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TRIPOTENCE FOR IRREDUCIBLE SIGN-PATTERN MATRICES

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ABSTRACT. A matrix whose entries consist of the symbols $+, -, 0$ is called a sign-pattern matrix. We characterize the $n \times n$ irreducible sign-pattern matrices that are sign tripotent.

I. Introduction

Qualitative matrix analysis involves the study of (required or allowed) properties that are based just upon knowledge of the signs of the entries of a matrix. A matrix whose entries consist of the symbols $+, -, 0$ is called a *sign-pattern matrix*. For a real matrix $\mathbf{B}$, by sgn $\mathbf{B}$ we mean the sign-pattern matrix in which each positive (respectively, negative, zero) entry is replaced by $+$ (respectively, $-, 0$). For each $n \times n$ sign-pattern matrix A, there is a natural class of real matrices whose entries have the sign indicated by A. If $A = [a_{ij}]$ is an $n \times n$ sign-pattern matrix, then the sign-pattern class of A is defined by

$$Q(A) = \{\mathbf{B} = [\mathbf{b}_{ij}] | \text{sgn } \mathbf{b}_{ij} = a_{ij} \text{ for all } i \text{ and } j \text{ in } \{i, \ldots, n\}\}.$$

If A and B are $n \times n$ sign-pattern matrices, then $A + B$ exists, i.e., $A+B$ is qualitatively defined if $a_{ij}b_{ij} \neq -$ for all i and j in $\{1, 2, \ldots, n\}$. If $a_{ij}b_{ij} = -$, then $a_{ij} + b_{ij}$ is $-$ or $+$. We cannot determine the sign of

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the entry $a_{ij} + b_{ij}$. That is, $A + B$ is undefined. Similarly, the product AB exists if no two terms in the sum

$$\sum_{k=1}^{n} a_{ik} b_{kj}$$

are oppositely signed for all i and j in $\{1, 2, \ldots, n\}$. For example, let

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Then, we have

$$\text{sgn } \mathbf{A} + \text{sgn } \mathbf{B} = \begin{bmatrix} - & + \\ + & - \end{bmatrix}, \quad \text{sgn } \mathbf{A} \cdot \text{sgn } \mathbf{B} = \begin{bmatrix} + & - \\ - & + \end{bmatrix}.$$

But both sgn $\mathbf{A}$ + sgn $\mathbf{C}$ and sgn $\mathbf{A}$ · sgn $\mathbf{C}$ are undefined.

If $A = [a_{ij}]$ is an $n \times n$ sign-pattern matrix, then a product of the form

$$\gamma = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k j}$$

is called a *path* of length k from i_1 to j. If the index $j = i_1$, then γ is called a *cycle* of length k. A cycle or a path is said to be *negative* (respectively, *positive*) if it contains an odd (respectively, even) number of negative entries and no enties equal to zero.

We say a powerful pattern A is $k - potent$ if $A = A^{m+1}$ for some $m > 0$ and k is the smallest such positive integer. A real matrix $\mathbf{B}$ of order n is said to be *idempotent* if $\mathbf{B} = \mathbf{B}^2$. Analogously, a square sign-pattern matrix A is said to be *sign idempotent* if $\mathbf{B}^2 \in Q(A)$ whenever $\mathbf{B} \in Q(A)$; henceforth we write $A = A^2$. Also, a matrix $\mathbf{B}$ is said to be *tripotent* if $\mathbf{B} = \mathbf{B}^3$. And, a square sign-pattern matrix A is said to be *sign tripotent* if $\mathbf{B}^3 \in Q(A)$ whenever $\mathbf{B} \in Q(A)$; henceforth we write $A = A^3$. One important reason for studying sign idempotent and sign tripotent is that powers of sign idempotent matrices and sign tripotent matrices preserve not only the sign-pattern, but also the cycle structure of the matrix.

In [1], Eschenbach characterized the sign idempotent matrices. In [3], the author characterized the irreducible sign tripotent matrices with no zero entries. In this paper, our objective is to characterize $n \times n$ irreducible sign-patterns that are sign tripotent.

