : \sum_{j=1}^{n} b_{ij} = 0 \text{ for } i = 1, 2, \ldots, n\}.$$

Then W is a cone in $gl(n, \mathbb{R})$ and $C \in W$.

$$C = \frac{d}{dt} e^{tC}|_{t=0} = \lim_{t \to 0^+} \frac{e^{tC} - I}{t}.$$

Since $C \in W$ and W is a cone, $\exp(tC) \in I + tW = I + W$ for $t \geq 0$. Since $\exp(tC)$ is non-singular, $\exp(tC) \in GL(n, \mathbb{R}) \cap (I + W) \subset T_n$. Thus $C \in \mathcal{D}(T_n)$. □

LEMMA 2.6. *Let S_n be the semigroup of all $n \times n$ non-singular totally positive matrices. Then A is an element of $\mathcal{D}(S_n)$ iff A is an $n \times n$ Jacobi matrix with non-negative off-diagonal elements.*

PROOF. See [4]. □

THEOREM 2.7. *Let S_n be the semigroup of all $n \times n$ non-singular stochastic totally positive matrices. Then A is an element of $\mathcal{D}(S_n)$ iff A is an $n \times n$ Jacobi intensity matrix.*

PROOF. Immediate from Proposition 2.2, Lemma 2.4, Lemma 2.5, and Lemma 2.6. □

LEMMA 2.8. *Any $n \times n$ non-singular stochastic totally positive matrix is represented as a finite product of exponentials of Jacobi intensity matrices.*

PROOF. See Theorem 1′ in [1]. □

THEOREM 2.9. *Each element in the semigroup S_n of all $n \times n$ non-singular stochastic totally positive matrices is generated from the infinitesimal elements of S_n, which form a cone consisting of all $n \times n$ Jacobi intensity matrices.*

PROOF. Immediate from Theorem 2.7 and Lemma 2.8. □

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