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AUTOMORPHISMS OF LOTKA-VOLTERRA ALGEBRAS[1)]

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ABSTRACT. The purpose of this paper is to give a characterization of automorphisms of the weighted Lotka-Volterra algebra (A, ω) at idempotent elements and to offer a condition that (A, ω) becomes a Jordan algebra.

1. Introduction and Preliminaries

Let K be a field A a commutative, not necessarily associative K-algebra. Recall that an algebra A over K is *baric* if it admits a nontrivial algebra homomorphism $\omega : A \longrightarrow K$, which is equivalent to say that, if there exists a surjective homomorphism $\omega : A \longrightarrow K$. The homomorphism ω is called *the weight function or weight homomorphism* Suppose that A is finite dimensional and $B = \{e_1, e_2, \cdots, e_n\}$ is a basis of A over K.

If (a_{ij}) is an anti-symmetric matrix with n rows and n columns where the entries a_{ij} are in the field K of characteristic not 2, then we can associate to this matrix a commutative K-algebra A of dimension n with the multiplication

$$e_i e_j = \Big(\frac{1}{2} + a_{ij}\Big)e_i + \Big(\frac{1}{2} + a_{ji}\Big)e_j \quad (i, j = 1, 2, \cdots, n)$$

relative to the basis $B = \{e_1, e_2, \cdots, e_n\}$. From the definition of multiplication, it can be easily seen that $e_i e_j = e_j e_i$, ${e_i}^2 = e_i$ and $e_i(e_j e_k) \neq$

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$(e_i e_j)e_k$ for $i, j, k = 1, 2, \cdots, n$. Such a commutative, nonassociative K-algebra A is called *Lotka-Volterra algebra associated to the matrix* (a_{ij}). It is not difficult to show that the K-linear mapping $\omega : A \longrightarrow K$ defined by $\omega(e_i) = 1$ $(i = 1, 2, \cdots, n)$ is a weight function of A and the baric algebra (A, ω) is a Lotka-Volterra algebra. We call also that (A, ω) is *a Lotka-Volterra algebra associated to the matrix* (a_{ij}). Under the multiplication of the Lotka-Volterra algebra (A, ω), we can easily see that the following holds.

PROPOSITION 1.1. *For* $x = \sum_{i=1}^{n} \lambda_i e_i$ *and* $y = \sum_{i=}^{n} \mu_i e_i (\lambda_i, \mu_i, \in K)$ *in* (A, ω), *we have* $xy = \frac{1}{2}(\omega(x)y + \omega(y)x) + \sum_{i=1}^{n} (\lambda_i \omega_i(y) + \mu_i \omega_i(x)) e_i$, *where* $\omega_i : A \longrightarrow K$ *is defined by the* K*-linear mapping* $e_i \mapsto a_{ij}$ $(i, j = 1, 2, \cdots, n)$.

To study the idempotent elements of an algebra A of dimension $n \geq 1$ is to find a condition on $x_i \in K (i = 1, 2, \cdots, n)$ such that $\Big(\sum_{j=1}^{n} x_i e_i\Big)^2 = \sum_{i=1}^{n} x_i e_i$, i.e., $x_k = x_k \Big(\sum_{j=1}^{n} (1 + 2a_{jk}) x_j\Big)^2$.

Since it can be rewritten by

$$x_k \Big(\sum_{j=1}^{n} (1 + 2a_{jk}) x_j - 1\Big) = 0 \quad (k = 1, 2, \cdots, n),$$

and each of this quadratic is the intersection of two planes of equation $x_k = 0$ and of equation $\sum_{j=1}^{n} (1 + 2 + a_{jk}) x_j - 1 = 0$, the study of idempotent elements is reduced to solve the 2^n systems of n linear equations with n unknowns.

From [9] and [13] , letting A the anti-symmetric matrix associated to the Lotka-Volterra K-algebra A, A_{ij} the (i, j)-minor of A, A_i the (i, i)-minor of A and $Pf(A)$ the Phaffian of A with size n [5]. we have the following.