2. Results

Let the index set $\{1, 2, \dots, n\}$ be represented by N, and let IST be the class of $n \times n$ irreducible sign-patterns that are sign tripotent matrices.

The first lemma is clear, and we state it without proof.

LEMMA 1. *The class IST is closed under the following operators:*

(i) *signature similarity;*

(ii) *permutation similarity; and*

(iii) *transposition.*

The next theorem is very useful for the characterization of tripotence.

THEOREM 2. [3]. *If $A \in IST$, then A is a symmetric sign-pattern matrix.*

Let $A = [a_{ij}]$ be an irreducible sign tripotent matrix with no zero entries and let $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D} \in Q(A)$. Then the following theorem holds.

THEOREM 3. [3]. $\mathbf{BCD} \in Q(A)$ *if and only if*

(i) $a_{ij}a_{ji} > 0$
(ii) $a_{ik}a_{kl}a_{lj}a_{ji} > 0$
(iii) $a_{ii} < 0$
(iv) $a_{ik}a_{kl}a_{li} < 0$

for $i, j, k, l \in N$.

Now, we can characterize the sign tripotent matrices with no zero entries, by using the above Lemma 1, Theorems 2 and 3.

THEOREM 4. [3]. *Let A be a sign-pattern matrix with no zero entries. Then, $A \in IST$ if and only if every odd cycle of A is negative and every even cycle is positive.*

For example, let

$$A = \begin{bmatrix} - & + & - \\ + & - & + \\ - & + & - \end{bmatrix}.$$

Then $A^3 = A$, that is, A is an irreducible sign tripotent matrix with no zero. Clearly, every odd cycle of A is negative and every even cycle is positive.

Now, we consider that A is an irreducible sign tripotent matrix with zero entries. Suppose that $A = [a_{ij}]$ is an irreducible sign tripotent matrix of order 2. Let $a_{ij} = 0$ for some $i, j \in \{1, 2\}$. Since A is irreducible sign tripotent, $i = j$. That is, $a_{ii} = 0$ for $i = 1$ or $i = 2$. Without loss of generality, let $a_{11} = 0$. Then, we may assume that $A = \begin{bmatrix} 0 & a \\ a & c \end{bmatrix}$, where $a \neq 0$. Then

$$A^3 = \begin{bmatrix} a^3c & a^3 + ac^2 \\ a^3 + ac^2 & a^2c + a^2c + a^3 \end{bmatrix}.$$

Since $A = A^3$ and $a \neq 0$, $c = 0$. So,

$$A^3 = \begin{bmatrix} 0 & a^3 \\ a^3 & 0 \end{bmatrix}.$$

Thus, $a = +$ or $a = -$. Therefore,

$$A = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \text{ or } A = \begin{bmatrix} 0 & - \\ - & 0 \end{bmatrix}.$$

Now, we prove that if $A \in IST$ with zero entries, then the diagonal entries of A are all zeros.

LEMMA 5. *Let $A \in IST$ with zero entries. If $a_{ij} = 0$ for some $i, j \in N$, there is a 2-path from i to j.*

PROOF. Suppose that $a_{ij} = 0$ for some $i, j \in N$. Since A is irreducible, there is a path from i to j, say

$$\gamma = a_{ik_1} a_{k_1 k_2} \cdots a_{k_m j} \neq 0, \quad k_h \in N, \quad h = 1, 2, \ldots, m.$$

Since A is sign tripotent and $a_{ij} = 0$, $a_{ij} = a_{ip} a_{pq} a_{qj} = 0$ for all $p, q \in N$. Thus, $m \neq 2$.

