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IDENTITIES INVOLVING THE HURWITZ AND RIEMANN ZETA FUNCTIONS

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ABSTRACT. We give some interesting identities involving the Hurwitz and Riemann zeta functions.

We give some interesting identities involving the Hurwitz and Riemann zeta functions using the theories of the Gamma function and the Hurwitz (and also the Riemann) zeta functions.

The Gamma function Γ can be defined by

$$\{\Gamma(z+1)\}^{-1} = e^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}, \tag{1}$$

where γ denotes the Euler-Mascheroni constant given by

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) \cong 0.577\,215\,664\ldots.. \tag{2}$$

There is a well-known relationship between the Gamma function and the circular functions (see [2, p.3], Equations (1) and (6)):

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}. \tag{3}$$

The ψ-function is defined as the logarithmic derivative of simple Gamma function, i.e.,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{4}$$

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which is often called the Digamma function [2, pp. 15-16].

From (3) and (4) we find

$$\psi(1+z)-\psi(1-z)=-\frac{d}{dz}\ln\left[\frac{\sin(\pi z)}{\pi z}\right], \tag{5}$$

The Riemann zeta function $\zeta(s)$ and the generalized (or Hurwitz) zeta function $\zeta(s,a)$ are defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1-2^{-s}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s} \qquad \mathrm{Re}(s)>1, \tag{6}$$

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \qquad \mathrm{Re}(s)>1 \qquad a\neq 0,\ -1,\ -2,\ \ldots. \tag{7}$$

It should be remarked in passing that both $\zeta(s)$ and $\zeta(s,a)$ are meromorphic functions everywhere in the complex s-plane except for a simple pole at $s=1$ with their residue 1.

The special values of $\zeta(s,a)$ leads to an interesting class of functions known as Bernoulli polynomials (see [1, p. 264]): For every integer $n\geq 0$ we have

$$\zeta(-n,a)=-\frac{B_{n+1}(a)}{n+1}, \tag{8}$$

where $B_n(a)$ are nth Bernoulli polynomials.

We can easily deduce the following (see Whittaker and Watson [4, p. 276]; see also Gradshteyn and Ryzhik ([3, p. 1074], Entry 9.532):

$$\sum_{k=2}^{\infty}(-1)^k\zeta(k,\alpha)t^{k-1}=\psi(t+\alpha)-\psi(\alpha) \qquad |t|<|\alpha|, \tag{9}$$

which yields

$$\sum_{k=2}^{\infty}\frac{(-1)^k a^{k+\beta}}{k+\beta}\zeta(k,\alpha)=\int_0^a t^{\beta}[\psi(t+\alpha)-\psi(\alpha)]dt \qquad \beta\geq 0. \tag{10}$$

In particular, setting $\alpha = 1$ and $\beta = 0$ in (10) yields

$$\sum_{k=2}^{\infty} \frac{(-1)^k a^k}{k} \zeta(k) = \int_0^a [\psi(t+1) - \psi(1)] dt \qquad |a| < 1. \tag{11}$$

For an arbitrary (real or complex) parameter λ, define a binomial coefficient by

$$\binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} \quad (n = 1, 2, 3, \dots). \tag{12}$$

By using elementary property of the Gamma function Γ, we have

$$\binom{-\lambda}{n} = (-1)^n \frac{\Gamma(\lambda+n)}{n!\Gamma(\lambda)} = (-1)^n \binom{\lambda+n-1}{n}. \tag{13}$$

Using (13) we can obtain

$$\begin{aligned} \zeta(s, 1+a) &= \sum_{n=1}^{\infty} \frac{1}{(n+a)^s} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(s+k) a^k}{k!\Gamma(s)} \zeta(s+k). \end{aligned} \tag{14}$$

Now we can divide (14) into two parts which are convergent one and divergent one:

$$\zeta(s, 1+a) = \zeta(s) - s\zeta(s+1)a + D(s, 1+a), \tag{15}$$

where

$$\begin{aligned} D(s, 1+a) &= \sum_{n=1}^{\infty} \left[\frac{1}{(n+a)^s} - \frac{1}{n^s} + \frac{sa}{n^{s+1}} \right] \\ &= \sum_{k=2}^{\infty} \frac{(-1)^k \Gamma(s+k)}{k!\Gamma(s)} \zeta(s+k) a^k, \end{aligned} \tag{16}$$

which gives the analytic continuation for $\zeta(s, 1+a)$ to $s = 0$.

