J. Korean Math. Soc. **34** (1997), No. 4, pp. 895–909

IDENTIFICATION PROBLEM FOR DAMPING PARAMETERS IN LINEAR DAMPED SECOND ORDER SYSTEMS

JONG-YEOUL PARK*, JUN-HONG HA AND HYO-KEUN HAN

ABSTRACT. We state the necessary conditions on optimizing the parameters which the damping differential operators contain in abstract linear damped second order evolution equation on the Gelfand five fold.

1. Introduction

A widely used approach to the identification problem for any system is to estimate the unknown parameters appear in the system by minimizing a quadratic function of the difference between observed value and desired value, so-called output least-square identification problem (OLSIP). We consider the system given by linear damped second order evolution equations, which we refer to Dautray and Lions[5] as a model, of the forms

$$\frac{d^2y}{dt^2} + A_2(q,t)\frac{dy}{dt} + A_1(q,t)y = f \quad \text{in} \quad (0,T) \tag{1.1}$$

with the initial values, and the cost functional given by the quadratic form

$$J(q) = \|Cy - z_d\|^2_{\mathcal{M}}, \tag{1.2}$$

where $A_1(t,q), A_2(t,q)$ are differential operators containing unknown parameter $q \in Q$ which are given by some bilinear forms on Hilbert spaces, C is an observation operator defined on an observation space $\mathcal{M}$, z_d is a desired value. The OLSIP suject to (1.1) with (1.2) is to find an element

Received April 8, 1997.

1991 Mathematics Subject Classification: 35L10, 49K20, 49N50.

Key words and phrases: identification, optimal control.

* This work was partial supported by the Basic Science Research Institute Program, Ministry Education, 1996, Project No. BSRI-96-1410.

$\overline{q} \in Q$ such that $\inf_{q \in Q} J(q) = J(\overline{q})$ and to give a characterization of such $\overline{q}$. In this paper we will study the OLSIP to the system (1.1) with (1.2). It is not easy to find the elements $\overline{q}$ belonging to an admissible set Q of parameters subject to (1.1) with (1.2). That is, we have no general answer for it. Hence we will show the existence of such $\overline{q}$ when Q is a compact subset of a topological space, and then we are going to concentrate on giving a characterization of such $\overline{q}$. Recently, inspired by the optimal control theoretical studies of Euler-Bernoulli Beam Equations with Kelvin-Voigt Damping, and Love-Kirchoff plate equations with various damping terms, there appeared numerous papers studying optimal control theory and identification problem for the autonomous case of (1.1) on the Gelfand triple spaces.
In Banks, Ito and Wang [3], Banks and Kunisch[4], they treated the existence of the optimal controls (or minimizing parameters) by using the method of approximations, but they didn't deal with the necessary conditions (or characterizations) on them. When $A_1(t,q) \equiv \gamma A_2(t,q), \gamma > 0$ in (1.1), the identification problem estimating q via OLSIP is studied by Ahmed[1, 2] based on the transposition method.

The aim of this paper is to study the identification problems through the method of OLSIP to (1.1) on the Gelfand five fold which will be stated in preliminary. As using the Gelfand five fold structure we may have some advantages that the operators $A_1(t,q)$ and $A_2(t,q)$ can be defined with free differential orders in spatial sense. This paper is composed of the parts of preliminaries as section 2, necessary conditions as section 3 and applications to partial differential equations as section 4.

2. Preliminaries

First we explain the notations used in this paper. Let X be a Hilbert space. $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ denote the inner product and the induced norm on X. X' denotes the dual space of X and $\langle \cdot,\cdot \rangle_{X',X}$ denotes the dual pairing between X' and X. Let us introduce underlying Hilbert spaces to describe damped second order evolution equations. Let H be a real pivot Hilbert space, its norm $\|\cdot\|_H$ is denoted simply by $|\cdot|_H$. For $i = 1, 2$, let V_i be a real separable Hilbert space. Assume that each pair (V_i, H) is a Gelfand triple space with the notation, $V_i \hookrightarrow H \equiv H' \hookrightarrow V_i'$. From now on, we write $V_1 = V$ for notational convenience. We shall give an exact description of damped second order evolution equations. We suppose that Q is algebraically contained in a linear topological vector space with topology τ and $Q_\tau = (Q,\tau)$ is closed. Let $T > 0$ be fixed.