Assume that $m \geq 3$. Then we have the three cases; $m = 3l$, $m = 3l+1$, $m = 3l+2$, where l is a positive integer. Since A is a sign tripotent matrix, in any cases, we have two cases, by tripotence of A,

$$\gamma = a_{ik_1} a_{k_1 k_2} \cdots a_{k_m j} = \begin{cases} a_{ik_m} a_{k_m j}; & \text{or} \\ a_{ik_{m-1}} a_{k_{m-1} k_m} a_{k_m j}. & \end{cases}$$

Since $a_{ij} = a_{ip}a_{pq}a_{qj} = 0$ for all $p, q \in N$, $\gamma = a_{ik_m}a_{k_m j}$.
Therefore, there is a 2-path from i to j. ∎

THEOREM 6. *If $A = [a_{ij}] \in IST$ with zero entries, then $a_{ii} = 0$ for all $i \in N$.*

PROOF. If $n = 2$, then the theorem holds. Suppose that $n \geq 3$. Let $a_{ij} = 0$ for some $i, j \in N$. Then $a_{ip}a_{pq}a_{qj} = 0$ for all $p, q \in N$.

First, suppose that $i \neq j$. Since A is irreducible, there is a $k \in N$ such that $a_{ik}a_{kj} \neq 0$. Since $a_{ij} = 0$,

$$a_{ii}a_{ik}a_{kj} = a_{ik}a_{kj}a_{jj} = a_{ik}a_{kk}a_{kj} = 0.$$

That is, $a_{ii} = a_{jj} = a_{kk} = 0$. So, we have two cases;

Case 1. $k = i$ or $k = j$.

Since the tripotence of A is permutation similar, without loss of generality, we may assume that $a_{11} = a_{22} = 0$. Assume that there is an $l \in N$ such that $a_{ll} \neq 0$. Without loss of generality, let $a_{33} \neq 0$. Since A is irreducible sign tripotent, there is a $p \in N$ such that $a_{3p} = a_{p3} \neq 0$. Then we have two cases; $1 \leq p \leq 2$ or $4 \leq p \leq n$.

If $1 \leq p \leq 2$, then, without loss of generality, we may assume $p = 1$. That is, $a_{31} \neq 0$. Since A is sign tripotent, $a_{13} \neq 0$ and $(A^3)_{11} \neq 0$. So, $p \neq 1$. Thus, $4 \leq p \leq n$. Without loss of generality, we may assume that $a_{3q} = a_{q3} \neq 0$, $4 \leq q \leq n$. Then, by tripotence of A, $a_{qq} \neq 0$. If $a_{q1} \neq 0$ (respectively, $a_{q2} \neq 0$), then $a_{11} \neq 0$ (respectively, $a_{22} \neq 0$). So, $a_{q1} = 0$ and $a_{q2} = 0$. Since A is irreducible, there is an r, $3 \leq r \leq n$, such that $a_{r1} \neq 0$ or $a_{r2} \neq 0$. If $a_{r1} \neq 0$ (respectively $a_{r2} \neq 0$), then, by tripotence of A, $a_{11} \neq 0$ (respectively. $a_{22} \neq 0$). This is a contradiction. Thus, $a_{ii} = 0$ for all $i \in N$.

Case 2. $k \neq i$ and $k \neq j$.

In this case, $a_{ii} = 0$, $a_{kk} = 0$, $a_{jj} = 0$, $a_{ij} = 0$ and $a_{ji} = 0$. And the proof is similar to Case 1.

Now, suppose that $i = j$, that is $a_{ii} = 0$ for some $i \in N$. Then, without loss of generality, by irreducibility of A, let $a_{11} = 0$, $a_{12} = a_{21} \neq 0$. Let $a_{22} = *$, where $*$ is a something-sign. Then $(A^2)_{11} = +$, $(A^2)_{12} = (A^2)_{21} = a_{12} \cdot *$, $(A^2)_{22} = +$. So,

$$\begin{aligned}(A^3)_{11} &= (A^2)_{12} \cdot a_{12} \\ &= a_{12} \cdot a_{12} \cdot * = 0.\end{aligned}$$

Since $a_{12} \neq 0$, $* = 0$. Thus, $a_{22} = 0$.

The rest of the proof is similar to the case of $i \neq j$.

