Comm. Korean Math. Soc. **12** (1997), No. 1, pp. 69–73

APPROXIMATE JORDAN MAPPINGS ON NONCOMMUTATIVE BANACH ALGEBRAS

YOUNG WHAN LEE AND GWANG HUI KIM

ABSTRACT. We show that if T is an $\varepsilon-$approximate Jordan functional such that $T(a) = 0$ implies $T(a^2) = 0$ $(a \in A)$ then T is continuous and $\|T\| \leq 1 + \varepsilon$. Also we prove that every $\varepsilon-$near Jordan mapping is an $g(\varepsilon)-$approximate Jordan mapping where $g(\varepsilon) \to 0$ as $\varepsilon \to 0$ and for every $\varepsilon > 0$ there is an integer m such that if T is an $\frac{\varepsilon}{m}-$approximate Jordan mapping on a finite dimensional Banach algebra then T is an $\varepsilon-$near Jordan mapping.

1. Introduction

A Jordan functional on a Banach algebra A is a nonzero linear functional ϕ such that $\phi(a^2) = \phi(a)^2$. Every Jordan functional ϕ on A is multiplicative[1]. A linear map T from A into a Banach algebra B is an $\varepsilon-$homomorphism if for every a, b in A

$$\|T(ab) - T(a)T(b)\| \leq \varepsilon\|a\|\|b\|.$$

In [2, Proposition 5.5], Jarosz proved that every $\varepsilon-$homomorphism from a Banach algebra into a continuous function space $C(X)$ is necessarily continuous. In[3], Johnson proved that other theorems about continuity of homomorphisms extend to generalized homomorphisms.

DEFINITION 1. A linear mapping T from a Banach algebra A into a Banach algebra B is an $\varepsilon-$approximate Jordan mapping if for all a in A

$$\|T(a^2) - T(a)^2\| \leq \varepsilon\|a\|^2.$$

If B is the complex field, then T is called the $\varepsilon-$approximate Jordan functional. Note that if T is an approximate Jordan functional on a commutative Banach algebra, then T is an approximate homomorphism.

Received August 31, 1996. Revised October 19, 1996.
1991 AMS Subject Classification: 46H40, 46H99.
Key words and phrases: Approximate Jordan mapping, Banach algebra.

THEOREM 2. *Let T be an ε-approximate Jordan functional on Banach algebra A such that $T(a) = 0$ implies $T(a^2) = 0$ for all a in A. Then $\|T\| \leq 1 + \varepsilon$.*

PROOF. Let A be a Banach algebra and T a generalized Jordan functional on A. If A does not posses a unit, then we can extend T to $A \oplus \{\lambda e\}$ by putting $T(a + \lambda e) = T(a) + \lambda$ and the extended T is still a generalized Jordan functional. Thus without loss of generality we may assume that A has a unit.

Suppose that T is discontinuous. Then $Ker(T)$ is a dense subset of A . Since the unit element 1 is in the closure of $Ker(T)$, we can choose $c \in Ker(T)$ such that $\|c - 1\| < \frac{1}{3}$. Then c is invertible, and $c^{-1} = 1 + \sum_{n=1}^{\infty}(1-c)^n$. And so $\|c^{-1}\| \leq \frac{1}{1-\|c-1\|} \leq \frac{3}{2}$. Let $b = \frac{c}{\|c\|} \in Ker(T)$. Then $b^{-1} = \|c\|c^{-1}$ and $\|b^{-1}\| \leq 2$. Put $\mid T(b^{-1}) \mid = \alpha$ and $\mid T(b^{-2}) \mid = \beta$. Note that for every x, y in A,

$$\begin{aligned} \mid T\left((x+y)^2\right) - (T(x+y))^2 \mid &= \mid T(xy + yx) - 2T(x)T(y) \mid \\ &\leq \varepsilon(\|x\|^2 + 2\|x\|\|y\| + \|y\|^2) \end{aligned}$$

If b^{-1} is not in $Ker(T)$, then for every a in A with $\|a\| = 1$,

$$\begin{aligned} \mid T(a) \mid &= \frac{1}{2\alpha} \mid 2T(a)T(b^{-1}) \mid \\ &\leq \frac{1}{2\alpha}(\mid 2T(a)T(b^{-1}) - T(ab^{-1} + b^{-1}a) \mid \\ &+ \mid T(bb^{-1}ab^{-1} + b^{-1}ab^{-1}b) - 2T(b^{-1}ab^{-1})T(b) \mid) \\ &\leq \frac{17}{\alpha}\varepsilon. \end{aligned}$$

Thus T is bounded and it is a contradiction. Therefore b^{-1} is in $Ker(T)$.

By assumption, b^{-2} is in $Ker(T)$. Then for every a in A with $\|a\| = 1$,

$$\begin{aligned}| T(a) | \leq & \frac{1}{2}(| T(a + b^{-1}ab) | + | T(a + bab^{-1}) | \\ & + | T(b^{-1}ab + bab^{-1}) |) \\ = & \frac{1}{2}(| T(a + b^{-1}ab) - 2T(b^{-1}a)T(b) | \\ & + | T(a + bab^{-1}) - 2T(ab^{-1})T(b) | \\ & + | T(b^{-1}ab + bab^{-1}) - 2T(bab)T(b^{-2}) |) \\ \leq & \frac{39}{2}\varepsilon.\end{aligned}$$

Thus T is continuous.

