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ASYMPTOTIC BEHAVIOR OF AN ASYMPTOTICALLY NONEXPANSIVE SEMIGROUP IN BANACH SPACES WITH OPIAL'S CONDITION

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ABSTRACT. In this paper, we study the asymptotic behavior of orbits $\{S(t)x\}$ of an asymptotically nonexpansive semigroup $\mathcal{S} = \{S(t) : t \in G\}$ for a right reversible semitopological semigroup G, defined on a weakly compact convex subset C of Banac spaces with Opial's condition for any $x \in C$.

1. Introduction

Opial ([21]) proved the weak convergence theorem in a Hilbert space: Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive asymptotically regular mapping for which the set $\mathcal{F}(T)$ of fixed points is nonempty. Then, for any x in C, the sequence $\{T^n x\}$ is weakly convergent to an element of $\mathcal{F}(T)$ (cf. [2], [22]). Similar results were also obtained by Bruck ([3]), Emmanuele ([4]), Gornicki ([6]), Hirano ([7]), Kobayashi ([14]) and Miyadera ([19]) in uniformly convex Banach spaces. Corresponding theorems for asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups were investigated by many mathematicians ([1], [18], [20], [24], [25], [27]).

And also, Lau-Takahashi ([15]) established the following theorem: Let C be a closed convex subset of a uniformly convex Banach space X with Fréchet differentiable norm, G a right reversible semitopological

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semigroup, and $\mathcal{S} = \{S(t) : t \in G\}$ a nonexpansive semigroup on C. If $\mathcal{F}(\mathcal{S}) \neq \varnothing$ and $\mathcal{W}(x) \subseteq \mathcal{F}(\mathcal{S})$ for $x \in C$, then the net $\{S(t)x\}$ converges weakly to some $p \in \mathcal{F}(\mathcal{S})$, where $\mathcal{W}(x)$ is the set of all weak limits of subnets of the net $\{S(t)x\}$ (see Theorem 2 and 3 in [15]).

Recently, Lin-Tan-Xu ([16]) proved the convergence of $\{T^n x\}$ of an asymptotically nonexpansive mapping T in Banach spaces without the uniform convexity.

In this paper, we prove the results of Lau-Takahashi ([15]) for an asymptotically nonexpansive semigroup in a k-uniformly rotund space or Banach space without the uniform convexity. The results of this paper are also extensions of Lin-Tan-Xu ([16]) and Kim-Chun-Park ([11]).

2. Preliminaries and Notations

Let C be a nonempty closed convex subset of a real Banach space X and let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $s \in G$ the mappings $s \to as$ and $s \to sa$ from G to G are continous. G is called *right reversible* if any two closed left ideals of G have nonvoid intersection. In this case, $(G, \succcurlyeq)$ is a directed system when the binary relation " $\succcurlyeq$ " on G is defined by $t \succcurlyeq s$ if and only if

$$\{t\} \cup \overline{Gt} \subseteq \{s\} \cup \overline{Gs}, \quad t, s \in G.$$

Right reversible semitopological semigroup include all commutative semigroups which are right amenable as discrete semigroups ([8]). Left reversibility of G is defined similarly. G is called *reversible* if it is both left and right reversible.

A family $\mathcal{S} = \{S(t) : t \in G\}$ of mappings from C into itself is said to be a *continuous representation of* G on C if it satisfies the followings:

(1) $S(ts)x = S(t)S(s)x$ for all $t, s \in G$ and $x \in C$.

(2) For every $x \in C$, the mapping $(s, x) \to S(s)x$ from $G \times C$ into C is continuous when $G \times C$ has the product topology.

A continuous representation $\mathcal{S}$ of G on C is said to be an *asymptotically nonexpansive semigroup* on C if each $t \in G$, there exists $k_t > 0$ such that

$$\| S(t)x - S(t)y \| \leq (1 + k_t) \| x - y \|$$

for all $x, y \in C$, where $\lim_{t\in G} k_t = 0$ (see [5]). Let $\mathcal{F}(\mathcal{S})$ denote the set of all common fixed points of mappings $S(t)$, that is

$$\mathcal{F}(\mathcal{S}) = \bigcap_{t\in G} \mathcal{F}(S(t)).$$

Next recall a generalization of uniform convex Banach spaces which is due to Sullivan ([26]). Let $k \geq 1$ be an integer. Then a Banach space X is said to be *k-uniformly rotund* (briefly. k-UR) if for given any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $\{x_1, x_2, \cdots, x_{k+1}\} \subset B_X(1)$, the closed unit ball of X, satisfies $V(x_1, x_2, \cdots, x_{k+1}) \geq \varepsilon$, then

$$\frac{1}{(k+1)}\left\|\sum_{i=1}^{k+1} x_i\right\| \leq 1 - \delta(\varepsilon).$$