In order to define two main differential operators, diffusion and damping, we introduce two bilinear forms $a_i(t,q;\phi,\varphi), i = 1,2, q \in Q_\tau, t \in [0,T]$ on $V_i \times V_i$ satisfying

(2.1) $a_i(t,q;\phi,\varphi) = a_i(t,q;\varphi,\phi)$ for all $\phi,\varphi \in V_i$ and $t \in [0,T]$,
there exists $c_{i1} > 0$ such that

(2.2) $|a_i(t,q;\phi,\varphi)| \le c_{i1}\|\phi\|_{V_i}\|\varphi\|_{V_i}$ for all $\phi,\psi \in V_i$ and $t \in [0,T]$
and there exist $\alpha_i > 0$ and $\lambda_i \in \mathbf{R}$ such that

(2.3) $a_i(t,q;\phi,\phi) + \lambda_i|\phi|_H^2 \ge \alpha_i\|\phi\|_{V_i}^2$ for all $\phi \in V_i$ and $t \in [0,T]$,
the function $t \to a_i(t,q;\phi,\varphi)$ is continuously differentiable in $[0,T]$and there exists $c_{i2} > 0$ such that

(2.4) $|a_i'(t,q;\phi,\varphi)| \le c_{i2}\|\phi\|_{V_i}\|\varphi\|_{V_i}$ for all $\phi,\psi \in V_i$ and $t \in [0,T]$,

where $' = \frac{d}{dt}$. Then we can define the operator $A_1(t,q) \in \mathcal{L}(V_i,V_i')$ for $t \in [0,T]$ deduced by the relation

$$a_i(t,q;\phi,\varphi) = \langle A_i(t,q)\phi,\varphi\rangle_{V_i',V_i} \text{ for all } \phi,\varphi \in V_i. \tag{2.5}$$

We suppose that $V_1 \equiv V$ is continuously embedded in V_2. Then we see that $V \hookrightarrow V_2 \hookrightarrow H \equiv H' \hookrightarrow V_2' \hookrightarrow V'$ and the equalities $\langle\phi,\varphi\rangle_{V',V} = \langle\phi,\varphi\rangle_{V_2',V_2}$ for $\phi \in V_2', \varphi \in V$ and $\langle\phi,\varphi\rangle_{V',V} = (\phi,\varphi)_H$ for $\phi \in H, \varphi \in V$ hold.

We often use the notations to express the time derivatives as g', g'' instead of $\frac{dg}{dt}, \frac{d^2g}{dt^2}$, respectively. We define a Hilbert space, which will be a solution space, as

$$W(0,T) = \{g | g \in L^2(0,T;V),\ g' \in L^2(0,T;V_2),\ g'' \in L^2(0,T;V')\}$$

with an inner product

$$(g_1,g_2)_{W(0,T)} = \int_0^T \{(g_1(t),g_2(t))_V + (g_1'(t),g_2'(t))_{V_2} + (g_1''(t),g_2''(t))_{V'}\}dt$$

and the induced norm

$$\|g\|_{W(0,T)} = \left(\|g\|^2_{L^2(0,T;V)} + \|g'\|^2_{L^2(0,T;V_2)} + \|g''\|^2_{L^2(0,T;V')}\right)^{\frac{1}{2}}.$$

In OLSIP, many adjoint state equations have a variety of perturbed terms as depending on observations. Hence we will treat existence, uniqueness and regularity for the more generalized equation than (1.1), i.e., we consider the Cauchy problem for the perturbed linear damped second order evolution equation of the form

$$\left\{\begin{array}{l} y'' + A_2(t,q)y' + A_1(t,q)y = L(t)y + f \ \text{ in } \ (0,T), \\ y(0) = y_0 \in V, \\ y'(0) = y_1 \in H, \end{array}\right. \tag{2.6}$$

where $L(\cdot) \in L^\infty(0,T;\mathcal{L}(V_2,V_2'))$ and $f \in L^2(0,T;V_2')$. Then we can show that there is a unique weak solution $y \in W(0,T) \cap C([0,T];V) \cap C^1([0,T];H)$ to (2.6) in the sense that

$$\left\{\begin{array}{l} \langle y''(\cdot),\phi\rangle_{V',V} + a_2(\cdot,q;y'(\cdot),\phi) + a_1(\cdot,q;y(\cdot),\phi) \\ \quad = \langle L(\cdot)y(\cdot) + f(\cdot),\phi\rangle_{V_2',V_2} \text{for all } \ \phi \in V \ \text{ in the sense of } \ \mathcal{D}'(0,T) \\ \text{with the initial conditions} \\ y(q;0) = y_0 \in V, \quad y'(q;0) = y_1 \in H. \end{array}\right.$$

