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QUADRATURE FORMULAS FOR WAVELET COEFFICIENTS

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ABSTRACT. We derive quadrature formulas for approximating wavelet coefficients for smooth functions from equally spaced point values with arbitrarily high degree of accuracy. We also estimate the error of quadrature formulas.

1. Introduction

Let n be a fixed positive integer. Assume $f \in C^n(\mathbf{R})$, the set of n times continuously differentiable functions on $\mathbf{R}$. Let ϕ be a scaling function with M vanishing moments for the corresponding wavelet ψ. Let j be an integer translate and $h = 2^{-k}$, where $k \in \mathbf{Z}$ is a level number.

We denote the scaled translates of ϕ by

$$\phi_j^k(x) = 2^{k/2}\phi(2^k x - j). \tag{1}$$

The *continuous moments* of the scaling function ϕ is defined by

$$\mathcal{M}_p = \int_{-\infty}^{\infty} x^p \phi(x)\, dx. \tag{2}$$

for any nonnegative integer p.

We also define the *shifted continuous moments* of ϕ by

$$\mathcal{M}_{i,j} = \int_{-\infty}^{\infty} x^i \phi(x-j)\, dx = \int_{-\infty}^{\infty} (x+j)^i \phi(x)\, dx = \sum_{l=0}^{i} \binom{i}{l} j^l \mathcal{M}_{i-l}. \tag{3}$$

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We would like to find a highly accurate numerical approximation for the wavelet coefficients of f,

$$f_j^k := \langle f, \phi_j^k \rangle = \int_{-\infty}^{\infty} f(x)\phi_j^k(x)\, dx.$$

Applications for wavelet coefficients are:

- wavelet approximations

$$f(x) = \sum_{j=-\infty}^{\infty} \langle f, \phi_j^k \rangle \phi_j^k(x) + \mathcal{O}(h^M),$$

- solutions of differential equations using wavelet-Galerkin methods, and
- wavelet series expansion of a multiplication of two functions.

Now, by substitution,

$$\langle f, \phi_j^k \rangle = h^{1/2} \int_{-\infty}^{\infty} f(h(x+j))\phi(x)\, dx. \tag{4}$$

Consider the level $k = 0$ and the integer translate $j = 0$. Consider ϕ as a weight function with support $[0, L]$. Finally, consider n point quadrature formulas of type

$$\int_{-\infty}^{\infty} f(x)\phi(x)\, dx \approx \sum_{i=0}^{n-1} w_i f(x_i), \tag{5}$$

where x_i are specially chosen abscissae and w_i are undetermined weights. We can choose the number of abscissae n arbitrarily high to obtain the desired degree of accuracy.

Given equally spaced x_i, we find weights w_i so that the formula (5) is exact for all polynomials of degree up to $n-1$. We are allowed to choose some of the abscissae x_i from outside of the $[0, L]$ if n is sufficiently large.

Our ultimate purpose is to find n point quadrature formulas which are applicable in [7]. There, we require the abscissae to be of fixed step size for all $n \geq 2$.

The multiple point formulas in [8] are not appropriate because of the inconsistent distance between two adjacent abscissae for different n. Hence we derive new n point quadrature formulas with fixed step size h ($h = 1$ for $k = 0$) for all $n \geq 2$.

2. Algorithm for finding weights w_i and abscissae x_i

Consider first for the shift $\tau = 0$. We follow a standard approach from numerical analysis.

THEOREM 2.1. *Let the level number $k \in \mathbf{Z}$ and translate $j \in \mathbf{Z}$ be fixed. Let $h = 2^{-k}$. If we choose equally spaced abscissae $x_{i,j}^k = h(j+i)$, then there exist weights $w_{i,j}^k, i = 0, 1, \ldots, n-1$ such that for $f(x) = x^p,\ p = 0, 1, \ldots, n-1$*

$$\int_{-\infty}^{\infty} f(x)\phi_j^k(x)\,dx = h^{1/2}\sum_{i=0}^{n-1} w_{i,j}^k f(x_{i,j}^k). \tag{6}$$

Moreover, for any $j, k \in \mathbf{Z}$, $w_{i,j}^k = w_{i,0}^0, \quad i = 0, 1, \ldots, n-1$.

