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A CHARACTERIZATION OF REFLEXIVITY OF NORMED ALMOST LINEAR SPACES

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Abstract. In [6] we proved that if a *nals* X is reflexive, then $X = W_X + V_X$. In this paper we show that, for a split *nals* $X = W_X + V_X$, X is reflexive if and only if V_X and W_X are reflexive.

1. Preliminaries

G. Godini [3,4,5] introduced a normed almost linear space(*nals*), a concept which generalizes normed linear space. An example of a *nals* is the collection of all nonempty, bounded and convex subsets of a real normed linear space. In [6], we defined the notion of reflexivity of a *nals*. Also, we proved that if a *nals* X is reflexive then X is split as $X = W_X + V_X$. In this note, we characterizes the reflexivity of a *nals* X (without basis). First of all we recall some definitions and results which are needed in this paper.

An *almost linear space* (*als*) is a set X together with two mappings $s : X \times X \to X$ and $m : \mathbb{R} \times X \to X$ satisfying the conditions $(L_1) - (L_8)$ given below. For $x, y \in X$ and $\lambda \in \mathbb{R}$ we denote $s(x, y)$ by $x + y$ and $m(\lambda, x)$ by λx, when these will not lead to misunderstandings. Let $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$. (L_1) $x+(y+z) = (x+y)+z$; (L_2) $x+y = y+x$; (L_3) There exists an element $0 \in X$ such that $x+0 = x$ for each $x \in X$; (L_4) $1x = x$; (L_5) $\lambda(x+y) = \lambda x + \lambda y$; (L_6) $0x = 0$; (L_7) $\lambda(\mu x) = (\lambda\mu)x$; (L_8) $(\lambda + \mu)x = \lambda x + \mu x$ for $\lambda \geq 0$, $\mu \geq 0$. We denote $-1x$ by $-x$, if there is no confusion likely, and in the sequel $x - y$ means $x + (-y)$.

A nonempty subset Y of an *als* X is called an *almost linear subspace* of X, if for each $y_1, y_2 \in Y$ and $\lambda \in \mathbb{R}$, $s(y_1, y_2) \in Y$ and $m(\lambda, y_1) \in Y$.

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An almost linear subspace Y of X is called a *linear subspace* of X if $s : Y \times Y \to Y$ and $m : \mathbb{R} \times Y \to Y$ satisfy all the axioms of a linear space.

For an *als* X we introduce the following two sets;

$$V_X = \{x \in X : x - x = 0\}, \tag{1.1}$$

$$W_X = \{x \in X : x = -x\}. \tag{1.2}$$

Then, we have the following properties: (1) The set V_X is a linear subspace of X, and it is the largest one. (2) The set W_X is an almost linear subspace of X and $W_X = \{x - x : x \in X\}$. (3) An *als* X is a linear space $\Longleftrightarrow V_X = X \Longleftrightarrow W_X = \{0\}$, and $V_X \cap W_X = \{0\}$.

Let X and Y be two almost linear spaces. A mapping $T : X \to Y$ is called a *linear operator* if $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$ for all $\lambda_i \in \mathbb{R}$ and $x_i \in X$, $i = 1, 2$. An *isomorphism* T of an *als* X onto an *als* Y is a bijective mapping which preserves the two algebraic operations of an *als*; that is, $T : X \to Y$ is a bijective linear operator. Then Y is said to be *isomorphic* with X. The following is well known.

PROPOSITION 1.1. *Let T be a linear operator from an als X into an als Y. Then*

(1) $T(V_X) \subset V_Y$, $T(W_X) \subset W_Y$.
(2) *If $X = V_X + W_X$, then $T(X) = T(V_X) + T(W_X)$. In particular, if T is an isomorphism, then $Y = T(X) = V_Y + W_Y$.*

Let X be an *als*. A function $f : X \to \mathbb{R}$ is called an *almost linear functional* if the conditions (1.3) – (1.5) are satisfied.

$$f(x + y) = f(x) + f(y) \quad (x, y \in X) \tag{1.3}$$

$$f(\lambda x) = \lambda \cdot f(x) \quad (\lambda \geq 0,\ x \in X) \tag{1.4}$$

$$f(w) \geq 0 \quad (w \in W_X). \tag{1.5}$$

A functional $f : X \to \mathbb{R}$ is called a *linear functional* on X if it satisfies (1.3), and (1.4) for each $\lambda \in \mathbb{R}$. Then (1.5) is also satisfied. Note that an almost linear functional is not a linear operator from X to $\mathbb{R}$, but a linear functional is a linear operator.