Moreover, the solution y to (2.6) satisfies the energy equality for each $t \in [0,T]$

$$\begin{aligned} & a_1(t,q;y(t),y(t)) + |y'(t)|_H^2 + 2\int_0^t a_2(\sigma,q;y'(\sigma),y'(\sigma))d\sigma \\ & = a_1(0,q;y_0,y_0) + |y_1|_H^2 + \int_0^t a_1'(\sigma,q;y(\sigma),y(\sigma))d\sigma \\ & +2\int_0^t \langle L(\sigma)y(\sigma) + f(\sigma),y'(\sigma)\rangle_{V_2',V_2}d\sigma. \end{aligned} \tag{2.7}$$

For the proofs of these we refer to Ha [6], and Dautray and Lions [5] whose treatment of equations was done on the Gelfand triple structure by the energy methods.

3. Necessary conditions

In this paper we consider the case where all the parameters q related to the diffusion operator $A_1(t,q)$ has already known, i.e., the damping operator $A_2(t,q)$ contains unknown parameters only. Hence letting $A_1(t,q) \equiv A_1(t)$ the system (2.6) is written by

$$\left\{\begin{array}{l} y'' + A_2(t,q)y' + A_1(t)y = f \ \text{ in } \ (0,T), \\ y(q;0) = y_0 \in V, \\ y'(q;0) = y_1 \in H. \end{array}\right. \tag{3.1}$$

Note that since there is a unique solution y to (3.1) for given $q \in Q_\tau$, we can have a well-defined mapping $y = y(q)$ of Q_τ into $W(0,T)$. We often call (3.1) the state equation and $y(q)$ the state with respect to (3.1).

Let us consider a quadratic cost functional attached to (2.6) as

$$(3.2) \qquad J(q) = \|\mathcal{C}y(q) - z_d\|^2_{\mathcal{M}}, \quad q \in Q_\tau,$$

where $\mathcal{M}$ is a Hilbert space of observations, $\mathcal{C} \in \mathcal{L}(W(0,T), \mathcal{M})$ is an observer and z_d is a desired value belonging to $\mathcal{M}$. Our aim is to find $\overline{q} \in Q_\tau$ satisfying

$$(3.3) \qquad J(\overline{q}) = \min_{q \in Q_\tau} J(q)$$

and to give a characterization of such $\overline{q}$. We call $\overline{q}$ the optimal control (or the minimizing parameter) to the system (3.1) and (3.2). Furthermore, we give an assumption to $a_2(t, q; \phi, \varphi)$:

(3.4)

$$q \to a_2(t, q; \phi, \varphi) : Q_\tau \to \mathbf{R} \text{ is continuous for all } t \in [0,T], \phi, \varphi \in V_2.$$

Note that for each $q \in Q_\tau, \phi, \varphi \in V_2$ the following equalities hold:

$$\sup_{\|\varphi\|_{V_2}=1} |a_2(t,q;\phi,\varphi)| = \sup_{\|\varphi\|_{V_2}=1} |\langle A_2(t,q)\phi, \varphi\rangle_{V_2', V_2}| = \|A_2(t,q)\phi\|_{V_2'},$$

whence the assumption (3.4) and the above equality imply that $\|A_2(t,q)\phi\|_{V_2'}$ is continuous on q.

LEMMA 3.1. *Let us assume that* (2.1)-(2.4) *and* (3.4) *hold. Then* $y(q)$ *is strongly continuous on* q, *i.e.*, $y(q) \in C(Q_\tau, W(0,T))$.