Proof. (6) leads to a system of n linear equations for $w_{i,j}^k$

$$h^{p+1/2}\sum_{i=0}^{n-1} w_{i,j}^k (j+i)^p = h^{p+1/2}\mathcal{M}_{p,j} \qquad \text{for } p = 0, 1, \ldots, n-1. \tag{7}$$

Matrix form of (7) is

$$\mathcal{A}\vec{w} = \vec{b}, \tag{8}$$

where

$$\begin{aligned}
\mathcal{A} &= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ j & j+1 & j+2 & \cdots & j+n-1 \\ j^2 & (j+1)^2 & (j+2)^2 & \cdots & (j+n-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j^{n-1} & (j+1)^{n-1} & (j+2)^{n-1} & \cdots & (j+n-1)^{n-1} \end{pmatrix}, \\
\vec{w} &= (w_{0,j}^k, w_{1,j}^k, w_{2,j}^k, \ldots, w_{n-1,j}^k)^T, \\
\vec{b} &= (\mathcal{M}_{0,j}, \mathcal{M}_{1,j}, \mathcal{M}_{2,j}, \ldots, \mathcal{M}_{n-1,j})^T.
\end{aligned}$$

For the existence of weights $w_{i,j}^k, i = 0, 1, \ldots, n-1$, it suffices to show that $\det \mathcal{A} \neq 0$. By performing elementary row operatios on $\mathcal{A}$ we obtain

$$\mathcal{V}\vec{w} = \vec{d}, \tag{9}$$

where

$$\begin{aligned} \mathcal{V} &= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 1^2 & 2^2 & \cdots & (n-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1^{n-1} & 2^{n-1} & \cdots & (n-1)^{n-1} \end{pmatrix} \\ \vec{w} &= (w_{0,j}^k, w_{1,j}^k, w_{2,j}^k, \dots, w_{n-1,j}^k)^T, \\ \vec{d} &= (\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_{n-1})^T. \end{aligned}$$

Note that the transpose matrix $\mathcal{V}^T$ of $\mathcal{V}$ is the Vandermonde matrix with $0, 1, \dots, n-1$ and $\det \mathcal{V} = \prod_{i=1}^{n-1} i! \neq 0$ ([1]).

Since the determinant of a matrix does not change by elementary row operations,

$$\det \mathcal{A} = \det \mathcal{V} = \prod_{i=1}^{n-1} i! \neq 0 \tag{10}$$

This completes the existence of weights $w_{i,j}^k, i = 0, 1, \dots, n-1$.

Since (9) is the case $j = 0$ and $k = 0$ in (8), we obtain

$$w_{i,j}^k = w_{i,0}^0, \quad i = 0, 1, \dots, n-1.$$

This completes the proof. □

Consider the case with the shift τ.

THEOREM 2.2. *Let the shift τ be given. Let the level number $k \in \mathbf{Z}$ and translate $j \in \mathbf{Z}$ be fixed. Let $h = 2^{-k}$. If we choose equally spaced abscissae $x_{i,j}^k = h(\tau + j + i)$, then there exist weights $w_{i,j}^k, i = 0, 1, \dots, n-1$ such that for $f(x) = x^p$, $p = 0, 1, \dots, n-1$*

$$\int_{-\infty}^{\infty} f(x)\phi_j^k(x)\,dx = h^{1/2} \sum_{i=0}^{n-1} w_{i,j}^k f(x_{i,j}^k). \tag{11}$$

Moreover, for any $j, k \in \mathbf{Z}$, $w_{i,j}^k = w_{i,0}^0, \quad i = 0, 1, \dots, n-1$.

Theorem 2.2 can be proved in the same way as Theorem 2.1.

REMARK 2.3. One of the advantages of the above algorithm is that finding weights once for $w^0_{i,0}$ covers $w^k_{i,j}$ for any integer translate j and level number k.

REMARK 2.4. Above algorithm is adaptable and is easy to implement for any positive integer n.

The matrix $\mathcal{V}$ becomes ill-conditioned as n increases. See Table 1 for the condition numbers $\kappa(\mathcal{V})$.