Therefore, $a_{ii} = 0$ for all $i \in N$. ■

LEMMA 7. *Let A be an irreducible sign-pattern matrix. If every nonzero even cycle of A is positive and every odd cycle of A is zero, then $a_{ij} = a_{ji}$ for all $i, j \in N$*

PROOF. If $a_{ij}a_{ji} \neq 0$, then $a_{ij} = a_{ji}$. If $a_{ij}a_{ji} = 0$, then $a_{ij} = 0$ or $a_{ji} = 0$. Suppose that $a_{ij} = 0$ and $a_{ji} \neq 0$. Since A is irreducible, there is a path γ from i to j, say

$$\gamma = a_{ij_1}a_{j_1j_2}\cdots a_{j_kj} \neq 0.$$

Then the length of γ is $k+1$. Since $\gamma a_{ji} = +$, $k+1$ is an odd number. So,

$$a_{jj_1}a_{j_1j_2}\cdots a_{j_kj} = 0.$$

Thus, $a_{jj_1} = 0$ and there is a path from j to j_1 of which the length is odd. Let

$$a_{jp_1}\cdots a_{p_lj_1} \neq 0.$$

Then, $\delta = a_{j_1j_2}\cdots a_{j_kj}a_{jp_1}\cdots a_{p_lj_1} \neq 0$. But, the length of δ is $k+l+1$, which is odd. This is a contradiction.

Therefore, $a_{ij} = 0$ and $a_{ji} = 0$. ■

Suppose that the power of A is always defined. Then, we have the following theorem.

THEOREM 8. *Let A be a sign-pattern matrix with zero entries. If $A \in IST$, then every nonzero even cycle of A is positive and every odd cycle of A is zero.*

PROOF. Suppose that $A \in IST$. Let

$$\gamma = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}.$$

Suppose that k is an even number. If $k = 2$, then sgn $\gamma = +$ or 0 because $a_{ij} = a_{ji}$. If $k \geq 4$, then, by tripotence of A,

$$\text{sgn } \gamma = \text{sgn } a_{i_1i_k}a_{i_ki_1} = + \text{ or } 0.$$

Now, suppose that k is an odd number. Then, by tripotence of A,

$$\begin{aligned}\operatorname{sgn} \gamma &= \operatorname{sgn} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} \\ &= \operatorname{sgn} a_{i_1 i_{k-1}} a_{i_{k-1} i_k} a_{i_k i_1} \\ &= \operatorname{sgn} a_{i_1 i_1} = 0. \qquad \blacksquare\end{aligned}$$

In general, the converse of theorem 8 does not hold. However, if

$$\sum_{p=1}^{n} \sum_{q=1}^{n} a_{ip} a_{pq} a_{qj} = 0$$

whenever $a_{ij} = a_{ji} = 0$, then $A \in IST$. The following theorem shows that fact.

THEOREM 9. *Let A be a sign-pattern matrix with zero entries. If every nonzero even cycle of A is positive and every odd cycle of A is zero and*

$$\sum_{p=1}^{n} \sum_{q=1}^{n} a_{ip} a_{pq} a_{qj} = 0$$

whenever $a_{ij} = a_{ji} = 0$, then $A \in IST$.

PROOF. First, suppose that

$$\Big(\sum_{p=1}^{n} \sum_{q=1}^{n} a_{ip} a_{pq} a_{qj}\Big) a_{ji} \neq 0.$$

Then, $\sum_{p=1}^{n} \sum_{q=1}^{n} a_{ip} a_{pq} a_{qj} = a_{ji}$. That is, by lemma 7,

