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TRANSNORMAL SYSTEMS ON R_1^{n+1} *

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ABSTRACT. In this paper, we study on a classification of hypersurfaces given by transnormal functions on R_1^{n+1}. If M is a level set of a transnormal function on R_1^{n+1}, then it is one of a hyperplane, a cylinder around k-plane, a pseudo-sphere and a pseudo-hyperbolic space.

1. Introduction

The study of isoparametric hypersurfaces was initiated by E. Cartan. Isoparametric hypersurfaces can be expressed as regular level sets of isoparametric functions. On the other hand, transnormal function is a weak notion of isoparametric function. These two notions in Riemannian space forms were studied by many authors such as Nomizu, Münzner, Terng, Wang, West. And there were several attempts to generalize these concepts to other Riemannian manifolds. In 1980's, K. Nomizu ([6]) and J. Hahn ([3]) generalized the notion of isoparametric functions and hypersurfaces to semi-Riemannian spaces. The classification of isoparametric systems on R^{n+1} is quite simple. But on R_1^{n+1} we have found some interesting restrictions.

A nonconstant real-valued function f on a space of constant curvature is called *transnormal* if $|f|^2$ is a function of f. J. Bolton, Q. Wang and A. West investigated transnormal systems and functions for Riemannian case([1], [2], [7]). For the semi-Riemannian manifolds, nobody tried to

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investigate transnormal systems or transnormal functions. These objects would have interesting properties as isoparametric ones.

Let $<,>$ denote the inner prodouct on R_1^{n+1} of signature $(1, n)$ given by

$$< v, w >= -v_0 w_0 + v_1 w_1 + \cdots + v_n w_n.$$

2. Transnormal functions

In this section, we introduce some typical transnormal functions. Actually these examples represent all of those functions. Let us start with the definition of transnormal function.

DEFINITION. A non-constant real-valued function $f : R_1^{n+1} \longrightarrow R$ is said to be *transnormal*, provided there is a smooth function b such that $<\nabla f, \nabla f>= b(f)$. Here ∇f denotes the gradient of f.

We are going to prove the following

THEOREM. *Let M be a regular level set of a transnormal function on R_1^{n+1}. Then M is essentially one of the following 3 types of hypersurfaces.*

(i) Hyperplanes,

(ii) Cylinders around a k-plane,

(iii) Pseudo-spheres and pseudo-hyperbolic spaces(or their lower dimensional cylinders).

Before we prove the theorem, let us consider some examples

EXAMPLE 1. Let $f : R_1^{n+1} \longrightarrow R$ be given by $f(x_0, \cdots, x_n) = x_i$. Then f is isoparametric and has no critical values. The transnormal system given by f is a family of hyperplanes orthogonal (in Euclidean sense) to the i-th coordinate axis.

EXAMPLE 2. Let $f : R_1^{n+1} \longrightarrow R$ be given by $f(x_0, \cdots, x_n) = x_{i_1}^2 + \cdots + x_{i_k}^2$, where $1 \leq i_1 < \cdots < i_k \leq n$. Then f is isoparametric and 0 is the only critical value. The focal submanifold $f^{-1}(0)$ is an $(n-k+1)$-dimensional plane $x_{i_1} + \cdots + x_{i_k} = 0$. The transnormal system given by f is a family of cylinders around $f^{-1}(0)$.

EXAMPLE 3. Let $f : R_1^{n+1} \longrightarrow R$ be given by $f(x_0, \cdots, x_n) = -x_0^2 + x_{i_1}^2 + \cdots + x_{i_k}^2$, where $1 \leq i_1 < \cdots < i_k \leq n$. Then f is isoparametric. In this case, the transnormal system given by f has, so-called, a null-hypersurface N. Its focal set is F given in the following case (ii). This transnormal system is a family of parallel pseudo-spheres and parallel pseudo-hyperbolic spaces (or their cylinders).

Let $f : R_1^{n+1} \longrightarrow R$ be a transnormal function and b is a function such that $<\nabla f, \nabla f>= b(f)$. Then R_1^{n+1} has a partition $B^+ \cup B^- \cup B^0 \cup F$, where B^+ and B^- are the unions of regular level hypersurfaces with positive and negative values of b, B^0 is the subset of $b^{-1}(0)$ with nonzero gradient and F is the focal set of f, i.e., the set of gradient zero. To classify transnormal systems given by f, we consider the following two possible cases.

Case (i) f is regular.

Then the image of b is contained in either $(0, \infty)$, or $(-\infty, 0)$. If ∇f is not a line, the union of level sets of f would be a proper subset of R_1^{n+1}. It's a contradiction. Thus ∇f is a line. Since f has no focal points, the resulting transnormal system consists of parallel hyperplanes. This is a transnormal system which can be obtained from Example 1 by using suitable rotation and translation.

Case (ii) f has a focal point.

If b is nonnegative or nonpositive, i.e., $0 \in$ (image of b) $\subset [0, \infty)$, (or $(-\infty, 0]$). If ∇f is parallel to the x_0-axis, then the resulting transnormal system would be non-singular as in case (i). But 0 is in the image of b. So ∇f is orthogonal to the x_0-axis. Then f is independent of the variable x_0. Now this means every level set is a kind of cylinder and the focal set F is a k-plane parallel to the x_0-axis. But on each hyperplane $x_0 = t$ the function f would be isoparametric in Riemannian sense, and hence each level set is a sphere or cylinder on a sphere. So the resulting transnormal system can be obtained from Example 2 by using suitable rotation and translation.

Next if b satisfies $(-\epsilon, \epsilon)$ is contained in the image of b for some $\epsilon > 0$. We can choose a regular level set M with a normal vector v_x at some point $x \in M$ such that $< v, v >$ is negative. Then b is negative on M. Along the normal line l_x at x, we get parallel hypersurfaces M_α. Note that b is positive on every M_α. It is easy to see there is only one critical point on l_x, say y. Then there is a null-hypersurface at y such that the M_α's lie in components of the x_0-axis direction. Thus the focal set F is a point or a k-plane with $0 < k < n$. Thus the resulting transnormal system can be obtained from Example 3 by using suitable rotation and translation. Thus we proved the theorem.

For the cases of codimensions > 1 would be more interesting and the result of this paper can be applied to those cases. Also for R_p^{n+1}, the problem would be much difficult if $p > 1$.

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