Comm. Korean Math. Soc. **12** (1997), No. 1, pp. 101–108

CONSTANT SCALAR CURVATURE ON OPEN MANIFOLDS WITH FINITE VOLUME

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ABSTRACT. We let (M, g) be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with finite volume and positive scalar curvature. We show the existence of a conformal metric with constant positive scalar curvature on (M, g) by gluing solutions of Yamabe equation on each compact subsets K_i with $\cup K_i = M$.

I. Introduction

In this paper, we study noncompact complete Riemannian manifolds with finite volume which admit a conformal metric with constant positive scalar curvature. The existence of a conformal metric with constant scalar curvature on compact Riemannian manifolds was studied by Yamabe [16], Trudinger [15], Aubin [1] and Schoen [12]. Finally, Schoen [12] showed that every compact Riemannian n-manifold M has a conformal metric of constant scalar curvature. Aviles and McOwen [3], Jin [6], Schoen and Yau [14], Schoen [13], Mazzeo and Smale [11], Kim [7,8], Mazzeo and Pacard [10] and others studied conformal metrics with constant scalar curvature on noncompact open Riemannian manifolds.

Throughout this paper, we let (M, g) be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with finite volume and positive scalar curvature S. Here we look for a conformal deformation of the given metric g, which makes the scalar curvature of the deformed metric a positive constant. Finding a suitable conformal transformation satisfying the above condition turns out to be equivalent to finding a positive

Received August 16, 1996. Revised October 14, 1996.
1991 AMS Subject Classification: 53C21.
Key words and phrases: Scalar curvature, complete manifolds, conformal metric.
Author is supported in part by GARC at SNU and KOSEF 960701020013.

smooth function $u(x)$ satisfying the following partial differential equation

$$\text{(A)} \qquad -\Delta u + \frac{n-2}{4(n-1)} S u = q u^{(n+2)/(n-2)},$$

where q is a positive constant and $\Delta u = \frac{1}{\sqrt{detg_{ij}}} \partial_j (g^{ij} \sqrt{detg_{ij}} \partial_i u)$ in a local coordinate system.

We plan to study equation (A) by using variational methods. To do this we introduce some notation

$$Q(M,g) \equiv \inf_{u \in C_0^\infty(M)} \frac{\int_M |\nabla u|^2 + \frac{n-2}{4(n-1)} S\, u^2 \, dV_g}{\left(\int_M u^{2n/(n-2)} \, dV_g\right)^{(n-2)/n}}.$$

We also use the notation $Q(M)$ for $Q(M,g)$ and $Q(S^n)$ for $Q(S^n, g_0)$, where g_0 is the standard metric.

We state our theorem.

THEOREM 1. *Let (M,g) be a noncompact complete Riemannian manifold with finite volume and positive scalar curvature. Assume that $0 < Q(M) < Q(S^n)$. Then there exists a positive solution u of the partial differential equation (A). Therefore, there exists a conformal metric $(M, u^{4/(n-2)} g)$ with constant positive scalar curvature.*

To study equation (A) using variational method, we need Sobolev Embedding Theorem (see Aubin [2]) on (M,g). But, it requires the boundedness of curvatures and positive injective radius on (M,g), which are not common features of noncompact complete Riemannian manifolds with finite volume. Note that there are no restrictions on the boundedness of curvatures and the injective radius on given conditions of the Theorem 1. When (M,g) is not locally conformally flat, $Q(M) < Q(S^n)$, therefore, (M,g) satisfies given conditions of the Theorem 1 if $Q(M) > 0$.

For compact manifolds, the existence of a solution for (A) was studied by gluing positive solutions u_ϵ of the following equation (B) with subcritical nonlinear exponent

$$\text{(B)} \qquad -\Delta u + \frac{n-2}{4(n-1)} S u = q u^{(n+2)/(n-2)-\epsilon},$$

where ϵ is a small positive constant (see [16,15,1,12]). For the noncompact case, the existence of a positive solution of (B) is not always guaranteed. It is known that there is no positive solution of (B) on (R^n, δ_{ij}), even though there exists a positive solution on each compact subset of R^n (see [4]). In this paper, we show the existence of a positive solution of (A) by gluing solutions of equation (A) on each compact subset K_i where $\cup K_i = M$ and $K_i \subset K_{i+1}$.

2. Proof of Theorem 1

Since we assume that $0 < Q(M) < Q(S^n)$, there exists a sequence of smooth compact domains K_i such that $Q(K_i) < Q(S^n)$ with $K_i \subset K_{i+1}$ satisfying $\cup K_i = M$. By using the work on the Yamabe problem in the compact case (see Trudinger [15], Aubin [1,2] and Schoen [12]) we have a positive smooth solution u_i on each K_i with

$$(1) \qquad -\Delta u_i + \frac{n-2}{4(n-1)} S u_i = q_i u_i^{(n+2)/(n-2)} \text{ on } K_i$$

$u_i = 0$ on ∂K_i and $\int\limits_{K_i} u_i^{\frac{2n}{n-2}} dV_g = 1$, where $q_i \equiv Q(K_i) \to Q(M)$ as $i \to \infty$.