Proof. Let us suppose that $q_n \to q$ in Q_τ and let $y_n = y(q_n), y = y(q)$ be the solutions corresponding to q_n, q, respectively. Then by letting $z_n = y_n - y$ we obtain the equation

$$(3.5) \qquad z_n'' + A_2(t,q_n)z_n' + A_1(t)z_n = [A_2(t,q) - A_2(t,q_n)]y'.$$

Since $[A_2(t,q) - A_2(t,q_n)]y' \in L^2(0,T; V_2')$, we can apply (3.5) to (2.7). Hence we have from $z_n(0) = z_n'(0) = 0$

$$(3.6) \qquad a_1(t; z_n(t), z_n(t)) + |z_n'(t)|^2_H + 2\int_0^t a_2(\sigma, q_n; z_n'(\sigma), z_n'(\sigma))d\sigma$$
$$= \int_0^t a_1'(\sigma; z_n(\sigma), z_n(\sigma))d\sigma + 2\int_0^t [a_2(\sigma, q; y'(\sigma), z_n'(\sigma))$$
$$-a_2(\sigma, q_n; y'(\sigma), z_n'(\sigma))]d\sigma.$$

Denote $\lambda_i = |\lambda_i|, i = 1, 2$ for simplicity. If we estimate the above equality by using (2.3) and (2.4), then we have

$$\begin{aligned}(3.7)\qquad & \alpha_1\|z_n(t)\|_V^2 + |z_n'(t)|_H^2 + 2\alpha_2\int_0^t \|z_n'(\sigma)\|_{V_2}^2 d\sigma \\ & \le \lambda_1|z_n(t)|_H^2 + 2\lambda_2\int_0^t |z_n'(\sigma)|_H^2 d\sigma + c_{12}\int_0^t \|z_n(\sigma)\|_V^2 d\sigma \\ & +2\int_0^t \|[A_2(\sigma,q) - A_2(\sigma,q_n)]y'(\sigma)\|_{V_2'}\|z_n'(\sigma)\|_{V_2} d\sigma.\end{aligned}$$

Noting that $|z_n(t)|_H^2 \le T\int_0^t |z_n'(\sigma)|_H^2 d\sigma$ and using Cauchy-Schwarz inequality it follows from (3.7) that

$$\begin{aligned}(3.8)\qquad & \alpha_1\|z_n(t)\|_V^2 + |z_n'(t)|_H^2 + \alpha_2\int_0^t \|z_n'(\sigma)\|_{V_2}^2 d\sigma \\ & \le (\lambda_1 T + 2\lambda_2)\int_0^t |z_n'(\sigma)|_H^2 d\sigma + c_{12}\int_0^t \|z_n(\sigma)\|_V^2 d\sigma \\ & +\frac{1}{\alpha_2}\int_0^t \|[A_2(\sigma,q) - A_2(\sigma,q_n)]y'(\sigma)\|_{V_2'}^2 d\sigma.\end{aligned}$$

Put $\alpha = \min\{1, \alpha_1, \alpha_2\} > 0$ and $\beta = \alpha^{-1}\max\{\lambda_1 T + 2\lambda_2, c_{12}, \alpha_2^{-1}\}$ and

$$\varphi_n(t) = \|z_n(t)\|_V^2 + |z_n'(t)|_H^2.$$

It follows from (3.8) and Bellman-Gronwall's lemma that

$$(3.9)\qquad \varphi_n(t) \le \left(\int_0^T \|[A_2(\sigma,q) - A_2(\sigma,q_n)]y'(\sigma)\|_{V_2'}^2 d\sigma\right)\beta e^{\beta T}.$$

Using the continuity of $a_2(t,q;\phi,\psi)$ on q, the right hand side of (3.9) goes to zero, and so, we have $\varphi_n(t) \to 0$ for all $t \in [0,T]$. Applying this fact to (3.8) we conclude that $y_n \to y$ in $C([0,T],V)$, $y_n' \to y'$ in $C([0,T],H)$ and $y_n' \to y'$ in $L^2(0,T;V_2)$. In particular, we also have $y_n \to y$ in $W(0,T)$ if estimating (3.5). □

LEMMA 3.2. *Let us assume that* (2.1)-(2.4) *and* (3.4) *hold. There is at least one optimal control* $\overline{q}$ *if* Q_τ *is compact.*

Proof. It is clear from Lemma 3.1 and continuity of norm. □

Now we present the necessary condition (the minimizing condition) for the optimal controls $\overline{q} \in Q_\tau$ to the system (3.1) with the cost functional $J(p)$ given by (3.2). If $J(p)$ is Gâteaux differentiable at $\overline{q}$ in the direction