LEMMA 1.2. *$det(A_{ij}) = Pf(A_i)Pf(A_i)$ $(i = 1, 2, \cdots, n)$.*

PROOF. This lemma is clear if $i = j$.
Let $i \neq j$ and V be a vector space over K of dimension $2k + 1$. If $B \in \wedge^2 V^*$ and E, F are two distinct hyperplanes of V. Then we have a K-linear mapping $U_{E,F} : E \longrightarrow F^*$ defined by $U_{E,F}(x)(y) = B'(x, y) = B(x, y)$ for any $x \in E$ and $y \in F$, where $B_E = B|_E, B_F = B|_F$ and $B' = B|_{E \times F} : E \times F \longrightarrow K$. Hence this lemma can be obtained from $\wedge^{2k} U_{E,F} = Pf(B_E) \otimes Pf(B_F)$, where $Pf(B_E) \in \wedge^{2k} E^*, Pf(B_F) \in \wedge^{2k} F^*$ and $\wedge^{2k} U_{E.F} \in Hom_k(\wedge^{2k} F^*) \simeq \wedge^{2k} E^* \otimes \wedge^{2k} F^*$.

LEMMA 1.3. *If n is even, then $det(A_{ij}) = Pf(A)Pf(A_{(i,j)})$ $(i, j = 1, \cdots, n)$, where $A_{(i,j)}$ is a submatrix of A by deleting $n - i$ rows and $n - j$ columns from A.*

PROOF. Since this lemma means that $\wedge^{n-1} U_{E,F} = Pf(B_V) \otimes Pf$ $(B_{E \cap F})$ and from the exact sequence $0 \longrightarrow E \cap F \longrightarrow E \oplus F \longrightarrow V \longrightarrow 0$, we have $\wedge^{n-1} E^* \otimes_k \wedge^{n-1} F^* = \wedge^n V^* \otimes_k \wedge^{n-2} (E \cap F)$.

THEOREM 1.4. *In a Lotka-Volterra algebra A, the idempotent elements are as follows :*

$$i)\ x_i = \frac{(-1)^i Pf(A_i)}{\left[\sum_{j=1}^{n} (-1)^j Pf(A_k)\right]^2} \quad (i = 1, 2, \cdots, n) \quad \text{if } n \text{ is odd,}$$

$$ii)\ x_i = \frac{\sum_{j=1}^{n} (-1)^{i+j} Pf(A_{ij})}{Pf(A)} \quad (i = 1, 2, \cdots, n) \quad \text{if } n \text{ is even.}$$

COROLLARY 1.5. *In a Lotka-Volterra algebra A, there are exactly 2^n idempotent elements.*

2. The structure of $\mathbf{Aut}_K(A, \omega)$.

Since the Lotka-Volterra algebra A has 2^n idempotent elements, we can say that there exists a homomorphism of groups $\mathrm{Aut}_K(A) \longrightarrow S_{2^n}$ defined by $\sigma \mapsto \sigma|_{\mathrm{Idemp}(A)}$, where S_{2^n} is the symmetric group of 2^n letters and Idemp(A) is the set of all idempotent elements of A. In particular, since the elements of the basis $B = \{e_1, e_2, \cdots, e_n\}$ are idempotent, such homommorphism is injective.

Let $\mathrm{Idemp}_0(A,\omega)$ and $\mathrm{Idemp}_1(A,\omega)$ be the set of all idempotent elements of (A,ω) which has its weight 0 and 1, respectively. Then, in general, there exists 2^{n-1} idempotent elements of each spaces and all automorphims of A permutes between the idempotent elements of weight 0 and that of weight 1.