We extend the domain of u_i by defining $u_i = 0$ on the outside of K_i. We use the same notation u_i for this extension. Note that the extension u_i is in $H_1^2(M, g)$, the completion of $C_0^\infty(M)$ with the norm $||u|| \equiv \int\limits_M |\nabla u|^2 + \frac{n-2}{4(n-1)} S u^2 \, dV_g$.

By multiplying (1) by u_i^{1+2b} where $b > 0$ and integrating over K_i, we have

$$(2) \qquad q_i \int\limits_{K_i} u_i^{2n/(n-2)+2b} \, dV_g = \int\limits_{K_i} \frac{1+2b}{(1+b)^2} |\nabla u_i^{1+b}|^2 + C_n S u_i^{2+2b} \, dV_g,$$

where $C_n = \frac{n-2}{4(n-1)}$.

From Hölder's inequality we have

(3)

$$
\begin{aligned}
&\int_{K_i} u_i^{2n/(n-2)+2b}\, dV_g \\
&\leq \Big(\int_{K_i} u_i^{(1+b)2n/(n-2)}\, dV_g\Big)^{\frac{n-2}{n}} \Big(\int_{K_i} u_i^{(2n/(n-2)+2b-(1+b)2)\frac{n}{2}}\, dV_g\Big)^{\frac{2}{n}} \\
&\leq \Big(\int_{K_i} u_i^{(1+b)2n/(n-2)}\, dV_g\Big)^{\frac{n-2}{n}}.
\end{aligned}
$$

By the Sobolev Embedding Theorem on Riemannian manifolds (see Aubin[2]) for u_i in $C_0^\infty(K_i)$, $K_i \subset M$, (note K_i is compact) and (2), for any given $\epsilon > 0$, there exists $C(\epsilon)$ with

(4)

$$
\begin{aligned}
&\Big(\int_{K_i} u_i^{(1+b)2n/(n-2)}\, dV_g\Big)^{\frac{n-2}{n}} \\
&\leq (1+\epsilon)\frac{1}{Q(S^n)}\int_{K_i} |\nabla u_i^{1+b}|^2\, dV_g + C(\epsilon)\int_{K_i} u_i^{2+2b}\, dV_g \\
&\leq (1+\epsilon)\frac{1}{Q(S^n)}\frac{(1+b)^2}{(1+2b)}\Big(\int_{K_i} q_i u_i^{2n/(n-2)+2b} - C_n\int_{K_i} S\, u_i^{2+2b}\, dV_g\Big) \\
&\quad + C(\epsilon)\int_{K_i} u_i^{2+2b}\, dV_g,
\end{aligned}
$$

where C is a general positive constant which does not depend on i.

From (3), we have

$$(5)\qquad \Big(\int_{K_i} u_i^{(1+b)2n/(n-2)}\,dV_g\Big)^{\frac{n-2}{n}}$$
$$\le (1+\epsilon)\frac{1}{Q(S^n)}\frac{(1+b)^2}{(1+2b)}\Big(q_i\Big(\int_{K_i} u_i^{(1+b)2n/(n-2)}\,dV_g\Big)^{\frac{n-2}{n}}$$
$$- C_n\int_{K_i} S\,u_i^{2+2b}\,dV_g\Big) + C(\epsilon)\int_{K_i} u_i^{2+2b}\,dV_g.$$

Since $q_i < c < Q(S^n)$ for some c, we can take $\epsilon > 0$ and $0 < b < \frac{2}{n-2}$ so that

$$(6)\qquad (1+\epsilon)\frac{q_i}{Q(S^n)}\frac{(1+b)^2}{(1+2b)} < 1.$$

Therefore, we have

$$(7)\qquad \Big(1-(1+\epsilon)\frac{q_i}{Q(S^n)}\frac{(1+b)^2}{(1+2b)}\Big)\Big(\int_{K_i} u_i^{(1+b)2n/(n-2)}\,dV_g\Big)^{\frac{n-2}{n}}$$
$$+ C\int_{K_i} S\,u_i^{2+2b}\,dV_g$$
$$\le C(\epsilon)\int_{K_i} u_i^{2+2b}\,dV_g.$$

By using Hölder's inequality, we have

$$\int_{K_i} u_i^{2+2b}\,dV_g \le \Big(\int_{K_i} u_i^{2n/(n-2)}\,dV_g\Big)^{t_1}|K_i|^{1-t_1}$$
$$\le |K_i|^{1-t_1} < 1,$$

where $t_1 = \frac{(1+b)(n-2)}{n}$.