$q - \overline{q}$, the necessary condition on $\overline{q}$ is characterized by the following inequality

$$\delta J(\overline{q})(q - \overline{q}) \geq 0 \quad \text{for all} \quad q \in Q, \tag{3.10}$$

where $\delta J(\overline{q})(q - \overline{q})$ denotes the Gâteaux derivative at $\overline{q}$ in the direction $q - \overline{q}$. Note that since $J(q)$ is composed of the term $y(q)$, the Gâteaux differentiability of $J(q)$ follows from that of $y(q)$. Hence to obtain that of $y(q)$ we assume that $a_2(t, q; \phi, \varphi)$ satisfies the following condition:

(3.11)

$$\begin{aligned} &q \longrightarrow A_2(t,q) \quad \text{is weakly Gâteaux differentiable for all } t \\ &\text{and} \quad \delta A_2(t,q)(p) \equiv \delta A_2(t,q;p) \in L^2(0,T;\mathcal{L}(V_2,V_2')) \quad \text{for all} \quad p \in Q_T, \end{aligned}$$

where $\delta A_2(t, q; p)$ denotes the Gâteaux derivative at q in the direction of p.

LEMMA 3.3. *Let us assume that* (2.1)-(2.4), (3.4) *and* (3.11) *are satisfied. Then $y(q)$ is weakly Gâteaux differentiable at $\overline{q}$ in the direction $q - \overline{q}$, denote the Gâteaux derivative of $y(q)$ by $z = \delta y(t, \overline{q}; q - \overline{q})$, which satisfies the following Cauchy problem:*

$$\left\{ \begin{array}{l} z'' + A_2(t,\overline{q})z' + A_1(t)z = -\delta A_2(t,\overline{q};q-\overline{q})y(\overline{q})' \quad \text{in} \quad (0,T), \\ z(0) = z'(0) = 0. \end{array} \right. \tag{3.12}$$

Proof. For $\lambda \in (0,1)$ let $y_\lambda = y(q_\lambda)$ and $\overline{y} = y(\overline{q})$ be weak solutions to (3.1) for given parameters $q_\lambda = \overline{q} + \lambda(q - \overline{q})$ and $\overline{q}$, respectively. Then $z_\lambda = (y_\lambda - \overline{y})/\lambda$ satisfies the following equation

(3.13)

$$\left\{ \begin{array}{l} z_\lambda'' + A_2(t,q_\lambda)z_\lambda' + A_1(t)z_\lambda = \dfrac{A_2(t,\overline{q}) - A_2(t,q_\lambda)}{\lambda}\overline{y}' \quad \text{in} \quad (0,T), \\ z_\lambda(0) = z_\lambda'(0) = 0. \end{array} \right.$$

Since for each λ, $[(A_2(t,\overline{q}) - A_2(t,q_\lambda))/\lambda]\overline{y}' \in L^2(0,T;V_2')$, we can apply (3.13) to (2.7). Hence from $z_\lambda(0) = z_\lambda'(0) = 0$ we have

$$\begin{aligned} a_1(t; z_\lambda(t), z_\lambda(t)) + |z_\lambda'(t)|_H^2 + 2\int_0^t a_2(\sigma, q_\lambda; z_\lambda'(\sigma), z_\lambda'(\sigma))d\sigma \\ = \int_0^t a_1'(\sigma; z_\lambda(\sigma), z_\lambda(\sigma))d\sigma + \frac{2}{\lambda}\int_0^t [a_2(\sigma,\overline{q};\overline{y}'(\sigma), z_\lambda'(\sigma)) \\ -a_2(\sigma, q_\lambda; \overline{y}'(\sigma), z_\lambda'(\sigma))]d\sigma. \end{aligned} \tag{3.14}$$

Let us consider the last term in (3.14). From the assumption (3.11) and the homogeneous property $\delta A_2(t,q;\varepsilon p) = \varepsilon \delta A_2(t,q;p)$ there is $\nu \in (0,1)$ such that

$$\begin{aligned} \text{(3.15)} \qquad & \frac{1}{\lambda}\int_0^t [a_2(\sigma,\overline{q};\overline{y}'(\sigma),z_\lambda'(\sigma)) - a_2(\sigma,q_\lambda;\overline{y}'(\sigma),z_\lambda'(\sigma))]d\sigma \\ & = \int_0^t \langle -\delta A_2(\sigma,\overline{q}+\lambda\nu(q-\overline{q});q-\overline{q})\overline{y}', z_\lambda'(\sigma)\rangle_{V_2',V_2} d\sigma \end{aligned}$$