Let $X^{\#}$ be the set of all almost linear functionals defined on an *als* X. We define two operations $s : X^{\#} \times X^{\#} \to X^{\#}$ and $m : \mathbb{R} \times X^{\#} \to X^{\#}$ as follows:

$$s(f_1, f_2)(x) = f_1(x) + f_2(x) \quad (f_1, f_2 \in X^{\#}),$$

$$m(\lambda, f)(x) = f(\lambda x) \quad (\lambda \in \mathbb{R},\ f \in X^{\#})$$

for all $x \in X$. Clearly, $s(f_1, f_2) \in X^{\#}$, $m(\lambda, f) \in X^{\#}$, and s, m satisfy $(L_1) - (L_8)$ with $0 \in X^{\#}$ being the functional which is 0 at each $x \in X$. Therefore $X^{\#}$ is an *als*. $X^{\#}$ is called the *algebraic dual space* of an *als* X. We denote $s(f_1, f_2)$ by $f_1 + f_2$ and $m(\lambda, f)$ by $\lambda \circ f$.

PROPOSITION 1.2 ([6]). *Let X be an als. Then $X^{\#} = W_{X^{\#}} + V_{X^{\#}}$.*

PROPOSITION 1.3 ([3]). *If f is an almost linear functional on an als X, then $f \in V_{X^{\#}}$ if and only if $f|_{W_X} = 0$.*

PROPOSITION 1.4. *Let X be a split als as $X = W_X + V_X$. If f is an almost linear functional on X, then $f \in W_{X^{\#}}$ if and only if $f|_{V_X} = 0$.*

PROOF. Suppose that $f \in W_{X^{\#}}$. Then for each $v \in V_X$ we have

$$f(v) + f(v) = f(v) + (-1 \circ f)(v) = f(v) + f(-v) = f(v - v) = f(0) = 0,$$

since $-1 \circ f = f$. Therefore $f|_{V_X} = 0$.

Conversely, suppose that $f|_{V_X} = 0$ and $x = v + w \in X$ with $v \in V_X$, $w \in W_X$. Since $f(v) = f(-v) = 0$, we have

$$\begin{aligned}(-1 \circ f)(x) = f(-x) = f(w - v) &= f(w) + f(-v) \\ &= f(w) + f(v) = f(w + v) = f(x).\end{aligned}$$

Therefore $f \in W_{X^{\#}}$. □

2. Reflexivity of NALS

A *norm* on an *als* X is a functional $\|\cdot\| : X \to \mathbb{R}$ satisfying the conditions $(N_1)-(N_3)$ below. Let $x, y, z \in X$ and $\lambda \in \mathbb{R}$. (N_1) $\|x-z\| \leq \|x-y\| + \|y-z\|$; (N_2) $\|\lambda x\| = |\lambda|\|x\|$; (N_3) $\|x\| = 0$ iff $x = 0$.

Using (N_1) we get

$$\|x+y\| \leq \|x\| + \|y\| \quad (x, y \in X) \tag{2.1}$$

$$\|x-y\| \geq |\|x\| - \|y\|| \quad (x, y \in X). \tag{2.2}$$

By the above axioms it follows that $\|x\| \geq 0$ for each $x \in X$.

An *als* X together with $\|\cdot\| : X \to \mathbb{R}$ satisfying $(N_1)-(N_3)$ is called a *normed almost linear space* (*nals*).

When X is a *nals*, for $f \in X^{\#}$, we define, as in the case of a normed linear space,

$$\|f\| = \sup\{|f(x)| : x \in X, \|x\| \leq 1\}, \tag{2.3}$$

and let

$$X^* = \{f \in X^{\#} : \|f\| < \infty\}.$$

Then X^* is a *nals*[4], called the *dual space* of X. We denote the dual space $(X^*)^*$ of X^* by X^{**} and call it the *second dual space* of X.