TABLE 1. The condition numbers $\kappa(\mathcal{V})$.

n	$\kappa(\mathcal{V})$	n	$\kappa(\mathcal{V})$
3	1.3912e+01	8	5.2938e+07
4	1.5446e+02	9	2.0437e+09
5	2.5929e+03	10	9.0078e+10
6	5.7689e+04	11	4.4628e+12
7	1.5973e+06	12	2.4564e+14

To overcome the ill-conditioning problem, the Chebyshev polynomials were considered in [8]. Instead of using the standard polynomials $f(x) = x^p$, $p = 0, 1, \ldots, n-1$ in (6) and their continuous moments, they used the Chebyshev polynomials $f(x) = T_p(x)$, $p = 0, 1, \ldots, n-1$ in (6) and their continuous moments. The Chebyshev polynomial $T_p(x)$ of degree of p is defined by

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_p(x) &= 2xT_{p-1}(x) - T_{p-2}(x), \qquad \text{for } p \geq 2. \end{aligned}$$

We have not pursued this aspect very far, since the numerical problems only occur in the derivation of the weights, not in the application, and only for higher levels of accuracy.

3. Error estimates

Fix the number of abscissae n and the level k. Let τ be the shift. The error of n point quadrature formula for the monomial x^p at resolution

$h = 2^{-k}$ (for the integer translate $j = 0$) is

$$\begin{aligned} e_p(\tau) &:= \int_{-\infty}^{\infty} x^p \phi_0^k(x)\,dx - h^{1/2} \sum_{i=0}^{n-1} w_i x_i^p \\ &= h^{1/2} \int_{-\infty}^{\infty} (hx)^p \phi(x)\,dx - h^{p+(1/2)} \sum_{i=0}^{n-1} w_i (\tau + i)^p \\ &= h^{p+(1/2)} \left[\mathcal{M}_p - \sum_{i=0}^{n-1} w_i (\tau + i)^p \right] \end{aligned}$$

Let

$$\tilde{e}_p(\tau) = h^{-p-(1/2)} e_p(\tau) = \mathcal{M}_p - \sum_{i=0}^{n-1} w_i (\tau + i)^p.$$

By (11), $e_p \equiv 0 \equiv \tilde{e}_p$, $p = 0, 1, \ldots, n-1$.

If $f \in C^n$, then, by using the Taylor expansion,

$$f(xh + jh) = \sum_{l=0}^{n-1} \frac{f^{(l)}(jh)}{l!} x^l h^l + \frac{f^{(n)}(\xi)}{n!} x^n h^n,$$

for some ξ between jh and $xh + jh$. Now

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) \phi_j^k(x)\,dx &= h^{1/2} \int_{-\infty}^{\infty} f(xh + jh) \phi(x)\,dx \\ &= \sum_{l=0}^{n-1} \frac{f^{(l)}(jh)}{l!} h^{l+(1/2)} \mathcal{M}_l + \rho_n h^{n+(1/2)}, \end{aligned}$$

where

$$|\rho_n| \leq \frac{1}{n!} \left\{ \max_{\xi \in \mathbf{R}} |f^{(n)}(\xi)| \right\} \left\{ \int_{-\infty}^{\infty} |x|^n |\phi(x)|\,dx \right\}.$$

Also

$$\begin{aligned} h^{1/2} \sum_{i=0}^{n-1} w_i f(x_{i,j}^k) &= h^{1/2} \sum_{i=0}^{n-1} w_i f((\tau + i)h + jh) \\ &= \sum_{l=0}^{n-1} \frac{f^{(l)}(jh)}{l!} h^{l+(1/2)} \left[\sum_{i=0}^{n-1} w_i (\tau + i)^l \right] + \mathcal{O}(h^{n+(1/2)}). \end{aligned}$$

The error of the n point quadrature formula is then

$$\begin{aligned} E_n(f) &:= \int_{-\infty}^{\infty} f(x)\phi_j^k(x)\,dx - h^{1/2}\sum_{i=0}^{n-1} w_i f(x_{i,j}^k) \\ &= \sum_{l=0}^{n-1} \frac{f^{(l)}(jh)}{l!} h^{l+(1/2)} \left[\mathcal{M}_l - \sum_{i=0}^{n-1} w_i(\tau+i)^l\right] + \mathcal{O}(h^{n+(1/2)}) \\ &= \sum_{l=0}^{n-1} \frac{f^{(l)}(jh)}{l!} h^{l+(1/2)} \tilde{e}_l(\tau) + \mathcal{O}(h^{n+(1/2)}) \\ &= \mathcal{O}(h^{n+(1/2)}). \end{aligned}$$