depending on $z_\lambda \in L^2(0,T;V)$. By the similar calculations to Lemma 3.1 we have the following estimation

$$\begin{aligned} \|z_\lambda(t)\|_V^2 + |z_\lambda'(t)|_H^2 + \int_0^t \|z_\lambda'(\sigma)\|_{V_2}^2 d\sigma \le C\int_0^t [|z_\lambda'(\sigma)|_H^2 + \|z_\lambda(\sigma)\|_V^2] d\sigma \\ + C\int_0^T \|\delta A_2(\sigma,\overline{q}+\lambda\nu(q-\overline{q});q-\overline{q})\overline{y}'(\sigma)\|_{V_2'}^2 d\sigma (\equiv K), \end{aligned}$$

where $C > 0$ is a proper constant. Using Bellman-Gronwall's lemma we have

$$\|z_\lambda(t)\|_V^2 + |z_\lambda'(t)|_H^2 + \int_0^t \|z_\lambda'(\sigma)\|_{V_2}^2 d\sigma \le CTKe^{CT} - K, \quad \forall \lambda \in (0,1),$$

which implies that $\{z_\lambda : \lambda \in (0,1)\}$ is bounded in $L^2(0,T;V) \subset L^\infty(0,T;V)$ and $\{z_\lambda' : \lambda \in (0,1)\}$ is bounded in $L^2(0,T;V_2) \cap L^\infty(0,T;H)$. Since $L^2(0,T;V)$ and $L^2(0,T;V_2)$ are reflexive, we can extract subsequences $\{z_{\lambda_n}\} \subset \{z_\lambda\}$ and find $z \in L^2(0,T;V)$ such that

(3.16)
$$z_{\lambda_n} \to z \text{ weakly in } L^2(0,T;V), \quad z_{\lambda_n}' \to w \text{ weakly in } L^2(0,T;V_2).$$

Since $z_{\lambda_n}' \in C([0,T],H)$ and $z_{\lambda_n}(0) = 0$ in H, we have that for each $t \in [0,T]$

$$z_{\lambda_n}(t) = \int_0^t z_{\lambda_n}'(\sigma)d\sigma$$

in the H sense. Since by (3.16) $\int_0^t z_{\lambda_n}'(\sigma)d\sigma \to \int_0^t w(\sigma)d\sigma$ weakly in H, we obtain

$$z(t) = \int_0^t w(\sigma)d\sigma$$

in the weak H sense, which implies that $z'(t)$ exists in H sense and that $w(t) = z'(t)$ in H with $z(0) = 0$. Since $w \in L^2(0,T;V_2)$, we have also $w = z'$ in $L^2(0,T;V_2)$. It follows by multiplying (3.13) by $\phi = \zeta(t)\psi, \zeta \in \mathcal{D}(0,T)$ and integrating it by parts that

$$(3.17)\int_0^T [(-z'_{\lambda_n}(t), \phi'(t))_H + a_2(t, q_{\lambda_n}; z'_{\lambda_n}(t), \phi(t)) + a_1(t; z_{\lambda_n}(t), \phi(t))]dt$$
$$= \int_0^T \langle \frac{A_2(t,\overline{q}) - A_2(t, q_{\lambda_n})}{\lambda_n} \overline{y}'(t), \phi(t)\rangle_{V_2',V_2} dt.$$

Since the assumption (3.11) induce that for all $\nu \in [0,1]$

$$|a_2(t, q_{\lambda_n}; z'_{\lambda_n}(t), \phi(t)) - a_2(t, \overline{q}; z'_{\lambda_n}(t), \phi(t))|$$
$$\leq |\lambda_n(q - \overline{q})|_\tau \|\delta A_2(\sigma, \overline{q} + \lambda\nu(q - \overline{q}); q - \overline{q})\phi\|_{V_2'} \|z'_{\lambda_n}\|_{V_2},$$

we have from (3.16)

$$(3.18) \qquad \int_0^T a_2(t, q_{\lambda_n}; z'_{\lambda_n}(t), \phi(t))dt \to \int_0^T a_2(t, \overline{q}; z'(t), \phi(t))dt$$