Here, $V(x_1, x_2, \cdots, x_{k+1})$ is the volume enclosed by the set $\{x_1, x_2, \cdots, x_{k+1}\}$, i.e.,

$$V(x_1, \cdots, x_{k+1}) = \sup\left\{\left|\begin{matrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{matrix}\right|\right\},$$

where the supremum is taken over all $f_1, f_2, \cdots, f_k \in B_{X^*}(1)$. The *modulus of k-uniform rotundity* of X is the function $\delta_X^{(k)}(\cdot)$ defined by

$$\begin{aligned}&\delta_X^{(k)}(\varepsilon)\\ &= \inf\left\{1 - \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| : x_i \in B_X(1),\ V(x_1, x_2, \cdots, x_{k+1}) \geq \varepsilon\right\}.\end{aligned}$$

Then it is seen that X is k-UR if and only if $\delta_X^{(k)}(\varepsilon) > 0$ for $\varepsilon > 0$. It is obvious that the modulus of k-uniform rotundity $\delta_X^{(k)}(\varepsilon)$ is a nondecreasing function of ε (In fact, it is almost surely the case that $\delta_X^{(k)}$ is also continuous, but this require a detailed argument).

The norm of X is said to be *uniformly Kadec-Klee* if given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $\{x_\alpha\}_{\alpha\in G}$ is a net in $B_X(1)$ converging weakly to x and that $Sep(x_\alpha) := \inf\{\| x_\alpha - x_\beta \|: \alpha \succcurlyeq \beta\} \geq \varepsilon$, then $\| x \| \leq 1 - \delta(\varepsilon)$ (see [9] and [10] for the same notion for a sequence $\{x_n\}$). It is immediate that uniformly convex Banach spaces have this stronger property. It is known that the following implications hold ([13], [26]).

Uniform convexity $\Leftrightarrow 1 - UR \Rightarrow 2 - UR \Rightarrow \cdots \Rightarrow k - UR$.

In the sequel, we use the following notations; $\overline{\lim} = \limsup$, $\underline{\lim} = \liminf$, " $\rightharpoonup$ " for weak convergence, and " $\rightarrow$ " for strong convergence. Also, a space X is always understood to be an infinite dimensional Banach space without Schur's property, i.e., the weak and strong convergence doesn't coincide for nets.

A Banach space X is said to satisfy *Opial's condition* if for each net $\{x_\alpha\}_{\alpha\in G}$ in X, the condition $x_\alpha \rightharpoonup x$ implies that

$$\overline{\lim_{\alpha\in G}} \| x_\alpha - x \| < \overline{\lim_{\alpha\in G}} \| x_\alpha - y \|$$

for all $y \neq x$(see [21] for the same notion for a sequence $\{x_n\}$). Spaces possessing that property include the Hilbert spaces and the l^p spaces for $1 \leq p < \infty$. However, $L^p(p \neq 2)$ do not satisfy that property ([17]).

Recently, Prus([23]) introduced the notion of the uniform Opial condition for any sequence $\{x_n\}$ in X. A Banach space X is said to satisfy the *uniform Opial condition* if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \leq \underline{\lim_{\alpha\in G}} \| x + x_\alpha \|$$

for each $x \in X$ with $\| x \| \geq c$ and each net $\{x_\alpha\}_{\alpha\in G}$ in X such that $x_\alpha \rightharpoonup 0$ and $\underline{\lim}_{\alpha\in G} \| x_\alpha \| \geq 1$. We now define *Opial's modulus of* X, denoted by $r_X(\cdot)$, as follows

$$r_X(c) = \inf\{\underline{\lim_{\alpha\in G}} \| x + x_\alpha \| -1\},$$

where $c \geq 0$ and the infimum is taken over all $x \in X$ with $\| x \| \geq c$ and nets $\{x_\alpha\}_{\alpha\in G}$ in X such that $x_\alpha \rightharpoonup 0$ and $\underline{\lim}_{\alpha\in G} \| x_\alpha \| \geq 1$.