We therefore deduce

$$\zeta(0, 1+a) = \zeta(0) - a \tag{17}$$

and

$$\zeta'(0, 1+a) = -\frac{1}{2}\ln(2\pi) - \gamma a + D'(0, 1+a), \tag{18}$$

where we use $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$ and $s\zeta(s+1) = 1 + \gamma s + \cdots$.

Differentiating the right side of the first equation of (16) with respect to s term by term and evaluating at $s = 0$ in the resulting equation yields

$$\begin{aligned} D'(0, 1+a) &= -\sum_{n=1}^{\infty}\left[\ln\left(1 + \frac{a}{n}\right) - \frac{a}{n}\right] \\ &= \sum_{k=2}^{\infty}\frac{(-1)^k a^k}{k}\zeta(k). \end{aligned} \tag{19}$$

Thus using (11) we obtain

$$D'(0, 1+a) = \int_0^a [\psi(t+1) - \psi(1)]dt. \tag{20}$$

We therefore have

$$\zeta'(0, 1+a) = -\frac{1}{2}\ln(2\pi) + \ln\Gamma(1+a), \tag{21}$$

where we use $\psi(1) = -\gamma$.

The symmetrised sum of this function on using (3) can be expressed as the following:

$$\zeta'(0, 1+a) + \zeta'(0, 1-a) = -\ln\left(\frac{2\sin(\pi a)}{a}\right). \tag{22}$$

Differentiating each side of the relation $\zeta(s, 1+a) = \zeta(s,a) - a^{-s}$ with respect to s and evaluating at $s = 0$ yields

$$\zeta'(0, 1+a) = \zeta'(0,a) + \ln a. \tag{23}$$

Combining (22) and (23), we obtain

$$\zeta'(0,a) + \zeta'(0, 1-a) = -\ln[2\sin(\pi a)]. \tag{24}$$

For any positive integer p decomposing over conjugacy classes mod p, we obtain

$$\zeta(s,a) = p^{-s} \sum_{j=0}^{p-1} \zeta\left(s, \frac{j+a}{p}\right). \tag{25}$$

Setting $s = -n$ in (25), where n is a nonnegative integer, and considering (8), we have

$$B_{n+1}(a) = p^n \sum_{j=0}^{p-1} B_{n+1}\left(\frac{j+a}{p}\right). \tag{26}$$

Differentiating each side of (25) with respect to s and letting $s = 0$ in the resulting equation, and recalling (21), (23), $\Gamma(a+1) = a\Gamma(a)$ and $\zeta(0,a) = 1/2 - a$, we obtain a multiplication formula for Γ

$$\prod_{j=0}^{p-1} \Gamma\left(\frac{j+a}{p}\right) = (2\pi)^{\frac{p-1}{2}} p^{\frac{1}{2}-a} \Gamma(a), \tag{27}$$

which, in fact, implies the Gauss's multiplication formula for Γ (see [4, p. 240]).

We also have, for any positive integer p,

$$\zeta(s) = p^{-s} \sum_{j=0}^{p-1} \zeta\left(s, 1 - \frac{j}{p}\right). \tag{28}$$

Differentiating each side of (28) with respect to s and letting $s = 0$ in the resulting equation yields

$$\sum_{j=0}^{p-1} \zeta'\left(0, 1 - \frac{j}{p}\right) = \zeta'(0) + \zeta(0) \ln p. \tag{29}$$