If $f \in C^{n+1}$, we can take the Taylor series expansion one step further:

$$\begin{aligned} E_n(f) &:= \int_{-\infty}^{\infty} f(x)\phi_j^k(x)\,dx - h^{1/2}\sum_{i=0}^{n-1} w_i f(x_{i,j}^k) \\ &= \frac{f^{(n)}(jh)}{n!} h^{n+(1/2)} \tilde{e}_n(\tau) + \mathcal{O}(h^{n+(3/2)}). \end{aligned}$$

If the shift τ is a zero of $\tilde{e}_n(\tau)$, it is a *superconverging shift*, with error of $\mathcal{O}(h^{n+(3/2)})$ for the quadrature formula based on $x_{i,j}^k = h(\tau + i + j)$.

It is not obvious how to find the superconverging shift τ with $\tilde{e}_n(\tau)$, since weights w_i are functions of τ. So we follow the method introduced in [8].

Let the level $k = 0$ and the integer translate $j = 0$. The value of the superconverging shift τ can be determined using the product polynomial $\prod(x)$. This polynomial is defined as

$$\prod(x) = \prod_{i=0}^{n-1}(x - x_{i,0}^0) = \prod_{i=0}^{n-1}(x - \tau - i) =: \sum_{i=0}^{n} p_i(\tau)x^i$$

where $p_i(x)$ is a polynomial of degree $n - i$. Since the degree of accuracy is n, the quadrature formula gives the exact result for the product polynomial $\prod(x)$. Hence

$$0 = \int_{-\infty}^{\infty} \prod(x)\phi(x)\,dx - \sum_{i=0}^{n-1} w_i \prod(x_i) = \sum_{i=0}^{n} p_i(\tau)\mathcal{M}_i := q(\tau).$$

The latter expression is a polynomial of degree n in τ. The existence of a root τ is guaranteed for odd n, but not for even n. If there is no root, an arbitrary value for τ, e.g. $\tau = 0$, must be chosen and one degree of accuracy is lost. With the superconverging shift τ and weights w_i we obtain

$$\sum_{i=0}^{n-1} w_i(i+\tau)^n = \mathcal{M}_n. \tag{12}$$

4. Numerical examples

4.1. Examples of the shift τ and weights w_i

Let the level $k = 0$ and the integer translate $j = 0$. Given $\tau = 0$ we find weights w_i in terms of continuous moments.

- For $n = 2$ we obtain

$$\begin{aligned} w_0 &= 1 - \mathcal{M}_1, \\ w_1 &= \mathcal{M}_1. \end{aligned}$$

- For $n = 3$ we obtain

$$\begin{aligned} w_0 &= 1 - \frac{3}{2}\mathcal{M}_1 + \frac{1}{2}\mathcal{M}_2, \\ w_1 &= 2\mathcal{M}_1 - \mathcal{M}_2, \\ w_2 &= -\frac{1}{2}\mathcal{M}_1 + \frac{1}{2}\mathcal{M}_2. \end{aligned}$$

- For $n = 4$ we obtain

$$\begin{aligned} w_0 &= 1 - \frac{11}{6}\mathcal{M}_1 + \mathcal{M}_2 - \frac{1}{6}\mathcal{M}_3, \\ w_1 &= 3\mathcal{M}_1 - \frac{5}{2}\mathcal{M}_2 + \frac{1}{2}\mathcal{M}_3, \\ w_2 &= -\frac{3}{2}\mathcal{M}_1 + 2\mathcal{M}_2 - \frac{1}{2}\mathcal{M}_3, \\ w_3 &= \frac{1}{3}\mathcal{M}_1 - \frac{1}{2}\mathcal{M}_2 + \frac{1}{6}\mathcal{M}_3. \end{aligned}$$

We find the superconverging shift τ and weights w_i.

- For $n = 1$:

  Since no step size is involved in the one point formulas, the one point formulas are the same as those in [8]. Hence $\tau = \mathcal{M}_1$ and $w_0 = 1$.

  Note that the one point quadrature formula with the point $x_0 = \mathcal{M}_1$ was introduced at first in [2].