It is easy to see that the function $r_X(\cdot)$ is nondecreasing and that X satisfies the uniform Opial condition if and only if $r_X(c) > 0$ for all $c > 0$. Furthermore, we know that the Opial's modulus $r_X(\cdot)$ of X is continuous.

We now introduce the notion of the locally uniform Opial condition. A Banach space X is said to satisfy the *locally uniform Opial condition* if for any weak null net $\{x_\alpha\}_{\alpha\in G}$ in X with $\underline{\lim}_{\alpha\in G} \| x_\alpha \| \geq 1$ and any $c > 0$, there is an $r > 0$ such that

$$1 + r \leq \underline{\lim}_{\alpha\in G} \| x_\alpha + x \|$$

for all $x \in X$ with $\| x \| \geq c$ (see [16] for the same notion for a sequence $\{x_n\}$). We can easily see that each "$\underline{\lim}$" can be replaced by "$\overline{\lim}$" in the definition of the (locally) uniform Opial condition. Clearly, uniform Opial condition implies locally uniform Opial condition, which in turn implies Opial's condition ([16]).

Let $\mathcal{W}(x)$ denote the set of all weak limits of subnets $\{S(t_\alpha)x\}$ of the net $\{S(t)x\}$ for a semitopological semigroup G.

3. Main Results

In this section, we study the asymptotic behavior of the orbits $\{S(t)x\}$ for an asymptotically nonexpansive semigroup $\mathcal{S} = \{S(t) : t \in G\}$, under the Opial's condition. In [11] (Proposition 3.3), we proved the demiclosedness principle at zero for an asymptotically nonexpansive semigroup in a Banach space with the locally uniform Opial codition. The demiclosedness principle plays a crucial role in the proofs of our main theorems in this section. We need the following lemma in order to prove our main theorems.

LEMMA 3.1. *Let X be a Banach space satisfying the locally uniform Opial condition, C a nonempty weakly compact convex subset of X, and G a right reversible semitopological semigroup, and $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C. If $S(t)$ is asymptotically regular at some $x \in C$, i.e., $\lim_{t\in G} \| S(st)x - S(t)x \| = 0$ for all $s \in G$, then we have the following conclusions.*

(1) *$\mathcal{F}(\mathcal{S}) \subseteq E(x)$, where $E(x) = \{y \in C : \exists \lim_{t\in G} \| S(t)x - y \| \}$.*

(2) $\mathcal{W}(x) \subseteq \mathcal{F}(\mathcal{S})$ *and* $\mathcal{W}(x)$ *is a singleton.*

PROOF. (1). Let $t \succcurlyeq s$ for $t, s \in G$. Then $t \in \{s\} \cup \overline{Gs}$. We may assume that $t \in \overline{Gs}$. So there exists a net $\{g_\alpha\}$ in G such that $g_\alpha s \to t$ as $\alpha \in G$. Then, for $\alpha \in G$ and $y \in \mathcal{F}(\mathcal{S})$,

$$\begin{aligned} \| S(g_\alpha s)x - y \| &= \| S(g_\alpha)S(s)x - S(g_\alpha)y \| \\ &\leq (1 + k_{g_\alpha}) \| S(s)x - y \| . \end{aligned}$$

Hence, we have

$$\| S(t)x - y \| \leq \| S(s)x - y \|$$

for all $t \succcurlyeq s$ and $y \in \mathcal{F}(\mathcal{S})$. This proves that $\mathcal{F}(\mathcal{S}) \subseteq E(x)$ as desired.

(2). Let $\{S(t_\alpha)x\}$ be a subnet of $\{S(t)x\}$ converging weakly to $y \in C$ as $\alpha \in G$. Letting $x_\alpha = S(t_\alpha)x$. Then, since $S(s)$ is asymptotically regular, $\| x_\alpha - S(s)x_\alpha \| \to 0$ as $\alpha \in G$ for all $s \in G$. Since $I - S(s)$ is demiclosed at zero, from Proposition 3.3 in [11], $(I - S(s))y = 0$ for all $s \in G$. Hence $\mathcal{W}(x) \subseteq \mathcal{F}(\mathcal{S})$. Moreover, let y_1 and y_2 be two weak limits of subnets $\{S(t_\alpha)x\}$ and $\{S(t_\beta)x\}$ of the net $\{S(t)x\}$, respectively. Since $\mathcal{W}(x) \subseteq \mathcal{F}(\mathcal{S})$, there are $d_1, d_2 \geq 0$ by (1) such that

$$d_1 = \lim_{t \in G} \| S(t)x - y_1 \| \quad \text{and} \quad d_2 = \lim_{t \in G} \| S(t)x - y_2 \| .$$