$$a_{ij} = a_{ji} = \sum_{p=1}^{n} \sum_{q=1}^{n} a_{ip} a_{pq} a_{qj}.$$

Now, suppose that

$$\Big(\sum_{p=1}^{n} \sum_{q=1}^{n} a_{ip} a_{pq} a_{qj}\Big) a_{ji} = 0.$$

Then, $\sum_{p=1}^{n}\sum_{q=1}^{n} a_{ip}a_{pq}a_{qj} = 0$ or $a_{ji} = a_{ij} = 0$. If $\sum_{p=1}^{n}\sum_{q=1}^{n} a_{ip}a_{pq}a_{qj} = 0$, then $a_{ip}a_{pq}a_{qj} = 0$ for all $p, q \in N$. So, $a_{ij}a_{ji}a_{ij} = 0$. Thus,

$$a_{ij} = \sum_{p=1}^{n}\sum_{q=1}^{n} a_{ip}a_{pq}a_{qj} = 0.$$

Assume that $a_{ij} = a_{ji} = 0$. Then, by the hypothesis,

$$\sum_{p=1}^{n}\sum_{q=1}^{n} a_{ip}a_{pq}a_{qj} = 0.$$

Therefore, $A \in IST$. ■

Let $\alpha = \{i_1, \dots, i_s\}$ and $\beta = \{j_1, \dots, j_t\}$. For a matrix A, $A(\alpha|\beta)$ will denote the matrix obtained from A by striking out the rows numbered α and the columns numbered β and $A[\alpha|\beta]$ is the $s \times t$ matrix whose (p, q)-entry is the same as the (i_p, j_q)-entry of A. If $\alpha = \beta$, then $A[\alpha|\alpha]$ is called a principal submatrix of A and is abbreviated $A[\alpha]$.

THEOREM 10. *Let A be a sign-pattern matrix. Then, $A \in IST$ if and only if the all principal submatrices of A are sign tripotent.*

PROOF. If all the principal submatrices of A are sign tripotent, then $A \in IST$, clearly.

Suppose that $A \in IST$. If there do not exist zero entries, then $a_{ii} < 0$ and all 2×2 principal submatrices of A are contained in IST. If there exist zero entries in A, then $a_{ii} = 0$ for all $i \in N$ and all 2×2 principal submatrices of A are contained in IST. Without loss of generality, let $\alpha = \{1, 2, \dots, k\} \subseteq N$, $k \geq 3$.

Case 1. $A \in IST$ with no zero entries.

For each $i, j \in \alpha$,

$$\begin{aligned}(A[\alpha])_{ij} = a_{ij} &= \sum_{p=1}^{n}\sum_{q=1}^{n} a_{ip}a_{pq}a_{qj} \\ &= \sum_{p=1}^{k}\sum_{q=1}^{k} a_{ip}a_{pq}a_{qj} = ((A[\alpha])^3)_{ij}.\end{aligned}$$

Thus, $A[\alpha]$ is sign tripotent.

Case 2. $A \in IST$ with zero entries.

If $a_{ij} = 0$, then $a_{ip}a_{pq}a_{qj} = 0$ for all $p, q \in N$. So,

$$(A[\alpha])_{ij} = a_{ij} = \sum_{p=1}^{k}\sum_{q=1}^{k} a_{ip}a_{pq}a_{qj} = ((A[\alpha])^3)_{ij} = 0.$$

Assume that $a_{ij} \neq 0$. By induction,

$$\begin{aligned}(A[\alpha])_{ij} = a_{ij} &= \sum_{p=1}^{n}\sum_{q=1}^{n} a_{ip}a_{pq}a_{qj} = \sum_{p=1}^{k-1}\sum_{q=1}^{k-1} a_{ip}a_{pq}a_{qj} \\ &= \sum_{p=1}^{k}\sum_{q=1}^{k} a_{ip}a_{pq}a_{qj} = ((A[\alpha])^3)_{ij}.\end{aligned}$$

Therefore, by induction on n, the proof is completed. ■

In [1], Eschenbach characterized the irreducible sign idempotent matrices. Theorem 1.4 in [1] proves that if A is an $n \times n$ irreducible sign-pattern matrix then A is sign idempotent if and only if it is entrywise positive. But, the irreducible sign tripotent matrices can have either negative entries or zero entries. It is the difference between irreducible sign tripotent and irreducible sign idempotent.

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