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HOMOTOPY OF PROJECTIONS IN C^*-ALGEBRAS

SANG OG KIM

ABSTRACT. We show that if a simple C^*-algebra A satisfies certain K_1-group conditions, then two unitarily equivalent projections are homotopic. Also we show that the equivalence of projections determined by a dimension function is a homotopy.

Let p and q be projections in a C^*-algebra A. They are called (Murray-von Neumann) *equivalent,* written $p \sim q$, if there is an operator $x \in A$ with $x^*x = p, xx^* = q$. Such an x is called a *partial isometry from p to q*. The equivalence relation $\sim$ is not the same as unitary equivalence for if X is a proper infinite dimensional subspace of a separable Hilbert space H, then the projection P_X which maps H onto X is equivalent to the identity 1 in $B(H)$ but not unitarily equivalent to 1. In fact, p and q are unitarily equivalent if and only if $p \sim q$ and $1-p \sim 1-q$. Projections p and q are said to be *homotopic* if there is a path in A consisting of projections from p to q. It is well known that p is homotopic to q if and only if there exists a unitary u in the path component of the unitary group of A containing 1 such that $upu^* = q$.

A projection $p \in A$ is said to be *infinite* if it is equivalent to a proper subprojection of itself. A C^*-algebra A is said to be *purely infinite* if every nonzero hereditary C^*-subalgebra of A has an infinite projection. A is said to have *stable rank one* if the set of invertible elements of A is dense in A.

S. Zhang [7] has suggested the study of the following question for a particular projection p in a C^*-algebra A: Is every projection which is unitarily equivalent to p necessarily homotopic to p? It was shown by Effros and Kaminker [4], and Zhang [6] that the answer is yes if A is a purely infinite simple C^*-algebra and by Zhang [5] that the answer is

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yes if A has real rank zero and stable rank one. Also Brown [3] showed that the answer is yes if A has stable rank one.

The purpose of this note is to show that if the simple C^*-algebra A satisfies certain K_1-group conditions, which include the case when A is purely infinite or of stable rank one, then two unitarily equivalent projections are homotopic. Also we show that the equivalence of projections determined by a dimension function is a homotopy.

Let $M_n(A)$ be the C^*-algebra of $n \times n$ matrices over A. We will denote the unitary group of $M_n(A)$ by $U_n(A)$ and the path component of $U_n(A)$ containing the identity by $U_n^0(A)$. In particular we will denote $U_1(A) = U(A)$ and $U_1^0(A) = U_0(A)$.

PROPOSITION 1. *Let A be a unital simple C^*-algebra , p a projection of A such that $K_1(A) = U(A)/U_0(A), K_1(pAp) = U(pAp)/U_0(pAp)$. Then $G_p = \{upu^* | u \in U(A)\}$ is connected, i.e. any projection which is unitarily equivalent to p is homotopic to p.*

PROOF. Since A is simple, by [4] the inclusion $i : pAp \to A$ induces an isomorphism

$$K_1(i) : K_1(pAp) = U(pAp)/U_0(pAp) \to K_1(A) = U(A)/U_0(A).$$

Hence we may identify $K_1(i)$ with the map

$$U(pAp)/U_0(pAp) \to U(A)/U_0(A) \text{ induced by } u' \mapsto u' + (1-p).$$

Hence for any $u \in U(A)$, there exists a unitary $u' \in U(pAp)$ such that $u \simeq u' + (1-p)$, where $u \simeq v$ for two unitaries means that $uv^{-1} \in U_0(A)$. Suppose that $upu^* = f$ for some unitary $u \in U(A)$. Then there exists a unitary $w \in U(pAp)$ such that $w + 1 - p \simeq u$. Then $(w+1-p)u^*$ is in $U_0(A)$ and $(w+1-p)u^* f u(w+1-p)^* = ww^* = p$. This completes the proof. □

There are many C^*-algebras satisfying the hypothesis of Proposition 1. All purely infinite simple C^*-algebras, stable rank one simple C^*-algebras satisfy the hypothesis. So all simple approximately divisible C^*-algebras [2] satisfy it because any simple approximately divisible C^*-algebras are either purely infinite or of stable rank one.

COROLLARY 2. *Let A be a unital simple C^*-algebra and p a projection of A such that the stable rank of $pAp, sr(pAp)$, is finite. If p and q are unitarily equivalent projections in A, then $p \otimes 1$ is homotopic to $q \otimes 1$ in $M_n(A)$ for all sufficiently large n.*

PROOF. The natural map $U_n(pAp)/U_n^0(pAp) \to K_1(pAp)$ is a bijection for $n \geq sr(pAp)+2$ by [1] since $sr(pAp) < \infty$. Also since $sr(A) < \infty$, $K_1(A) = K_1(M_n(A)) = U_n(A)/U_n^0(A)$ for all sufficiently large n. Noting that $M_n(pAp) \cong (p \otimes 1)(A \otimes M_n)(p \otimes 1)$, we can see that $G_{p\otimes 1}$ is connected in $M_n(A)$ by the above Proposition. Hence $q \otimes 1$ is homotopic to $p \otimes 1$ in $M_n(A)$. □

A function d from the set of projections of A is called a *dimension function* if it has the following properties:

(1) $d(p) = d(q)$ if $p \sim q$

(2) $d(p+q) = d(p) + d(q)$ if $p \perp q$

It is well known that every factor has a dimension function d, which is unique up to normalization, such that $p \sim q$ if and only if $d(p) = d(q)$. Also a finite factor has a unique trace state τ, and $d(p) = \tau(q)$ for every projection p. The following proposition shows that the equivalence of projections determined by a dimension function is a homotopy.

Recall that a C^*-algebra A is said to have *real rank zero* if the set of self-adjoint invertible elements of A is dense in the set of self-adjoint elements of A.

PROPOSITION 3. *Let A be a unital C^*-algebra with real rank zero and τ a trace state of A such that any two projections of the same traces are equivalent. Then $G_p = \{upu^* | u \in U(M_n(A))\}$ is connected for any n and projection p in $M_n(A)$.*

PROOF. Assume that $q \in G_p$ in $M_n(A)$. Since $RR(M_n(A)) = 0$, we have that p is homotopic to $q_1 + q_2$, where $q_1 \leq q$ and $q_2 \leq 1 - q$ by [5, Theorem 3.2]. Then $\tau(q - q_1) = \tau(q_2)$, whence $q - q_1 \sim q_2$. Since $q - q_1 \perp q_2$ and equivalent orthogonal projections are homotopic, working in $(1-q_1)M_n(A)(1-q_1)$, we can find a unitary v in $U_0((1-q_1)M_n(A)(1-q_1))$ such that $vq_2v^* = q - q_1$. Set $u = q_1 + v$. Then u is a unitary in $U_0(M_n(A))$ and

$$u(q_1+q_2)u^* = (q_1+v)(q_1+q_2)(q_1+v)^* = q_1 + vq_2v^* = q_1 + (q - q_1) = q.$$

Therefore q is homotopic to $q_1 + q_2$. Since $q_1 + q_2$ is homotopic to p, the proof is completed. $\square$

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Department of Mathematics
Hallym University
Chuncheon 200-702, Korea