Finally, we have

$$(8)\quad \Big(\int_M u_i^{(1+b)2n/(n-2)}\, dV_g\Big)^{\frac{n-2}{n}} = \Big(\int_{K_i} u_i^{(1+b)2n/(n-2)}\, dV_g\Big)^{\frac{n-2}{n}} \le A_2$$

where A_2 does not depend on i.

Using the standard elliptic regularity theory we can show that u_i is $C^{2,\alpha}$ bounded on each compact subset. We refer details to Gilbarg and Trudinger [5], Lee and Parker [9] or Kim [7]. Therefore, we can find a subsequence $\{u_{i_k}\}$ which converges uniformly to its limit u on each compact subset by the Arzela-Ascoli Theorem.

We do not know whether u is a positive solution or not because u can be identically zero. Let us consider the noncompact complete Riemannian manifolds $M = M_1 \# M_2 \# M_3 \cdots$ with $Q(M_k) = 1 + \frac{1}{k}$. There may exist a minimizing sequence which escapes to infinity. This case happens when the support of u_i escapes to infinity. Therefore, we need to consider the outside of a compact subset. We claim that if M has a finite volume, there is no mass escape to infinity.

Let Q be a compact subset of M. By using (8), we have

$$(9)\quad \int_{M-Q} u_i^{2n/(n-2)}\, dV_g = \Big(\int_{M-Q} u_i^{\frac{2n}{n-2}(1+b)}\, dV_g\Big)^{\frac{1}{1+b}} |M-Q|^{\frac{b}{1+b}}$$
$$\le A_3 |M-Q|^{\frac{b}{1+b}},$$

where A_3 is a constant not depending on i.

Therefore, for any given $\epsilon > 0$, there exists Q with $\int_{M-Q} u_i^{2n/(n-2)}\, dV_g \le \epsilon$ for all u_i. Since u_i uniformly converges to u on a compact subset Q, there exists i_0 such that $|u_i - u| < \epsilon$ for all $i > i_0$. Finally, we have

(10)

$$\begin{aligned}\int_M u^{2n/(n-2)}\, dV_g &= \int_Q u^{2n/(n-2)}\, dV_g + \int_{M-Q} u^{2n/(n-2)}\, dV_g \\ &\geq \int_Q u_i^{2n/(n-2)}\, dV_g - \epsilon \\ &\geq 1 - \int_{M-Q} u_i^{2n/(n-2)}\, dV_g - \epsilon \\ &\geq 1 - 2\epsilon,\end{aligned}$$

because u_i converges uniformly on Q.

We conclude that $\int_M u^{2n/(n-2)}\, dV_g = 1$. In other words, a minimizer for $Q(M,g)$ exists. Finally we have a $H_1^2(M,g)$ solution by the above argument.

The regularity, which is a local property, comes from the result of Trudinger [15].

THEOREM (Trudinger). *Any $H_1^2(M,g)$ solution of the Yamabe equation (A) is smooth.*

The positivity of the solution comes from the Maximum Principle. Finally, we have a positive smooth solution of the Yamabe equation (A).

ACKNOWLEDGMENT. Author thanks to Patricio Aviles for his advice and encouragements on this work.

References

1. T. Aubin, *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976), 269-296.
2. ———, *Nonlinear analysis on manifolds, Monge-Ampere equations*, Springer-Verlag, New York, 1982.
3. P. Aviles and R. McOwen, *Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds*, J. Diff. Geom. **27** (1988), 225-239.

4. A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math **42** (1989), 271-297.
5. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, New York, 1983.
6. Z. Jin, *A counterexample to the Yamabe problem for complete noncompact manifolds*, Partial differential equations, Lect. Notes in Math. 1306, Springer-Verlag, New York, 1988, pp. 93-101.
7. S. Kim, *Scalar curvature on noncompact complete Riemannian manifolds*, Nonlinear Analysis **26** (1996), 1985-1993.
8. ———, *The Yamabe problem and applications on noncompact complete Riemannian manifolds*, Geometriae Dedicata (to appear).
9. L. M. Lee and T.K. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987), 37-91.
10. R. Mazzeo and N. Smale, *Conformally flat metrics of constant positive scalar curvature on subdomains of the sphere*, J. Diff. Geom. **34** (1991), 581-621.
11. R. Mazzeo and F. Pacard, *Constant scalar curvature metrics with isolated singularities*, preprint.
12. R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Diff. Geom. **20** (1984), 479-495.
13. ———, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure Appl. Math. **41** (1988), 317-392.
14. R. Schoen and S. T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), 47-71.
15. N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifold*, Ann. Scuola Norm. Sup. Pisa **3** (1968), 265-274.
16. H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21-37.

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