Now let x be any element of A with $\|x\| \leq 1$. We have

$$| T(x^2) - T(x)^2 | \leq \varepsilon.$$

Thus

$$| T(x) |^2 - \varepsilon \leq | T(x^2) | \leq \|T\|$$

and so

$$\|T\|^2 - \varepsilon \leq \|T\|.$$

This proves $\|T\| \leq 1 + \varepsilon$.

COROLLARY 3. *Let X be a compact Hausdorff space and $C(X)$ the set of all continuous complex valued functions. If T is an $\varepsilon-$ approximate Jordan mapping from a Banach algebra A into $C(X)$ such that $T(a)(x) = 0$ implies $T(a^2)(x) = 0$ $(a \in A,\ x \in X)$, then T is continuous and $\|T\| \leq 1 + \varepsilon$.*

PROOF. For every x in X we define a linear mapping $T_x : A \to \mathbb{C}$ by $T_x(a) = T(a)(x)$. Then for every a in A,

$$| T_x(a^2) - (T_x(a))^2 | \leq \|T(a^2) - (T(a))^2\| \leq \varepsilon\|a\|^2$$

and if $T_x(a) = 0$ then $T_x(a^2) = 0$.

By Theorem 2, $\|T_x\| \leq 1 + \varepsilon$. Thus

$$\|T_a\| = \sup_{x \in X} \|T(a)(x)\| = \sup_{x \in X} \|T_x(a)\| \leq (1 + \varepsilon)\|x\|$$

and so $\|T\| \leq 1 + \varepsilon$.

DEFINITION 4. A continuous linear map T between Banach algebras is an $\varepsilon-$near Jordan mapping if $\|T - J\| < \varepsilon$ for some Jordan mapping J.

PROPOSITION 5. *Every $\varepsilon-$near Jordan mapping between Banach algebras is a $g(\varepsilon)-$approximate Jordan mapping where $g(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

PROOF. If T is an $\varepsilon-$near Jordan mapping from a Banach algebra A into a Banach algebra B. Then there exists a Jordan mapping J such that $\|T - J\| < \varepsilon$. Then for every a in A

$$\begin{aligned}\|T(a^2)-(Ta)^2\| &\leq \|T(a^2) - J(a^2)\| + \|(Ta)^2 - (Ja)^2\| \\ &\leq \varepsilon\|a\|^2 + \|T(a) - J(a)\|\|T(a)\| + \|J(a)\|\|T(a) - J(a)\| \\ &\leq (\varepsilon + \varepsilon\|T\| + \varepsilon\|J\|)\|a\|^2.\end{aligned}$$

THEOREM 6.. *For every $\varepsilon > 0$ there exists an integer m such that every $\frac{\varepsilon}{m}-$approximate Jordan mapping on a finite dimensional Banach algebra is an $\varepsilon-$near Jordan mapping.*

PROOF. Let $J(A)$ be the set of all Jordan mapping on a finite dimensional Banach algebra A, $BL(A)$ the set of all bounded linear mappings on A, and let for each J in $BL(A)$

$$\begin{aligned}j(J) &= \inf\{\|T - J\| : T \in J(A)\}, \\ C &= \{J \in BL(A) : j(J) \geq \varepsilon\},\end{aligned}$$

and

$$G_n = \{J \in BL(A) : \sup_{\|a\| \leq 1} \|J(a^2) - J(a)^2\| > \frac{\varepsilon}{n}\}.$$

Since C is a closed subset of a finite dimensional space $BL(A)$, C is compact. Suppose that there is a sequence $\{J_n\}$ in $G_n{}^c$ such that $J_n \to J$. Let $\varepsilon' > 0$ be given. We can choose n such that $\|J - J_n\| < \frac{\varepsilon'}{2+\|J\|+\|J_n\|}$. Then

$$\begin{aligned}
&\sup_{\|a\|\le 1} \|J(a^2) - J(a)^2\| \\
&\quad \le \sup_{\|a\|\le 1} (\|J - J_n\|\|a\|^2 + \|J(a)\|\|J - J_n\|\|a\| + \|J_n(a)\|\|J \\
&\qquad - J_n\|\|a\| + \|J_n(a^2) - J_n(a)^2\|) \\
&\quad \le (1 + \|J\| + \|J_n\|)\|J - J_n\| + \sup_{\|a\|\le 1} \|J_n(a^2) - J_n(a)^2\| \\
&\quad < \varepsilon' + n\varepsilon
\end{aligned}$$

Since ε' was arbitrary, $\sup_{\|a\|\le 1} \|J(a^2) - J(a)^2\| \le n\varepsilon$ and so $J \in {G_n}^c$. Therefore G_n is open. Note that

$$C \subset BL(A) \setminus J(A) \subset \bigcup_{n=1}^{\infty} G_n.$$

Since C is compact there is m such that $C \subset G_m$. If $T \in G_m^c$ then $T \in C^c$. That is, if T is an $\frac{\varepsilon}{m}$−approximate Jordan mapping then T is an ε−near Jordan mapping.

References

1. F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin, 1973.
2. K. Jarosz, *Perturbations of Banach algebra*, Springer-Verlag, Berlin, 1980.
3. B. E. Johnson, *Continuity of generalised homomorphisms*, Bull London Math. Soc. **19** (1887), 67-71.

Young Whan Lee
Department of Mathematics
Taejon University
Taejon 300-716, Korea

Gwang Hui Kim
Department of Mathematics
kangnam University
Yongin 449-702, Korea