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GENUS NUMBERS AND AMBIGUOUS CLASS NUMBERS OF FUNCTION FIELDS

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ABSTRACT. Some formulas of the genus numbers and the ambiguous ideal class numbers of function fields are given and these numbers are shown to be the same when the extension is cyclic.

1. Ambiguous Class Number

Let k be a global function field with exact constant field $\mathbb{F}_q$ and ∞ be a fixed place of degree δ. For a finite extension K of k, we denote by S the set of places of K lying over ∞, by $\mathcal{O}_K$ the ring of functions in K which are regular away from S, and E_K the group of S-units of K. Let $\tilde{K}$ be the Hilbert class field of $\mathcal{O}_K$, that is, the maximal unramified abelian extension of K where any place in S splits completely. For a Galois extension K of k we fix the following notations throughout the paper.

P_K (P_k) : the group of principal fractional ideals of O_K (resp. O_k)
I_K (I_k) : the group of fractional ideals of $\mathcal{O}_K$ (resp. O_k)
Cl_K (Cl_k) : the quotient group I_K/P_K (resp. I_k/P_k)
A : the group of ambiguous ideal classes in K/k
A_0 : the group of ideal classes of K represented by ambiguous ideal in K/k
A_k : the group of ideal classes of K represented by ideals of k
N : the set of elements in $\mathbb{F}_q^*$ which are norms of elements of K
N' : the set of elements in $\mathbb{F}_q^*$ which are locally norms from K to k
J_K (J_k) : the group of ideles of K (k)

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We usually denote a finite place of k by $\mathfrak{p}$. For each finite place $\mathfrak{p}$ of k we denote by $e(\mathfrak{p})$ the ramification index of K/k at $\mathfrak{p}$, and by d_∞ the order of the decomposition group at ∞. Let

$$U(K) = \prod_{\tilde{\infty} \in S} K^*_{\tilde{\infty}} \prod_{\mathfrak{p}: finite} O^*_{\mathfrak{p}}.$$

Let $\tilde{h}_K = |Cl_K|$, $\tilde{h}_k = |Cl_k|$, $a = |A|$, $a_0 = |A_0|$ and h_0 be the number of ideal classes of k which become principal in K. Let h_k be the number of divisor classes of degree 0 of k. Then $\tilde{h}_k = \delta h_k$. The aim of this paper is to compare these numbers.

PROPOSITION 1.1. *We have*

$$a_0 = \tilde{h}_k \frac{\prod_{\mathfrak{p}} e(\mathfrak{p})}{|H^1(G, E_K)|}.$$

PROOF. $a_0 = |A_0| = [P_K \cdot I_K^G : P_K] = [I_K^G : P_K^G] = [I_K^G : P_k]/[P_K^G : P_k]$. We know from the proof of Theorem 1 [2] that $P_K^G/P_k \simeq H^1(G, E_K)$ and $[I_K^G : I_k] = \prod e(\mathfrak{p})$. Hence the result follows.

Similar process as in [3] gives;

LEMMA 1.2. *We have*

i) $[P_K^G \cap I_k : P_k] = h_0$.
ii) $[I_k : P_K^G \cap I_k] = [P_K^G \cdot I_k : P_K^G] = \tilde{h}_k/h_o$.
iii) $[P_K^G : P_K^G \cap I_k] = [P_K^G I_k : I_k] = |H^1(G, E_K)|/h_0$.
iv) $[I_K^G : P_K^G \cdot I_k] = \prod e(\mathfrak{p}) \cdot h_0/|H^1(G, E_K)|$.

In particular, h_0 is a common divisor of $\tilde{h}_k$ and $|H^1(G, E_K)|$.

PROOF. i) and ii) are clear from the definitions. iii) follows from i) and the fact that $P_K^G/P_k \simeq H^1(G, E_K)$. Since $[I_K^G : P_K^G \cdot I_k] = [I_K^G : P_K^G]/[P_K^G \cdot I_k : P_K^G]$, iv) follows from ii) and Proposition 1.1.