If $y_1 \neq y_2$, then we have

$$\begin{aligned} d_1 &= \lim_{t \in G} \| S(t)x - y_1 \| = \overline{\lim_{\alpha \in G}} \| S(t_\alpha)x - y_1 \| \\ &< \overline{\lim_{\alpha \in G}} \| S(t_\alpha)x - y_2 \| = \overline{\lim_{\beta \in G}} \| S(t_\beta)x - y_2 \| \\ &< \overline{\lim_{\beta \in G}} \| S(t_\beta)x - y_1 \| = \lim_{t \in G} \| S(t)x - y_1 \| \\ &= d_1 . \end{aligned}$$

This is a contradiction. Hence $\mathcal{W}(x)$ is a singleton. □

As a direct consequence of Lemma 3.1, we can prove the convergence theorem of orbits $\{S(t)x\}$ of an asymptotically nonexpansive semigroup $\mathcal{S} = \{S(t) : t \in G\}$ for a right reversible semitopological semigroup G.

THEOREM 3.2. *(see [11]) Let C be a nonempty weakly compact convex subset of a Banach space X satisfying the locally uniform Opial condition, G a right reversible semitopological semigroup, and $\mathcal{S} = \{S(t) : t \in G\}$ an asymptotically nonexpansive semigroup on C. If $S(t)$ is asymptotically regular at $x \in C$, then $\{S(t)x\}$ converges weakly to a point p in $\mathcal{F}(\mathcal{S})$.*

PROOF. From Lemma 3.1, it is easy to show that the orbits $\{S(t)x\}$ converges weakly to p in $\mathcal{F}(\mathcal{S})$. □

It is not clear whether the asymptotic regularity in Theorem 3.2 can be weakened to the weakly asymptotic regularity. But we improved the Theorem 3.2 when the space X is assumed to be satisfying the uniform Opial condition.

THEOREM 3.3. *(see [11]) Let C be a nonempty weakly compact convex subset of a Banach space X satisfying the uniform Opial condition and let G, S be as in Theorem 3.2. If $S(t)$ is weakly asymptotically regular at $x \in C$. i.e., w-$\lim_{t\in G} \| S(st)x - S(t)x \| = 0$ for all $s \in G$, then $\{S(t)x\}$ converges weakly to a point p in $\mathcal{F}(\mathcal{S})$.*

We don't know whether the conclusion of Theorem 3.2 is still true if the locally uniform Opial condition is weakened to Opial's condition, but we have the following partial answer. We first establish lemmas which are of interest in itselves. The following lemma is easily obtained in a similar way to the that of Proposition 4.2 in [12].

LEMMA 3.4. *Let G be a right reversible semitopological semigroup and let $\{x_\alpha\}_{\alpha\in G}$ be a bounded net in a Banach space X which has no strongly convergent subnet. Then given any positive integer k, there exist $\rho > 0$ and a subnet $\{z_\alpha\}$ of $\{x_\alpha\}$ such that if $\{z_{\alpha_1}, z_{\alpha_2}, \cdots, z_{\alpha_{k+1}}\}$ is any set of $k+1$ distinct points of $\{z_\alpha\}$, then $V(z_{\alpha_1}, z_{\alpha_2}, \cdots, z_{\alpha_{k+1}}) \geq \rho$.*

We can get the following proposition from the definition of the modulus of k - uniform rotundity of X.

PROPOSITION 3.5. *([13]) Let X be a k - uniformly rotund Banach space, and $r > 0$. If for $x_i \in B_X(r)(i = 1, 2, \cdots, k+1)$ and given $\varepsilon > 0, V(x_1, x_2, \cdots, x_{k+1}) \geq \varepsilon$, then*

$$\left\| \sum_{i=1}^{k+1} \frac{x_i}{k+1} \right\| \leq r\left(1 - \delta_X^{(k)}\left(\frac{\varepsilon}{r^k}\right)\right),$$

where $V(x_1, x_2, \cdots, x_{k+1})$ *is the volume enclosed by the set* $\{x_1, x_2, \cdots, x_{k+1}\}$.