  If ϕ is an orthogonal scaling function with $M \geq 2$ vanishing moments for the corresponding wavelet ψ, then $\mathcal{M}_2 = {\mathcal{M}_1}^2$ ([5, 8]). The one point quadrature formula $f(\mathcal{M}_1)$ for the wavelet coefficients of f has degree of accuracy 2 ([8]).

- For $n = 2$:

  The product function is

$$\prod(x) = (x - \tau)(x - \tau - 1) = x^2 - (2\tau + 1)x + \tau(\tau + 1).$$

The polynomial $q(\tau)$ is

$$q(\tau) = \tau^2 + (-2\mathcal{M}_1 + 1)\tau + (\mathcal{M}_2 - \mathcal{M}_1)$$

so that

$$\tau = \frac{2\mathcal{M}_1 - 1 \pm \sqrt{1 + 4{\mathcal{M}_1}^2 - 4\mathcal{M}_2}}{2}.$$

If ϕ is the Daubechies scaling function with one vanishing moment for the corresponding wavelet ψ, then $\mathcal{M}_1 = 1/2$ and $\mathcal{M}_2 = 1/3$. The discriminant is then $1 + 4{\mathcal{M}_1}^2 - 4\mathcal{M}_2 = 2/3 > 0$ so that $\tau = \pm\sqrt{6}/6$. The weights for the shift $\tau = -\sqrt{6}/6$ can be found by solving

$$\begin{pmatrix} 1 & 1 \\ -\sqrt{6}/6 & 1 - \sqrt{6}/6 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \mathcal{M}_1 \end{pmatrix}.$$

This yields

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 - \sqrt{6}/6 & -1 \\ \sqrt{6}/6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathcal{M}_1 \end{pmatrix} = \begin{pmatrix} (3 - \sqrt{6})/6 \\ (3 + \sqrt{6})/6 \end{pmatrix}$$

which implies that

$$\begin{aligned} \tau^2 w_0 + (\tau + 1)^2 w_1 &= \left(-\frac{\sqrt{6}}{6}\right)^2 \left(\frac{3 - \sqrt{6}}{6}\right) + \left(-\frac{\sqrt{6}}{6} + 1\right)^2 \left(\frac{3 + \sqrt{6}}{6}\right) \\ &= \frac{1}{3} = \mathcal{M}_2. \end{aligned}$$

Hence equation (12) is satisfied. The weights for the shift $\tau = \sqrt{6}/6$ are $w_0 = (3+\sqrt{6})/6$ and $w_1 = (3-\sqrt{6})/6$.

If ϕ is an orthogonal scaling function with $M \geq 2$ vanishing moments for the corresponding wavelet ψ, then $\mathcal{M}_2 = \mathcal{M}_1{}^2$ ([5, 8]). The discriminant is $1 + 4\mathcal{M}_1{}^2 - 4\mathcal{M}_2 = 1 > 0$. Hence there always exist two distinct real τ, and

$$\tau = \mathcal{M}_1 \text{ or } \mathcal{M}_1 - 1.$$

The weights for the shift $\tau = \mathcal{M}_1$ can be found by solving

$$\begin{pmatrix} 1 & 1 \\ \mathcal{M}_1 & \mathcal{M}_1 + 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \mathcal{M}_1 \end{pmatrix}.$$

This yields

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} \mathcal{M}_1 + 1 & -1 \\ -\mathcal{M}_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mathcal{M}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

which implies that

$$\tau^2 w_0 + (\tau+1)^2 w_1 = \mathcal{M}_1{}^2 1 + (\mathcal{M}_1 + 1)^2 0 = \mathcal{M}_1{}^2 = \mathcal{M}_2.$$

Hence equation (12) is satisfied. The weights for the shift $\tau = \mathcal{M}_1 - 1$ are $w_0 = 0$ and $w_1 = 1$.

Since one of the weights, w_1, is zero for the shift $\tau = \mathcal{M}_1$, two point quadrature formulas become one point quadrature formulas with degree of accuracy 2. This result is the same as that for one point quadrature formulas.

REMARK 4.1. *Let ϕ be an orthogonal scaling function with M vanishing moments for the corresponding wavelet ψ. If the two point quadrature formulas have degree of accuracy 2, then $M = 1$.*

Table 2 shows the shift τ and weights w_i for some M, where ϕ is a Daubechies scaling function with M vanishing moments for the corresponding wavelet ψ.