We now assume that K is a cyclic extension of k of degree n and let σ be a generator of the Galois group $G = G(K/k)$. It is well-known that the Herbrand quotient $Q(E_K)$ is equal to d_∞/n.

LEMMA 1.3. $\quad {}_N Cl_K \simeq N^{-1}(k^*U(k))/K^*U(K)$
and

$$Cl_K^{1-\sigma} \simeq N^{-1}(k^*)U(K)/K^*U(K).$$

PROOF. The first statement is clear from the definitions. Let $C_K = J_K/K^*$, where J_K is the group of ideles. Since $H^1(G, C_K) = 1$, $C_K^{1-\sigma} = Ker(N : C_K \to C_k) = N^{-1}(k^*)/K^*$. Hence the result follows.

LEMMA 1.4. $\quad [A : A_0] = [A : I_K^G \cdot P_K/P_K] = [N : NE_K]$.

PROOF. For an ideal $\mathfrak{a}$ in an ambiguous class, $\mathfrak{a} - \mathfrak{a}^\sigma$ is a principal ideal (θ). Since $N(\mathfrak{a} - \mathfrak{a}^\sigma) = 0$, $N(\theta)$ is a unit in O_k, and so $N(\theta) \in N$. Under this correspondence an ambiguous ideal corresponds to an element in NE_K.

THEOREM 1.5. *Let* $n_1 = \frac{\tilde{h}_k}{[Cl_K : {}_N Cl_K]}$ *and* $n_2 = \frac{d_\infty \prod e(\mathfrak{p})}{[E_k : N][{}_N Cl_K : Cl_K^{1-\sigma}]}$. *Then* n_1 *and* n_2 *are integers, and we have*

$$a = \frac{\tilde{h}_k}{n_1} \times \frac{d_\infty \prod e(\mathfrak{p})}{n_2[E_k : N]}.$$

Moreover, $n_1 n_2 = n$.

PROOF. Since $[Cl_K : {}_N Cl_K] = |NCl_K|$ divides $\tilde{h}_k$, n_1 is an integer. By Lemma 1.3
$[{}_N Cl_K : Cl_K^{1-\sigma}]$ equals $|k^*U(k) \cap N(J_K)/k^*N(U(K)) \cap N(J_K)|$, which divides

$$\begin{aligned} |k^*U(k)/k^*N(U(K))| &= [U(k) : N(U(K))]/[k^* \cap U(k) : k^* \cap N(U(K))] \\ &= \frac{d_\infty \prod e(\mathfrak{p})}{[E_k : N']}. \end{aligned}$$

Since G is cyclic, Hasse norm theorem implies that $N' = N$. Hence n_2 is an integer. Since

$$a = [Cl_K : Cl_K^{1-\sigma}] = [Cl_K : {}_N Cl_K][{}_N Cl_K : Cl_K^{1-\sigma}],$$

we have

$$a = \frac{\tilde{h}_k}{n_1} \times \frac{d_\infty \prod e(\mathfrak{p})}{n_2[E_k : N]}. \tag{1}$$

By Lemma 1.4

$$a = [A : I_K^G \cdot P_K/P_K][I_K^G \cdot P_K : P_K] = [A : I_K^G \cdot P_K/P_K]a_0 = a_0[N : NE_K].$$

Thus by Proposition 1.1 we have

$$a = \tilde{h}_k \frac{\prod e(\mathfrak{p})}{|H^1(G, E_K)|}[N : NE_K]. \tag{2}$$