For a *nals* X and $f \in X^*$, an equivalent formula for the norm of f is

$$\|f\| = \sup_{\|x\|=1} |f(x)| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}, \tag{2.4}$$

hence

$$|f(x)| \leq \|f\|\|x\|.$$

In the theory of a normed linear space an important tool is the Hahn-Banach theorem. An analogous theorem is no longer true in a *nals* [3, 4.5. Example]. But we have the following Propositions.

Proposition 2.1 ([5]). *Let $(X, \|\cdot\|)$ be a nals. Then for each $x \in X$ there exists $f_x \in X^*$ such that $\|f_x\| = 1$ and $f_x(x) = \|x\|$.*

PROPOSITION 2.2 ([4]). *Let $(X, \|\cdot\|)$ be a nals. Then for each $f \in (W_X)^*$ there exists $f_1 \in W_{X^*}$ such that $f_1|_{W_X} = f$ and $\|f_1\| = \|f\|$.*

PROPOSITION 2.3 ([4]). *Let $(X, \|\cdot\|)$ be a nals and split as $X = W_X + V_X$. Then for each $f \in (V_X)^*$ there exists $f_1 \in V_{X^*}$ such that $f_1|_{V_X} = f$ and $\|f_1\| = \|f\|$.*

PROPOSITION 2.4. *For any x in a nals X, we have*

$$\|x\| = \sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, f \neq 0\right\}.$$

PROOF. For any $x \in X$, by Proposition 2.1, there exists $f_x \in X^*$ such that $\|f_x\| = 1$ and $f_x(x) = \|x\|$. So, we have

$$\|x\| = \frac{|f_x(x)|}{\|f_x\|} \leq \sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, f \neq 0\right\}.$$

From $|f(x)| \leq \|f\|\|x\|$, we have

$$\sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, f \neq 0\right\} \leq \|x\|$$

for each $f \in X^*$. Hence $\|x\| = \sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, f \neq 0\right\}$. □

An *isomorphism* T of a *nals* X onto a *nals* Y is a bijective linear operator $T : X \to Y$ which preserves the norm, that is, for all $x \in X$,

$$\|T(x)\| = \|x\|.$$

Then X is called *isomorphic* with Y.

For $x \in X$ let Q_x be the functional on X^* defined, as in the case of a normed linear space, by

$$Q_x(f) = f(x) \quad (f \in X^*). \tag{2.5}$$

Then Q_x is an almost linear functional on X^* and

$$\|Q_x\| \leq \|x\|. \tag{2.6}$$

Hence Q_x is an element of X^{**}, by definition of X^{**}. This defines a mapping

$$C : X \to X^{**} \tag{2.7}$$

by $C(x) = Q_x$. C is called the *canonical mapping* of X into X^{**}.

If the canonical mapping C of a *nals* X into X^{**} defined by (2.7) is an isomorphism, then X is said to be *reflexive.*

PROPOSITION 2.5. *For a nals X, the canonical mapping C defined by (2.7) is a linear operator and preserves the norm.*

PROOF. By (2.4) and Proposition 2.4, we have

$$\|Q_x\| = \sup_{f\neq 0} \frac{|Q_x(f)|}{\|f\|} = \sup_{f\neq 0} \frac{|f(x)|}{\|f\|} = \|x\|$$

for each $x \in X$. Hence C preserves the norm.

Let $x, y \in X$ and $\alpha \in \mathbb{R}$. For each $f \in X^*$, we have

$$Q_{x+y}(f) = f(x+y) = f(x) + f(y) = Q_x(f) + Q_y(f),$$

$$Q_{\alpha x}(f) = f(\alpha x) = (\alpha \circ f)(x) = (\alpha \circ Q_x)(f).$$

Thus, $C(x+y) = C(x) + C(y)$ and $C(\alpha x) = \alpha \circ C(x)$. Therefore C is a linear operator. □

THEOREM 2.6 ([6]). *If a nals X is reflexive, then $X = W_X + V_X$.*

THEOREM 2.7. *If a nals X splits as $X = W_X + V_X$, then*

(1) *V_{X^*} is isomorphic with $(V_X)^*$,*
(2) *W_{X^*} is isomorphic with $(W_X)^*$.*