- For $n = 3$:

The product function is

$$\begin{aligned} \prod(x) &= (x-\tau)(x-\tau-1)(x-\tau-2) \\ &= x^3 - 3(\tau+1)x^2 - (3\tau^2 + 6\tau + 2)x - (\tau^3 + 3\tau^2 + 2\tau). \end{aligned}$$

TABLE 2. The superconverging shift τ and weights w_i for $n = 2$.

n	M	τ	w_0	w_1
2	1	-4.0825e-01	9.1752e-02	9.0825e-01
		4.0825e-01	9.0825e-01	9.1752e-02
	2	6.3397e-01	1.0000e+00	0.0000e+00
		-3.6603e-01	0.0000e+00	1.0000e+00
	3	8.1740e-01	1.0000e+00	0.0000e+00
		-1.8260e-01	0.0000e+00	1.0000e+00
	4	1.0054e+00	1.0000e+00	0.0000e+00
		5.3932e-03	0.0000e+00	1.0000e+00
	5	1.1939e+00	1.0000e+00	0.0000e+00
		1.9391e-01	0.0000e+00	1.0000e+00

The polynomial $q(\tau)$ is

$$q(\tau) = -\tau^3 + 3(\mathcal{M}_1 - 1)\tau^2 + (3\mathcal{M}_2 + 6\mathcal{M}_1 - 2)\tau + (\mathcal{M}_3 - 3\mathcal{M}_2 + 2\mathcal{M}_1).$$

Table 3 shows the shift τ and weights w_i for some M, where ϕ is a Daubechies scaling function with M vanishing moments for the corresponding wavelet ψ.

4.2. Examples of error and convergence order

We fix the shift $\tau = 0$ for all the examples in this section.

EXAMPLE 4.2. In order to compare our results to those in [8], we take ϕ and f as in [8], i.e., let ϕ be the Daubechies scaling function with $M = 3$ vanishing moments for the corresponding wavelet ψ, and $f(x) = \sin(x)$. Table 4 shows the weights for $n = 5$ and $n = 10$. As we can see from Table 4, w_1 is always the biggest for any n, because $\mathcal{M}_1 = 0.8174 \approx 1$.

Let Q_n be the n point quadrature formula introduced in [8]. Table 5 shows the error of the n point quadrature formulas with $\tau = 0$ for $n = 5$ and $n = 10$, and the error of Q_5 and Q_{10} which are in Table 2.1 of [8].

The absolute error of our formula for $n = 5$ is smaller than Q_5 for $k \geq 2$, but greater than Q_5 for $k \leq 1$. The absolute error for $n = 10$ is greater than Q_{10} for $2 \leq k \leq 5$. This is because some of the abscissae

TABLE 3. The superconverging shift τ and weights w_i for $n = 3$.

n	M	τ	w_0	w_1	w_2
3	1	-1.3660e+00	-1.6346e-02	1.6667e-01	8.4968e-01
		-5.0000e-01	4.1667e-02	9.1667e-01	4.1667e-02
		3.6603e-01	8.4968e-01	1.6667e-01	-1.6346e-02
	2	-1.4229e+00	3.0074e-02	-1.1706e-01	1.0870e+00
		5.6518e-01	8.9917e-01	1.3286e-01	-3.2031e-02
		-2.4032e-01	7.0753e-02	9.8420e-01	-5.4951e-02
	3	-1.2296e+00	2.4593e-02	-9.6165e-02	1.0716e+00
		7.6264e-01	9.1936e-01	1.0651e-01	-2.5879e-02
		-8.0864e-02	5.6043e-02	9.8965e-01	-4.5693e-02
	4	-1.0452e+00	2.6555e-02	-1.0367e-01	1.0771e+00
		9.4570e-01	9.1224e-01	1.1582e-01	-2.8064e-02
		1.1564e-01	6.1200e-02	9.8785e-01	-4.9046e-02
	5	1.1265e+00	9.0110e-01	1.3035e-01	-3.1450e-02
		-8.6208e-01	2.9562e-02	-1.1511e-01	1.0855e+00
		3.1734e-01	6.9337e-02	9.8476e-01	-5.4100e-02

fall outside of the support of ϕ_j^k for $n = 10$ and also because we use the shift $\tau = 0$.