Now $H^1(G, E_K) = H^0(G, E_K)/Q(E_K) = [E_k : NE_K]\frac{d_\infty}{n} = [E_k : N][N : NE_K]\frac{d_\infty}{n}$. Hence we have

$$a = \tilde{h}_k \frac{d_\infty \prod e(\mathfrak{p})}{[E_k : N]n}. \tag{3}$$

Comparing (1) and (2) we get

$$n_1 n_2 = n.$$

COROLLARY 1. *$a = \tilde{h}_k$ if and only if $\frac{\prod e(\mathfrak{p})}{[E_k : N]} = \frac{n}{d_\infty}$.*

COROLLARY 2. *We have*

$$[I_K^G \cdot P_K : I_k \cdot P_K] = \frac{h_0 \prod e(\mathfrak{p})}{|H^1(G, E_K)|}.$$

PROOF. Divide a into

$$a = [A : I_K^G \cdot P_K/P_K][I_K^G \cdot P_K : I_k \cdot P_K][I_k \cdot P_K : P_K].$$

Then the equality follows from Lemma 1.4, equation (2) above, and the fact that $[I_k \cdot P_K : P_K] = \frac{[I_k : P_k]}{[P_K \cap I_k : P_k]} = \frac{\tilde{h}_k}{h_0}$.

Exactly the same process as in [3] would give

PROPOSITION 1.6. *Let K/k be a Galois extension of degree n. If $(\tilde{h}_k, n) = 1$, then*

i) $A_k = I_k \cdot P_K/P_K \simeq Cl_k$, $\tilde{h}_k = |A_k|$, *and* $h_0 = n_1 = 1$.

ii) *the sum* $Cl_K = A_k + {}_N Cl_K$ *is direct*

iii) $\prod e(\mathfrak{p}) = |H^1(G, E_K)|[I_K^G \cdot P_K : I_k \cdot P_K]$.

PROPOSITION 1.7. *Let K/k be a cyclic extension of degree n. Let $a_1 = |A \cap Cl_K^{1-\sigma}|$. Then we have*

i) *the sum* $Cl_K = A + Cl_K^{1-\sigma}$ *is direct if and only if* $a_1 = 1$.

ii) a *is not prime to* n *if* $a_1 \neq 1$.

PROPOSITION 1.8. *Let K/k be a cyclic extension of degree n and assume that $(a, n) = 1$. Then we have*

i) *the sum* $Cl_K = A + Cl_K^{1-\sigma}$ *is direct,*

ii) $a = \frac{\tilde{h}_k}{h_0}$, $h_0 = n_1$, *and* $a_1 = 1$.

iii) $|H^1(G, E_K)| = h_0 \prod e(\mathfrak{p})$, $|H^0(G, E_K)| = \frac{d_\infty h_0 \prod e(\mathfrak{p})}{n}$ *and*

$$|H^r(G, Cl_K)| = 1 \quad \text{for any} \quad r > 0.$$

PROOF. $|H^0(G, E_K)| = |H^1(G, E_K)|\frac{d_\infty}{n}$. From the decomposition

$$a = \frac{\tilde{h}_k}{n_1} \times \frac{d_\infty \prod e(\mathfrak{p})}{n_2[E_k : N]} = \frac{\tilde{h}_k}{h_0} \frac{h_0 \prod e(\mathfrak{p})}{|H^1(G, E_K)|}[N : NE_K],$$

the factors $\frac{d_\infty \prod e(\mathfrak{p})}{n_2[E_k:N]}$, $\frac{h_0 \prod e(\mathfrak{p})}{|H^1(G,E_K)|}$, and $[N : NE_K]$ are integers composed of prime factors of n. But since $(a, n) = 1$, they must be 1. Hence $|H^1(G, E_K)| = h_0 \prod e(\mathfrak{p})$, $[N : NE_K] = 1$ and $|H^0(G, E_k)| = \frac{d_\infty h_0 \prod e(\mathfrak{p})}{n}$. We also have $a = \frac{\tilde{h}_k}{n_1} = \frac{\tilde{h}_k}{h_0}$. Hence $h_0 = n_1$, and the result follows as in [3].