PROOF. Since $x^*|_{V_X} \in (V_X)^*$ for each $x^* \in V_{X^*}$, we can define an operator

$$T : V_{X^*} \to (V_X)^*$$

by $T(x^*) = x^*|_{V_X}$ for each $x^* \in V_{X^*}$. For $x^*, y^* \in V_{X^*}$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} T(\alpha \circ x^* + \beta \circ y^*)(v) &= (\alpha \circ x^* + \beta \circ y^*)(v) \\ &= x^*(\alpha v) + y^*(\beta v) \\ &= T(x^*)(\alpha v) + T(y^*)(\beta v) \\ &= (\alpha \circ T(x^*))(v) + (\beta \circ T(y^*))(v) \\ &= [\alpha \circ T(x^*) + \beta \circ T(y^*)](v) \end{aligned}$$

for each $v \in V_X$. Hence T is a linear operator.

If $x^* \neq y^* \in V_{X^*}$, then $x^*(v) \neq y^*(v)$ for some $v \in V_X$ by Proposition 1.3. So, $T(x^*) \neq T(y^*)$. Hence T is injective.

For each $v^* \in (V_X)^*$, there exists $x^* \in V_{X^*}$ such that $x^*|_{V_X} = v^*$ by Proposition 2.3. Hence T is surjective.

For any $v^* \in V_{X^*}$, $\|v^*\| \geq \|v^*|_{V_X}\| = \|T(v^*)\|$. Also, if $x = v + w \in X$, $v \in V_X$, $w \in W_X$ with $\|x\| \leq 1$, then $\|v\| \leq 1$ and $v^*(x) = v^*(v)$. So we have

$$\begin{aligned} \|v^*\| &= \sup\{|v^*(x)| : x \in X, \|x\| \leq 1\} \\ &\leq \sup\{|v^*(v)| : v \in V_X, \|v\| \leq 1\} \\ &= \sup\{|T(v^*)(v)| : v \in V_X, \|v\| \leq 1\} \\ &= \|T(v^*)\| \end{aligned}$$

Hence T preserves the norm. Therefore, V_{X^*} is isomorphic with $(V_X)^*$.

Similarly, applying Proposition 1.4 and Proposition 2.2, we can show that an operator $T' : W_{X^*} \to (W_X)^*$, $T'(x^*) = x^*|_{W_X}$ ($x^* \in W_{X^*}$), is an isomorphism. □

COROLLARY 2.8. *If a nals X splits as $X = W_X + V_X$, then*

(1) *$V_{X^{**}}$ is isomorphic with $(V_X)^{**}$,*
(2) *$W_{X^{**}}$ is isomorphic with $(W_X)^{**}$.*

An arbitrary normed almost linear subspace of a *nals* X need not be reflexive even if X is reflexive. But, we have the following result:

THEOREM 2.9. *If a nals X is reflexive, then V_X and W_X are reflexive.*

PROOF. By Theorem 2.6, $X = W_X + V_X$ since X is reflexive. Let $C : X \to X^{**}$ be the canonical isomorphism, and let $C' : V_X \to (V_X)^{**}$ be the canonical mapping. We will show that C' is bijective. Let $v^{**} \in (V_X)^{**}$. By Theorem 2.7, $T : V_{X^*} \to (V_X)^*$, $T(v^*) = v^*|_{V_X}$ $(v^* \in V_{X^*})$, is an isomorphism. Since $x^*|_{V_X} \in (V_X)^*$ for each $x^* \in X^*$, we can define a functional

$$\overline{v}^{**} : X^* \to \mathbb{R}$$

by $\overline{v}^{**}(x^*) = v^{**}(x^*|_{V_X})$ for each $x^* \in X^*$. Then $\overline{v}^{**} \in V_{X^{**}}$. Since C is an isomorphism of X onto X^{**}, there exists $v \in V_X$ such that $C(v) = \overline{v}^{**}$. For this $v \in V_X$, $C'(v) = v^{**}$. Indeed, for each $v^* \in (V_X)^*$, there exists $\overline{v}^* \in V_{X^*}$ such that $\overline{v}^*|_{V_X} = v^*$ by Proposition 2.3. So, we have $v^{**}(v^*) = v^{**}(\overline{v}^*|_{V_X}) = \overline{v}^{**}(\overline{v}^*) = C(v)(\overline{v}^*) = \overline{v}^*(v) = v^*(v) = C'(v)(v^*)$. Hence C' is surjective.

