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ON p-ADIC ANALOGUE OF HYPERGEOMETRIC SERIES

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ABSTRACT. In this paper we will study a p-adic analogue of Kummer's theorem[6], [7], which gives the value at $x = -1$ of a well-piosed ${}_2F_1$ hypergeometric series.

1. Introduction

We let $F(a, b; c; x)$ be the hypergeometric series defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!} \tag{1}$$

for c neither zero nor a negative integer. In (1) the notation $(\alpha)_n$ is given by

$$(\alpha)_n = \begin{cases} 1, & \text{if } n = 0 \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, 3, \cdots\}. \end{cases}$$

Kummer [6,7] obtained

$$F(a, b; 1 + a - b; -1) = \frac{\Gamma(1 + a - b)\Gamma(\frac{1}{2})}{2^a \Gamma(1 + \frac{1}{2}a - b)\Gamma(\frac{1}{2} + \frac{1}{2}a)}.$$

N. Koblitz[4] proved that for $a, b \in \mathbb{Z}_p$ the value of the continuation of $F(a, b; 1; x)/F(a', b'; 1; x^p)$ at $x = 1$ is analytic $\Gamma_p(a)\Gamma_p(b)/\Gamma_p(a + b)$,

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where $\mathbb{Z}_p$ is the ring of p-adic integers and Γ_p is the p-adic gamma function [5]. In this article, we will prove that the ratio

$$H_p(2a, b; c; x) = \frac{F(2a, b; c; x)}{F(2a', b'; c'; x^p)}$$

has an analytic continuation to $x = -1$ if certain conditions on a, b and c are satisfied, and show that the value of the continuation of $F(2a, b; 2a - b + 1; x)/F(2a', b'; 2a' - b' + 1; x^p)$ at $x = -1$, for any appropriate a, b and c in $\mathbb{Z}_p$, has $\Gamma_p(1 + 2a - b)\Gamma_p(1 + a)/\Gamma_p(1 + 2a)\Gamma_p(1 + a - b)$.

2. Hypergeometric Series with p-adic Parameters

Let p be an odd prime, n a natural number, and $|p| = p^{-1}$. The p-adic gamma function Γ_p is defined by setting $\Gamma_p(0) = 1$, and for positive integer n by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{t<n \\ (t,p)=1}} t.$$

This can be extended to a continuous function from $\mathbb{Z}_p$ to $\mathbb{Z}_p$.

THEOREM 1. *Let $n \in \mathbb{N}$. Then*

$$\Gamma_p(n + 1) = (-1)^{n+1} n!/[n/p]!p^{[\frac{n}{p}]}$$

where $[\,.\,]$ is the Gauss symbol.

This can be immediately proved from the above definition.

Let $-a = a_0 + a_1p + a_2p^2 + \cdots \in \mathbb{Z}_p$. Let $a \longmapsto a'$ be the map induced by shifting the p-adic expansion of $-a$:

$$-a' = a_1 + a_2p + a_3p^2 + \cdots ,$$

Let $a^{(0)} = a, a^{(i)} = (a^{(i-1)})'$. Then $a \in \mathbb{Q} \cap [0, 1)$ if and only if $a^{(i)} = a$ for some i.

The following theorem was well-known due to Kummer[6,p.68].

THEOREM 2. *If $1+2a-b$ is neither zero nor a negative integer, and* $\mathrm{Re}(b)<1$ *for convergence,*

$$F(2a,b;1+2a-b;-1)=\frac{\Gamma(1+2a-b)\Gamma(1+a)}{\Gamma(1+a-b)\Gamma(1+2a)}.$$

If a,b and c are in $\mathbb{Z}_p$, then the hypergeometric series

$$F(2a,b;c;x)=\sum_{n\geq 0}\frac{(2a)_n(b)_n x^n}{(c)_n n!}$$

does not converge at $x=-1$ unless the series terminates.

In [2], B. Dwork has shown that for $a,b,c\in\mathbb{Z}_p$, a certain ratio of the hypergeometric series can be extended as an analytic element (i.e., uniform limit of rational functoins) to a domain larger than the disk of convergence ($|x|<1$) of the series.

a' is defined as $(a+\bar{a})/p$, where $\bar{a}$ is the least nonnegative integer $\equiv -a \pmod p$. $\bar{a}^{(i)}$ is $\overline{a^{(i)}}$.