TABLE 4. Weights w_i for $n = 5$ and $n = 10$.

weights	$n = 5$	$n = 10$
w_0	9.0735e-02	7.1852e-02
w_1	1.0230e+00	1.1499e+00
w_2	-1.4013e-01	-5.2157e-01
w_3	3.1030e-02	7.0958e-01
w_4	-4.5979e-03	-7.9913e-01
w_5		6.3929e-01
w_6		-3.5404e-01
w_7		1.2961e-01
w_8		-2.8267e-02
w_9		2.7845e-03

EXAMPLE 4.3. Let ϕ be the Daubechies scaling function with M vanishing moments for the corresponding wavelet ψ. Let $f(x) = \cos(2\pi x)$.

TABLE 5. Errors of the quadrature formulas for $n = 5$ and $n = 10$.

k	$n = 5$ (Q_5)	$n = 10$ (Q_{10})
0	3.0239e-03 (2.15e-03)	-
1	3.9833e-05 (4.40e-05)	4.4891e-07 (1.03e-08)
2	2.6513e-07 (6.51e-07)	1.0228e-10 (1.11e-12)
3	1.5325e-09 (9.38e-09)	9.7172e-14 (4.21e-15)
4	8.5614e-12 (1.38e-10)	2.0747e-15 (9.99e-16)
5	4.7431e-14 (2.09e-12)	-
6	2.6173e-16 (3.19e-14)	-
7	1.5179e-18 (1.11e-16)	-

Then $f(x+1) = f(x)$. Fix level $k = 3$. Let $M = 2$. We test for some n. Errors of quadrature formulas in norms l^1, l^2, and l^∞ are in Table 6. As we expect, the errors become smaller as n increases.

TABLE 6. Errors of quadrature formulas with fixed level k.

k	M	n	error		
			l^1	l^2	l^∞
3	2	1	1.1476e-01	1.2536e-01	1.7142e-01
		2	1.5528e-02	1.7056e-02	2.3508e-02
		3	3.3179e-03	3.7507e-03	5.2812e-03
		4	1.6155e-03	1.7763e-03	2.4756e-03
		5	8.7432e-04	9.5906e-04	1.3125e-03
		6	4.7909e-04	5.3749e-04	7.6567e-04

EXAMPLE 4.4. Let ϕ be the Daubechies scaling function with M vanishing moments for the corresponding wavelet ψ. Let $f(x) = \cos(2\pi x)$. Then $f(x+1) = f(x)$. Fix $n = 3$. Let $M = 2$. We test for some k. Errors and convergence orders for quadrature formulas in norms l^1, l^2, and l^∞ are in Table 7. As we expect, the convergence orders for n point quadrature formulas approach $n + (1/2)$ as k increases.

TABLE 7. Errors and convergence orders for quadrature formulas with fixed n.

n	M	k	error			convergence order		
			l^1	l^2	l^∞	l^1	l^2	l^∞
3	2	3	3.3179e-03	3.7507e-03	5.2812e-03			
		4	3.0608e-04	3.4148e-04	4.8238e-04	3.4383	3.4573	3.4526
		5	2.7343e-05	3.0404e-05	4.2987e-05	3.4847	3.4894	3.4882
		6	2.4234e-06	2.6924e-06	3.8074e-06	3.4961	3.4973	3.4970

5. Summary

For the numerical calculation of the wavelet coefficients of f,

$$f_j^k := \langle f, \phi_j^k \rangle = \int_{-\infty}^{\infty} f(x)\phi_j^k(x)\, dx,$$

we can choose a positive integer n arbitrarily large to obtain the desired degree of accuracy. We then use n point quadrature formulas of type

$$\int_{-\infty}^{\infty} f(x)\phi_j^k(x)\, dx \approx h^{1/2} \sum_{i=0}^{n-1} w_i f(x_{i,j}^k).$$

We choose the abscissae

$$x_{i,j}^k = h(\tau + i + j),$$

in order to guarantee that the abscissae are equally spaced with fixed step size $h = 2^{-k}$ for all $n \geq 2$.
The error of n point quadrature formulas with the superconverging shift τ is $\mathcal{O}(h^{n+(3/2)})$. With all shifts τ other than superconverging τ the error is $\mathcal{O}(h^{n+(1/2)})$. Hence we achieve the same degree of accuracy and convergence order as in [8] from our derivations for quadrature formulas. Moreover, we obtain the same weights for any integer translate j and level k.

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