THEOREM 2.1. *In a Lotka-Volterra algebra (A,ω), if an automorphism σ in $Aut_K(A,\omega)$ leaves all idempotent elements of weight 0 fixed, then $\sigma = 1_{(A,\omega)}$.*

PROOF. To show it if $f = \frac{1}{2^{n-1}} \sum_{e \in \mathrm{Idemp}_1(A,\omega)} e$, then $\sigma(f) = \frac{1}{2^{n-1}} \sum_{e \in \mathrm{Idemp}_i(A,\omega)} \sigma(e) = f$ for all σ in $\mathrm{Aut}_K(A,\omega)$ and $\sigma(f) = 1$. Using the fact that $\omega(f) = \frac{1}{2^{n-1}} \sum_{e \in \mathrm{Idemp}_1(A,\omega)} \omega(e) = 1$, we have a Peirce decomposition of A in direct sum of K-vector spaces as follows : $A = Kf \oplus \mathrm{Ker}(\omega)$. Consequently, if an automorphism σ in $\mathrm{Aut}_K(A,\omega)$ leaves all elements of $\mathrm{Idemp}_0(A,\omega)$ fixed, then it leaves also the basis $\frac{-1}{2a_{12}}(e_1 - e_2), \cdots, \frac{-1}{2a_{1n}}(e_1 - e_n)$ of $\mathrm{Ker}(\omega)$ fixed. So, we have $\sigma = 1_{(A,\omega)}$.

COROLLARY 2.2. *Let K be a field of characteristic not 2 and (A,ω) a Lotka-Volterra K-algebra of dimension n. If the weight homomorphism ω is unique, then there exists an injective homomorphism of groups $Aut_K(A) \hookrightarrow S_{2^{n-1}-1}$.*

PROOF. If we defined a mapping $\mathrm{Aut}_K(A,\omega) \longrightarrow S_{2^{n-1}-1}$ by $\sigma \mapsto \sigma|_{\mathrm{Idem}_0(A)-\{0\}}$, then by theorem 2.1, we have the required result.

THEOREM 2.3. *Let K be a field of characteristic not 2 and let (a_{ij}) be an anti-symmetric matrix with coefficients in K. If A is a Lotka-Volterra algebra associated to the matrix (a_{ij}), then the following conditions are equivalent :*

i) A is a Jordan algebra.

ii) If $a_{ij} = \frac{1}{2}$ and $a_{jk} = \frac{1}{2}$, then $a_{ik} = \frac{1}{2}$.

PROOF. i) $\Rightarrow$ ii). Let B be a subalgebra of A generated by $\{e_i, e_j, e_k\}$. Then the multiplication allows us that B is not a power-associative algebra if ii) does not hold. Since it is known that any Jordan algebra is power-associative [12], if ii) is not hold, then (A, ω) is not a Jordan algebra.

ii) $\Rightarrow$ i). Since the condition ii) implies that $e_i e_j = e_{\min(i,j)}$ $(i, j = 1, 2, \cdots, n), (A, \omega)$ is an associative algebra.

EXAMPLE 2.4. Consider the Lotka-Volterra algebra A of dimension 3 with a basis $B = \{e_i, e_j, e_k\}$. Assume that the matrix $\begin{vmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{vmatrix}$ is associated to the Lotka-Volterra algebra. Then we have the multiplication as follow :

$$\begin{aligned}
{e_i}^2 &= e_i \qquad (i = 1, 2, \cdots, n), \\
e_1 e_2 &= \left(\frac{1}{2} + a_{12}\right) e_1 + \left(\frac{1}{2} - a_{12}\right) e_2 \\
e_1 e_3 &= \left(\frac{1}{2} + a_{13}\right) e_1 + \left(\frac{1}{2} - a_{13}\right) e_3 \\
e_2 e_3 &= \left(\frac{1}{2} + a_{23}\right) e_2 + \left(\frac{1}{2} - a_{23}\right) e_3
\end{aligned}$$

Under these multiplication, we can see that $(x^2)^2 \neq x^4$ for a vector $x = e_1 - e_2 + e_3$. Therefore, A is not power-associative and not a Jordan algebra.

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