We have the following interesting result which is concerned with the minimum of any functional.

LEMMA 3.6. *Let* C *be a nonempty closed convex subset of a* k*-uniformly rotund Banach space* X *for some* $k \geq 1$, G *a right reversible semitopological semigroup,* $\{x_\alpha\}_{\alpha \in G}$ *a bounded net in* X, *and* F *the functional on* X *defined by*

$$F(y) = \overline{\lim_{\alpha \in G}} \parallel x_\alpha - y \parallel$$

for each $y \in X$. *Let* $\{y_\beta\}_{\beta \in G}$ *be the net which satisfies the condition*

$$\lim_{\beta \in G} F(y_\beta) = \inf_{y \in C} F(y).$$

Then $\{y_\beta\}$ *has a strongly convergent subnet in* C.

PROOF. Let $r = \inf_{y \in C} F(y)$ and $\{y_\beta\}$ a net in C such that $\lim_{\beta \in G} F(y_\beta) = r$. If $r = 0$, then for all $\varepsilon > 0$ there is a $\beta_\circ \in G$ such that $\overline{\lim}_{\alpha \in G} \parallel x_\alpha - y_\beta \parallel < \frac{\varepsilon}{2}$ for all $\beta \succcurlyeq \beta_\circ$. Since for such $\beta_\circ \in G$ and $\beta, \gamma \geq \beta_\circ$,

$$\begin{aligned} \parallel y_\beta - y_\gamma \parallel &\leq \overline{\lim_{\alpha \in G}} \parallel x_\alpha - y_\beta \parallel + \overline{\lim_{\alpha \in G}} \parallel x_\alpha - y_\gamma \parallel \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

the whole net $\{y_\beta\}$ is Cauchy in C. Hence $\{y_\beta\}$ has a strongly convergent subnet in C. Next, we suppose that $r > 0$ and $\{y_\beta\}$ has no strongly convergent subnet in C. Then, by Lemma 3.4, there exist $\rho > 0$ and subnet $\{z_\beta\}_{\beta \in G}$ of $\{y_\beta\}$ in C such that if $\{z_{\beta_1}, z_{\beta_2}, \cdots, z_{\beta_{k+1}}\}$ is any set of $k+1$ distinct points of $\{z_\beta\}$, then for $\{x_\alpha\} \subset C$

$$V(x_\alpha - z_{\beta_1}, x_\alpha - z_{\beta_2}, \cdots, x_\alpha - z_{\beta_{k+1}}) \geq \rho.$$

Let $\delta_X^{(k)}(\cdot)$ be the modulus of k - uniform rotundity of X. Then we can choose an $\varepsilon > 0$ so small that

$$(r + \varepsilon)\left[1 - \delta_X^{(k)}\left(\frac{\rho}{(r+\varepsilon)^k}\right)\right] < r.$$

Since $r = \lim_{\beta\in G} F(y_\beta)$, there exists a $\beta_\circ \in G$ such that $F(y_\beta) < r + \varepsilon$ for all $\beta \succcurlyeq \beta_\circ$. Let $\{z_{\beta_1}, z_{\beta_2}, \cdots, z_{\beta_{k+1}}\}$ be any distinct points such that $\beta_j \succcurlyeq \beta_\circ$ for $1 \le j \le k+1$. Then, there is an $\alpha_\circ \in G$ such that $\| x_\alpha - z_{\beta_j} \| < r + \varepsilon$ for all $\alpha \succcurlyeq \alpha_\circ$ and $1 \le j \le k+1$. From Proposition 3.5,

$$\left\| x_\alpha - \sum_{j=1}^{k+1} \frac{z_{\beta_j}}{k+1} \right\| = \left\| \frac{\sum_{j=1}^{k+1}(x_\alpha - z_{\beta_j})}{k+1} \right\| \le (r+\varepsilon)\left[1 - \delta_X^{(k)}\left(\frac{\rho}{(r+\varepsilon)^k}\right)\right],$$

which shows that $F\left(\sum_{j=1}^{k+1} \frac{z_{\beta_j}}{k+1}\right) < r$. Since $\sum_{j=1}^{k+1} \frac{z_{\beta_j}}{k+1} \in C$, it is a contradiction. This completes the proof. □

Finally, we need the following lemma in order to prove the Theorem 3.8. The following lemma is more beautiful than the Theorem 3.5 in [16].