as $\lambda_n \to 0$. Hence if we take $\lambda_n \to 0$ in (3.17) by using (3.16), (3.11) and (3.18) then we have

$$(3.19) \quad \int_0^T [(-z'(t), \phi'(t))_H + a_2(t, \overline{q}; z'(t), \phi(t)) + a_1(t; z(t), \phi(t))]dt$$
$$= \int_0^T \langle -\delta A_2(t, \overline{q}; q - \overline{q})\overline{y}'(t), \phi(t)\rangle_{V_2',V_2} dt,$$

which implies

$$(3.20) \quad \int_0^T [\langle z''(t), \phi(t)\rangle_{V',V} + a_2(t, \overline{q}; z'(t), \phi(t)) + a_1(t; z(t), \phi(t))]dt$$
$$= \int_0^T \langle -\delta A_2(t, \overline{q}; q - \overline{q})\overline{y}'(t), \phi(t)\rangle_{V_2',V_2} dt.$$

On the other hand, it follows from taking $\phi(t) = \zeta(t)\psi$ such that $\zeta \in C^1[0,T]$, $\zeta(T) = 0$ in (3.19), that

$$(3.21) \quad \int_0^T [\langle z''(t), \psi\rangle_{V',V} + a_2(t, \overline{q}; z'(t), \psi) + a_1(t; z(t), \psi)]\zeta(t)dt$$
$$= (z'(0), \phi(0))_H + \int_0^T \langle -\delta A_2(t, \overline{q}; q - \overline{q})\overline{y}'(t), \psi\rangle_{V_2',V_2} \zeta(t)dt$$
$$\text{for all } \phi \in V.$$

Substituting (3.20) into (3.21) we have $z'(0) = 0$. □

By Lemma 3.3, the cost functional $J(q)$ is weakly Gâteaux differentiable at $\overline{q}$ in the direction $q-\overline{q}$, and so, the condition (3.10) is rewritten by

(3.22)
$$(\mathcal{C}y(\overline{q}) - z_d, \mathcal{C}z)_{\mathcal{M}} = \langle \mathcal{C}^*\Lambda_{\mathcal{M}}(\mathcal{C}y(\overline{q}) - z_d), z\rangle_{W(0,T)',W(0,T)} \geq 0, \quad \forall q \in Q_\tau,$$

where z is a unique weak solution to (3.12), $\mathcal{C}^* \in \mathcal{L}(\mathcal{M}', W(0,T)')$ is the adjoint operator of $\mathcal{C}$ and $\Lambda_{\mathcal{M}}$ is the canonical isomorphism of $\mathcal{M}$ onto $\mathcal{M}'$ in the sense that

$$\text{(i)} \quad \langle \Lambda_{\mathcal{M}}\phi, \phi\rangle_{\mathcal{M}',\mathcal{M}} = \|\phi\|^2_{\mathcal{M}},$$
$$\text{(ii)} \quad \|\Lambda_{\mathcal{M}}\phi\|_{\mathcal{M}'} = \|\phi\|_{\mathcal{M}} \quad \text{for all} \quad \phi \in \mathcal{M}.$$

In order to avoid the complexity of setting up observation spaces, we consider the following two types of distributive and terminal value observations in time sense. That is, the following cases:

(i) We take $\mathcal{C}_1 \in \mathcal{L}(L^2(0,T;V_2), \mathcal{M})$ and observer $z(q) = \mathcal{C}_1 y(q)$;

(ii) We take $\mathcal{C}_2 \in \mathcal{L}(H, \mathcal{M})$ and observer $z(q) = \mathcal{C}_2 y(T,q)$.

3.1. The case where $\mathcal{C}_1 \in \mathcal{L}(L^2(0,T;V_2), \mathcal{M})$

In this case the cost functional is given by

$$J(q) = \|\mathcal{C}_1 y(q) - z_d\|^2_{\mathcal{M}}, \quad q \in Q_\tau,$$

and then the necessary condition (3.22) is equivalent to

$$\int_0^T \langle \mathcal{C}_1^*\Lambda_{\mathcal{M}}(\mathcal{C}_1 y(t,\overline{q}) - z_d), z(t)\rangle_{V_2',V_2} dt \geq 0, \quad \forall q \in Q_\tau. \tag{3.23}$$