2. Genus Numbers

Define the (*relative*) *genus field* $\bar{K}$ to be the maximal extension of K inside $\tilde{K}$ which is a composite of K and some abelian extension of k, and the degree $g = [\bar{K} : K]$ is called the *(relative) genus number* of K over k. Let $\bar{K}_0$ (resp. K_0) be the maximal subfield of $\bar{K}$ (resp. K) which is abelian over k. Then we have;

PROPOSITION 2.1. *Let $\mathcal{L}$ be the subgroup of the idele group J_k of k corresponding to $\bar{K}_0$. Then we have*

$$\mathcal{L} = k^* \prod_{\mathfrak{p}} N(U_{\mathfrak{P}}) \cdot N(K^*_{\bar{\infty}}),$$

where the product is taken over all finite places $\mathfrak{p}$, $\mathfrak{P}$ is any place lying over $\mathfrak{p}$, and $\bar{\infty}$ is any place over ∞.

PROOF. The result follows from the exactly same process as in [1] except that the subgroup of J_K associated to $\tilde{K}$ is $K^* \prod_{\mathfrak{P}} U_{\mathfrak{P}} \cdot \prod_{\bar{\infty}|\infty} K^*_{\bar{\infty}}$.

THEOREM 2.2. *The genus number g of a Galois extension K over k is equal to*

$$\frac{\delta h_k d'_{\infty} \prod_{\mathfrak{p}:finite} e'_{\mathfrak{p}}}{[K_0 : k][\mathbb{F}^*_q : N']}$$

where $e'_{\mathfrak{p}}$ is the ramification index of the maximal abelian subfield of $K_{\mathfrak{P}}$ over $k_{\mathfrak{p}}$, d'_{∞} is the degree of the maximal abelian subfield of $K_{\bar{\infty}}$ over k_{∞}.

PROOF. Since $K\bar{K}_0 = \bar{K}$ and $K \cap \bar{K}_0 = K_o$, the genus number $[\bar{K} : K]$ is equal to $[\bar{K}_0 : K_0]$. We have

$$[\bar{K}_0 : K_0] = \frac{[\bar{K}_0 : k]}{[K_0 : k]} = \frac{[J_k : \mathcal{L}]}{[K_0 : k]} = \frac{[J_k : k^*U][k^*U : \mathcal{L}]}{[K_0 : k]},$$

where $U = \prod_{\mathfrak{p}:finite} U_{\mathfrak{p}} \cdot k^*_{\infty}$. Since $[J_k : k^*U] = \delta h_k$, we have

$$g = \frac{\delta h_k [k^*U : \mathcal{L}]}{[K_0 : k]}.$$

Also we have

$$[k^*U : \mathcal{L}] = [\mathcal{L}U : \mathcal{L}] = [U : \mathcal{L} \cap U] = \frac{[U : \prod_{\mathfrak{p}} NU_{\mathfrak{P}} \cdot NK^*_{\bar{\infty}}]}{[\mathcal{L} \cap U : \prod_{\mathfrak{p}} NU_{\mathfrak{P}} \cdot NK^*_{\bar{\infty}}]}.$$

It is known that $[U_{\mathfrak{p}} : NU_{\mathfrak{P}}] = e'_{\mathfrak{p}}$ and $[k^*_{\infty} : NK^*_{\bar{\infty}}] = d'_{\infty}$. Hence it remains to show that $[\mathcal{L} \cap U : \prod_{\mathfrak{p}} NU_{\mathfrak{P}} \cdot NK^*_{\bar{\infty}}] = [\mathbb{F}^*_q : N']$. This follows from exactly the same method as in [1], since $k^* \cap U = \mathbb{F}^*_q$ in the function field case.

COROLLARY. *Suppose that K/k is cyclic. Then we have*

i) $a = g$

ii) *if ∞ is totally ramified, then $\tilde{h}_k \mid a = g$,*

iii) *if, moreover, K is unramified outside ∞, then $a = \tilde{h}_k$.*

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