LEMMA 3.7. *If X is a k-uniformly rotund Banach space for some $k \ge 1$ and satisfies the Opial's condition, then it satisfies the uniform Opial condition.*

PROOF. Let $\{x_\alpha\}_{\alpha\in G}$ be a net in X such that $x_\alpha \rightharpoonup 0$ and $\overline{\lim}_{\alpha\in G} \| x_\alpha \| \ge 1$. For the functional

$$F(y) = \overline{\lim_{\alpha\in G}} \| x_\alpha - y \|$$

for $y \in X$, it suffices to show that

$$\overline{\lim_{\alpha\in G}} \| x_\alpha \| < \inf\{F(y) : \| y \| \ge c,\ c > 0\},$$

in order to get the goal of this lemma. Let $\{y_\beta\}_{\beta\in G}$ be a net in X with $\| y_\beta \| \ge c$ and

$$\lim_{\beta\in G} F(y_\beta) = \inf\{F(y) : \| y \| \ge c, c > 0\}.$$

Then $\{y_\beta\}$ is bounded, so there exists an $M > 0$ such that $\sup_{\beta\in G} \| y_\beta \| \le M$. Suppose that we can choose $c > 0$ such that

$$\inf\{F(y) : \| y \| \ge c,\ c > 0\} = \overline{\lim_{\alpha\in G}} \| x_\alpha \| .$$

Then, since X satisfies the Opial's condition, we have

$$\lim_{\beta\in G} F(y_\beta) = \inf_{y\in B_X(M)} F(y),$$

where $B_X(M) = \{x \in X :\| x \|\leq M\}$. Hence,by Lemma 3.6, there exists a subnet $\{y_{\beta_\gamma}\}$ of $\{y_\beta\}$ and $z \in B_X(M)$ such that $\lim_{\gamma\in G} y_{\beta_\gamma} = z$. Clearly, $\| z \|\geq c(> 0)$, and also we have

$$\begin{aligned}\overline{\lim_{\alpha\in G}} \| x_\alpha - z \| = F(z) = \lim_{\gamma\in G} F(y_{\beta_\gamma}) = \inf\{F(y) :\| y \|\geq c, c > 0\}\\ = \overline{\lim_{\alpha\in G}} \| x_\alpha \| .\end{aligned}$$

This is a contradiction for the Opial's condition. Hence, for all $c > 0$,

$$\overline{\lim_{\alpha\in G}} \| x_\alpha \|< \inf\{F(y) :\| y \|\geq c, c > 0\}.$$

Hence, for $x_\alpha \rightharpoonup 0$ and $\| y \|\geq c$, $c > 0$,

$$1 \leq \overline{\lim_{\alpha\in G}} \| x_\alpha \|< \overline{\lim_{\alpha\in G}} \| x_\alpha + (-y) \| .$$

Therefore, there is an $r > 0$ such that

$$1 + r \leq \overline{\lim_{\alpha\in G}} \| x_\alpha + x \|$$

for all $\| x \|\geq c, c > 0$. This completes the proof. □

We are now in a position to prove the Theorem 3.8.

THEOREM 3.8. *Let X be a k-uniformly rotund Banach space for some $k \geq 1$ with the Opial's condition and let $C, G, \mathcal{S}$ be as in Theorem 3.2. If $S(t)$ is weakly asymptotically regular at $x \in C$, then $\{S(t)x\}$ converges weakly to a point p in $\mathcal{F}(\mathcal{S})$.*

PROOF. We know that X satisfies the uniform Opial condition from the Lemma 3.7. And hence, by Theorem 3.3, $\{S(t)x\}$ converges weakly to a point p in $\mathcal{F}(\mathcal{S})$. □

Clearly, $V(x_1, x_2) =\| x_1 - x_2 \|$ and thus the 1-UR space simply the uniformly convex Banach space. So, we have the next corollary.

COROLLARY 3.9. *(cf. [4],[6] and [15]) Let X be a uniformly convex Banach space with the Opial's condition, and let $C, G, \mathcal{S}$ and $S(t)$ be as in Theorem 3.3. Then the conclusion of Theorem 3.8 holds.*

And also, we have another convergence theorem in a Banach space X with uniformly Kadec-Klee norm and satisfying the Opial's condition.