The following result is partially derived from the method of Diamond [1,p.267].

THEOREM 3. *For $a,b,c\in\mathbb{Z}_p$,$|c^{(i)}|=1$ and if $2a_i=c_i$ with $b_i\leq 2a_i$ for all $i\geq 0$, then $|F_1(2a^{(i)},b^{(i)};c^{(i)};-1)|=1$ where a_i,b_i and c_i are the i-th digits in the p-adic expansion of $-a,-b$ and $-c$,*

$$F_s(2a,b;c;x)=\sum_{n=0}^{p^s-1}\frac{(2a)_n(b)_n x^n}{(c)_n n!}.$$

PROOF. It is easy to see that $\bar{a}^{(i)}\equiv -a^{(i)}\equiv a_i \pmod p$. Hence the conditions $a_i+b_i-1<c_i$ are the same as $\bar{a}^{(i)}+\bar{b}^{(i)}-1<\bar{c}^{(i)}$. To prove Theorem 3 it is sufficient to work with $i=0$. The given condition that $|c'|=1$ implies that $c+\bar{c}\neq 0 \pmod{p^2}$. Let $\bar{b}\leq\bar{a}$. Then

$$F_1(2a,b;c;-1)\equiv\sum_{j=0}^{\bar{b}}\frac{(-\bar{b})_j(2a)_j}{j!(c)_j}(-1)^j \pmod p.$$

If $2\bar{a} = \bar{c}$, then $\bar{b} \neq 0$ leads to

$$F_1(2a, b; c; -1) = \sum_{j=0}^{\bar{b}} \binom{\bar{b}}{j} = 2^{\bar{b}} \not\equiv 0 \pmod{p}.$$

So

$$|F_1(2a^{(i)}, b^{(i)}; c^{(i)}; -1)| = 1$$

holds.

Let $\mathcal{D}$ be a quasi-connected subset of $\bar{Q_p}$ (Q_p is the field of p-adic numbers) such that for all $x \in \mathcal{D}$ and all $i \geq 0$

$$|F_1(2a^{(i)}, b^{(i)}; 1 + 2a^{(i)} - b^{(i)}; -1)| = 1.$$

Dwork Theorem : For $r \geq s$ there are formal congruences

$$F_{r+1}(x)F_s(x^p) \equiv F_r(x^p)F_{s+1}(x) \ (\text{mod } p^{s+1}\mathbb{Z}_p(x)).$$

The following theorem can be easily proved by Dwork theorem [2,p.37-42].

THEOREM 4. *If $a, b, c \in \mathbb{Z}_p$ and if the following conditions are satisfied for $i = 0, 1, 2, \cdots$*

(2) $|c^{(i)}| = 1$,
(3) *if* $c \neq 1$, *then* $2\bar{a}^{(i)}, \bar{b}^{(i)} < \bar{c}^{(i)}$,
(4) $|F_1(2a^{(i)}, b^{(i)}; c^{(i)}; -1)| = 1$,

$$H_p(2a, b; c; x) = \frac{F(2a, b; c; x)}{F(2a', b'; c'; x^p)} = \lim_{s\to\infty} \frac{F_{s+1}(2a, b; c; x)}{F_s(2a', b'; c'; x^p)}$$

has an analytic continuation to $x = -1 \in \mathcal{D}$.

Putting $c_i = 1 + 2a_i - b_i$ in Theorem 3, $b_i = 1$ holds.
So $b_i - a_i < p - 1$. Therefore we get a corollary as follows ;

COROLLARY 5. *Let T denote the set of all $(a, b) \in {\mathbb{Z}_p}^2$ such that the conditions of Theorem 3 are satisfied for the series $F(2a, b; 1+2a-b; -1)$. If the following conditions are satisfied for $i = 0, 1, 2, \cdots$;*

(5) $2a_i \geq b_i$ *for all* $i \geq 0$;
(6) *If* $2a \neq b$, *then* $b_i - a_i < p-1$ *for all* $i \geq 0$, *then* $(a,\ b) \in T$, *where* a_i, b_i *are the i-th digits in the p-adic expansion of* $-a, -b$.