If $v_1 \neq v_2$ in V_X, then $C(v_1) \neq C(v_2)$ in X^{**} since C is an isomorphism. Choose $f \in X^*$ such that $C(v_1)(f) \neq C(v_2)(f)$, i.e, $f(v_1) \neq f(v_2)$. For this $f \in X^*$, $f|_{V_X} \in (V_X)^*$. And $f|_{V_X}(v_1) \neq f|_{V_X}(v_2)$. So, we have $C'(v_1) \neq C'(v_2)$. Hence C' is injective. Therefore C' is an isomorphism. Similarly, we can show that W_X is reflexive. □

THEOREM 2.10. *Let X be a split nals as $X = W_X + V_X$. If V_X and W_X are reflexive, then X is reflexive.*

PROOF. Note that $X^* = W_{X^*} + V_{X^*}$ and $X^{**} = W_{X^{**}} + V_{X^{**}}$. Let $C' : V_X \to (V_X)^{**}$ and $C'' : W_X \to (W_X)^{**}$ be the canonical isomorphism, and let $C : X \to X^{**}$ be the canonical map. We will show that C is bijective. Let $v^{**} \in V_{X^{**}}$. By Proposition 1.3, we have $v^{**}(x^*) = v^{**}(v^*)$ for each $x^* = v^* + w^* \in X^*$, $v^* \in V_{X^*}$, $w^* \in W_{X^*}$. And $v^{**}|V_{X^*} \in (V_{X^*})^*$. Recall that $T : V_{X^*} \to (V_X)^*$, $T(v^*) = v^*|_{V_X}$ $(v^* \in V_{X^*})$, is an isomorphism. Define a functional

$$\overline{v}^{**} : (V_X)^* \to \mathbb{R}$$

by $\overline{v}^{**}(v^*|_{V_X}) = v^{**}(v^*)$, for each $v^*|_{V_X} \in (V_X)^*$. Then $\overline{v}^{**} \in (V_X)^{**}$. Since C' is an isomorphism of V_X onto $(V_X)^{**}$, there exists $v \in V_X$ such that $C'(v) = \overline{v}^{**}$. For this $v \in V_X$, $C(v) = v^{**}$. Indeed, $v^{**}(x^*) = v^{**}(v^*)$

$= \overline{v}^{**}(v^*|_{V_X}) = C'(v)(v^*|_{V_X}) = v^*|_{V_X}(v) = v^*(v) = x^*(v) = C(v)(x^*)$ for each $x^* = v^* + w^* \in X^*$ with $v^* \in V_{X^*}$, $w^* \in W_{X^*}$.

Similarly, for each $w^{**} \in W_{X^{**}}$, there exists $w \in W_X$ such that $C(w) = w^{**}$. Hence, for each $x^{**} = v^{**} + w^{**} \in X^{**}$ with $v^{**} \in V_{X^{**}}$, $w^{**} \in W_{X^{**}}$, there exists $x = v + w \in X$ with $v \in V_X$, $w \in W_X$ such that

$$C(x) = C(v) + C(w) = v^{**} + w^{**} = x^{**}.$$

Hence C is surjective.

If $w_1 \neq w_2$ in W_X, then $C''(w_1) \neq C''(w_2)$ in $(W_X)^{**}$ since C'' is an isomorphism. Choose $f \in (W_X)^*$ such that $C''(w_1)(f) \neq C''(w_2)(f)$, i.e., $f(w_1) \neq f(w_2)$. By Proposition 2.2, there exists $f_1 \in X^*$ such that $f_1|_{W_X} = f$ and $\|f_1\| = \|f\|$. For this f_1, we have $C(w_1)(f_1) \neq C(w_2)(f_2)$ since $f_1(w_1) \neq f_1(w_2)$. Hence $C(w_1) \neq C(w_2)$. Similarly, $C(v_1) \neq C(v_2)$ for $v_1 \neq v_2$ in V_X. Therefore C is injective since C is a linear operator. □

From Theorem 2.9 and Theorem 2.10, we have the following theorem:

THEOREM 2.11. *Let X be a split nals as $X = W_X + V_X$. Then X is reflexive if and only if V_X and W_X are reflexive.*

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