THEOREM 3.10. *Let X be a Banach space satisfying the Opial's condition with a uniformly Kadec-Klee norm and let $C, G, \mathcal{S}$ be as in Theorem 3.3. If $S(t)$ is weakly asymptotically regular at $x \in C$, then $\{S(t)x\}$ converges weakly to a point p in $\mathcal{F}(\mathcal{S})$.*

PROOF. We must prove the results of Lemma 3.1 under the assumptions of this theorem. We can easily prove the result (1) of Lemma 3.1. And so, we have to show only (2). For our purpose, it suffices to show that $I - S(t)$ is demiclosed at zero. Let $\{x_\alpha\}_{\alpha \in G}$ be a net in C such that $x_\alpha \rightharpoonup x$ and $x_\alpha - S(t)x_\alpha \to 0$ for all $t \in G$. By Lemma 3.2 in [11], we have $S(t)x \rightharpoonup x$, as $t \in G$. Now let

$$r = \overline{\lim_{t \in G}} \parallel S(t)x - x \parallel \text{ and } r(s) = \overline{\lim_{t \in G}} \parallel S(t)x - S(s)x \parallel .$$

Since $S(t)$ is weakly asymptotically regular at x, $r < r(s) \leq (1 + k_s)r$ from the Opial's condition. Hence we have $\lim_{s \in G} r(s) = r$. Now, we must prove that $r = 0$. Suppose that $r > 0$, then $\{S(t)x\}$ has no strongly convergent subnet. Therefore, there exists a subnet $\{S(t_\alpha)x\}$ of $\{S(t)x\}$ such that

$$Sep(S(t_\alpha)x) > 0.$$

Let $\varepsilon_\circ = \dfrac{1}{2r} Sep(S(t_\alpha)x)$ and $\delta_\circ(\varepsilon_\circ)$ the number satisfying the definition of uniformly Kadec-Klee norm corresponding to $\varepsilon_\circ$. Then we can choose a $\eta > 0$ with $0 < \eta < 1$ so small that $(1 + \eta)(1 - \delta_\circ(\varepsilon_\circ)) < 1$. Since $\lim_{s \in G} k_s = 0$, there is an $s_\circ \in G$ such that $k_s < \eta$ for all $s \succcurlyeq s_\circ$. Fix $s \succcurlyeq s_\circ$. Since

$$r(s) = \overline{\lim_{t \in G}} \parallel S(t)x - S(s)x \parallel \leq (1 + k_s)r < (1 + \eta)r$$

there is an $\alpha_\circ = \alpha_\circ(s, \eta)$ such that

$$\parallel S(t_\alpha)x - S(s)x \parallel < (1 + \eta)r$$

for all $\alpha \succcurlyeq \alpha_\circ$. Let $y_\alpha = \dfrac{S(t_\alpha)x - S(s)x}{(1+\eta)r}$ for all $\alpha \succcurlyeq \alpha_\circ$. Then $\| y_\alpha \| < 1$,

$$y_\alpha \rightharpoonup \frac{x - S(s)x}{(1+\eta)r}$$

and

$$Sep(y_\alpha) = \frac{Sep(S(t_\alpha)x)}{(1+\eta)r} > \varepsilon_\circ.$$

Since the norm of X is uniformly Kadec-Klee, it follows that

$$\frac{\| x - S(s)x \|}{(1+\eta)r} \leq 1 - \delta_\circ(\varepsilon_\circ)$$

for all $s \succcurlyeq s_\circ$. Hence, we have

$$r = \overline{\lim_{s\in G}} \| x - S(s)x \| \leq (1+\eta)(1-\delta_\circ(\varepsilon_\circ))r < r.$$

This is a contradiction. Therefore, $r = 0$ and so $\lim_{s\in G} S(s)x = x$. From the continuity of $S(t)$, $S(t)x = x$ for all $t \in G$. This completes the proof. □

PROPOSITION 3.11. *(Theorem 1 in [13]) If X is a k- uniformly rotund Banach space for some $k \geq 1$, then X has a uniformly Kadec-Klee norm.*

REMARK. Using the Proposition 3.11, then the Theorem 3.8 can be easily proved as corollary of Theorem 3.10. And also, since uniformly convex Banach space X has a uniformly Kadec-Klee norm, the Corollary 3.9 is obvious as corollary of Theorem 3.10.

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