For $p = 2r + 1$, (29) gives

$$\begin{aligned}
-\frac{1}{2}\ln p &= \sum_{j=1}^{2r} \zeta'\left(0, 1 - \frac{j}{p}\right) \\
&= \sum_{\ell=1}^{r} \left\{ \zeta'\left(0, 1 - \frac{2\ell}{p}\right) + \zeta'\left(0, 1 - \frac{2\ell - 1}{p}\right) \right\} \\
&= \sum_{\ell=1}^{r} \left\{ \zeta'\left(0, \frac{2\ell}{p}\right) + \zeta'\left(0, 1 - \frac{2\ell}{p}\right) \right\} \\
&= -\sum_{\ell=1}^{r} \ln\left[2 \sin\left(\frac{2\ell\pi}{p}\right)\right]
\end{aligned}$$

where we use $\{2\ell \mid \ell = 1, 2, \ldots, r\} = \{2r + 2 - 2\ell \mid \ell = 1, 2, \ldots, r\}$ for the third equality and (24) for the fourth one.

We therefore have

$$\sum_{\ell=1}^{(p-1)/2} \ln\left[2 \sin\left(\frac{2\ell\pi}{p}\right)\right] = \frac{1}{2} \ln p, \tag{30}$$

where p is an odd positive integer.

Similarly we have

$$\sum_{\ell=0}^{(p-3)/2} \ln\left[2 \sin\left(\frac{(2\ell + 1)\pi}{p}\right)\right] = \frac{1}{2} \ln p, \tag{31}$$

where p is an odd positive integer.

For $p = 2r$ even using (29) we have

$$\zeta'(0) + \zeta(0)\ln(2r) = \sum_{\ell=0}^{r-1}\left[\zeta'\left(0, 1 - \frac{\ell}{r}\right) + \zeta'\left(0, 1 - \frac{2\ell+1}{2r}\right)\right]$$
$$= \zeta'(0) + \zeta(0)\ln r + \sum_{\ell=0}^{r-1}\zeta'\left(0, 1 - \frac{2\ell+1}{2r}\right)$$

and therefore we find

$$\sum_{\ell=0}^{r-1}\zeta'\left(0, 1 - \frac{2\ell+1}{2r}\right) = -\frac{1}{2}\ln 2. \tag{32}$$

For $p = 4t + 2\,(= 2r)$ using (32), we have

$$-\frac{1}{2}\ln 2 = \sum_{\ell=0}^{2t}\zeta'\left(0, 1 - \frac{2\ell+1}{2r}\right)$$
$$= \sum_{\ell=0}^{t-1}\zeta'\left(0, 1 - \frac{2\ell+1}{p}\right) + \sum_{\ell=t+1}^{2t}\zeta'\left(0, 1 - \frac{2\ell+1}{p}\right) + \zeta'\left(0, \frac{1}{2}\right).$$

Since $\zeta'(0, 1/2) = -\frac{1}{2}\ln 2$, we obtain

$$0 = \sum_{\ell=0}^{t-1}\zeta'\left(0, 1 - \frac{2\ell+1}{p}\right) + \sum_{\ell=t+1}^{2t}\zeta'\left(0, 1 - \frac{2\ell+1}{p}\right)$$
$$= \sum_{\ell=0}^{t-1}\left\{\zeta'\left(0, 1 - \frac{2\ell+1}{p}\right) + \zeta'\left(0, \frac{2\ell+1}{p}\right)\right\}$$
$$= -\sum_{\ell=0}^{t-1}\ln\left[2\sin\left(\frac{(2\ell+1)\pi}{p}\right)\right],$$

where we use (24) and

$$\{2\ell+1 \mid \ell = 0, 1, \dots, t-1\} = \{4t - 2\ell + 1 \mid \ell = t+1, t+2, \dots, 2t\}.$$

We thus have

$$\sum_{\ell=0}^{(p-6)/4}\ln\left[2\sin\left(\frac{(2\ell+1)\pi}{p}\right)\right] = 0, \tag{33}$$

where p is an even positive integer of the form $4t + 2$.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, 1976.
[2] H. Bateman, *Higher Transcendental Functions*, McGraw-Hill Book Company, 1953.
[3] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Corrected and Enlarged Ed. prepared by A. Jeffrey), Academic Press, 1980.
[4] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (4th Ed.), Cambridge University Press, 1963.

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