Corollary 5 means $H_p(2a, b; 1+2a-b; x)$ has an analytic continuation at $x = -1$ if $2a_i \geq b_i$ and $b_i - a_i < p-1$ for all $i \geq 0$.
Conclusively, $H_p(2a, b; 1+2a-b; x)$ is continuous on $T \times \mathbb{Z}_p \times \mathcal{D}$ by Dwork Theorem.
The generalized hypergeometric series

$$ {}_kF_{k-1}(\alpha_1, \cdots, \alpha_k; \gamma_1, \cdots, \gamma_{k-1}; x) = \sum_{s=0}^{\infty} \frac{(\alpha_1)_s \cdots (\alpha_k)_s x^s}{(\gamma_1)_s \cdots (\gamma_{k-1})_s s!}, $$

for all values of x for which it converges.

If the parameters satiafy

$$ 1 + \alpha_1 = \gamma_1 + \alpha_2 = \cdots = \gamma_{k-1} + \alpha_k, $$

ths series is said to be *well-poised*. We finally prove a p-adic analogue of Kummer's theorem, which gives the value at $x = -1$ of well-poised ${}_2F_1$ series.

THEOREM 6. *For $a, b \in \mathbb{Z}_p$ with $2a_i \geq b_i$ for all $i \geq 0$; If $2a \neq b$, then $b_i - a_i < p - 1$,*

$$ H_p(2a, b; 1+2a-b; -1) = \frac{\Gamma_p(1+2a-b)\Gamma_p(1+a)}{\Gamma_p(1+2a)\Gamma_p(1+a-b)}. \tag{7} $$

PROOF. By continuity of Γ_p, it suffices to prove (7) when a and b are non-positive integers. Since p is odd prime, we obtain

$$ \begin{aligned} H_p(2a, b; 1+2a-b; -1) &= \frac{F(2a, b; 1+2a-b; -1)}{F(2a', b'; 1+2a'-b'; -1)} \\ &= \frac{\Gamma(2a-b+1)\Gamma(a+1)}{\Gamma(2a+1)\Gamma(a-b+1)} \Big/ \frac{\Gamma(2a'-b'+1)\Gamma(a'+1)}{\Gamma(2a'+1)\Gamma(a'-b'+1)} \\ &= \frac{(2a-b)!a!}{(2a)!(a-b)!} \Big/ \frac{(2a'-b')!a'!}{(2a')!(a'-b')!} \\ &= \frac{(2a-b)!a!}{[\frac{2a-b}{p}]!p^{[\frac{2a-b}{p}]}[\frac{a}{p}]!p^{[\frac{a}{p}]}} \Big/ \frac{(2a)!(a-b)!}{[\frac{2a}{p}]!p^{[\frac{2a}{p}]}[\frac{a-b}{p}]!p^{[\frac{a-b}{p}]}}. \end{aligned} $$

By using Theorem 1, we get

$$ \begin{aligned} H_p(2a, b; 1+2a-b; -1) &= \frac{(-1)^{2a-b+1}\Gamma_p(2a-b+1)(-1)^{a+1}\Gamma_p(a+1)}{(-1)^{2a+1}\Gamma_p(2a+1)(-1)^{a-b+1}\Gamma_p(a-b+1)} \\ &= \frac{\Gamma_p(2a-b+1)\Gamma_p(a+1)}{\Gamma_p(2a+1)\Gamma_p(a-b+1)}, \end{aligned} $$

as desired.

By using Theorem 1, we get

$$H_p(2a, b; 1+2a-b; -1) = \frac{(-1)^{2a-b+1}\Gamma_p(2a-b+1)(-1)^{a+1}\Gamma_p(a+1)}{(-1)^{2a+1}\Gamma_p(2a+1)(-1)^{a-b+1}\Gamma_p(a-b+1)}$$
$$= \frac{\Gamma_p(2a-b+1)\Gamma_p(a+1)}{\Gamma_p(2a+1)\Gamma_p(a-b+1)},$$

as desired.

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