Let us introduce an adjoint state $p(\overline{q})$ satisfying

(3.24)
$$\begin{cases} \eta''(\overline{q}) - A_2(t,\overline{q})\eta'(\overline{q}) + (A_1(t) - A_2'(t,\overline{q}))\eta(\overline{q}) = \mathcal{C}_1^*\Lambda_{\mathcal{M}}(\mathcal{C}_1 y(\overline{q}) - z_d), \\ \eta(T,\overline{q}) = \eta'(T,\overline{q}) = 0. \end{cases}$$

Since $\mathcal{C}_1^*\Lambda_{\mathcal{M}}(\mathcal{C}_1 y(\overline{q}) - z_d) \in L^2(0,T;V_2')$ and $A_2'(\cdot,\overline{q}) \in L^\infty(0,T;\mathcal{L}(V_2,V_2'))$, the equation (3.24) is well-posed and permits a unique weak solution $\eta(\overline{q}) \in W(0,T)$ if we consider the change of the time variable as $t \to T-t$. Multiplying (3.24) by z, which is a weak solution to (3.12), integrating it by parts after integrating it on $[0,T]$, we obtain

$$\int_0^T \langle \eta(t,\overline{q}), z'' + A_2(t,\overline{q})z' + A_1(t)z\rangle_{V,V'}dt$$
$$= \int_0^T \langle \eta(t,\overline{q}), -\delta A_2(t,\overline{q};q-\overline{q})y'(t,\overline{q})\rangle_{V,V'}dt \geq 0.$$

Here we used the inequality (3.23). Summarizing these we have the following theorem.

THEOREM 3.1. *Let us assume that* (2.1)-(2.4), (3.4) *and* (3.11) *hold. Then the optimal control* $\overline{q}$ *is characterized by state and adjoint equations and inequality:*

$$\begin{cases} y(\overline{q})'' + A_2(t,\overline{q})y(\overline{q})' + A_1(t)y(\overline{q}) = f \quad in \quad (0,T), \\ y(0,\overline{q}) = y_0 \in V, \\ y'(0,\overline{q}) = y_1 \in H, \end{cases}$$

$$\begin{cases} \eta''(\overline{q}) - A_2(t,\overline{q})\eta'(\overline{q}) + (A_1(t) - A_2'(t,\overline{q}))\eta(\overline{q}) \\ \quad = \mathcal{C}_1^*\Lambda_{\mathcal{M}}(\mathcal{C}_1 y(\overline{q}) - z_d) \quad in \quad (0,T), \\ \eta(T,\overline{q}) = \eta'(T,\overline{q}) = 0, \end{cases}$$

$$\int_0^T \langle \eta(t,\overline{q}), \delta A_2(t,\overline{q};q-\overline{q})y'(t,\overline{q})\rangle_{V,V'}dt \leq 0, \quad \forall q \in Q_\tau.$$

3.2. The case where $\mathcal{C}_2 \in \mathcal{L}(H,\mathcal{M})$

In this case the cost functional is given by

$$J(q) = \|\mathcal{C}_2 y(T,q) - z_d\|^2_{\mathcal{M}}, \quad q \in Q_\tau,$$

and then the necessary condition (3.22) is equivalent to

$$(\mathcal{C}_2^*\Lambda_{\mathcal{M}}(\mathcal{C}_2 y(T,\overline{q}) - z_d), z(T))_H \geq 0, \quad \forall q \in Q_\tau. \tag{3.25}$$

Let us introduce an adjoint state $\eta(\overline{q})$ satisfying

$$\begin{cases} \eta''(\overline{q}) - A_2(t,\overline{q})\eta'(\overline{q}) + (A_1(t) - A_2'(t,\overline{q}))\eta(\overline{q}) = 0, \\ \eta(T,\overline{q}) = 0, \\ \eta'(T,\overline{q}) = \mathcal{C}_2^*\Lambda_{\mathcal{M}}(\mathcal{C}_2 y(T,\overline{q}) - z_d). \end{cases} \tag{3.26}$$

It follows by the same reason as the case 3.1 that there is a unique weak solution $\eta(\overline{q}) \in W(0,T)$, because $\mathcal{C}_2^*\Lambda_{\mathcal{M}}(\mathcal{C}_2 y(T,\overline{q}) - z_d) \in H$.

THEOREM 3.2. *Let us assume that* (2.1)-(2.4), (3.4) *and* (3.11) *hold. Then the optimal control* $\overline{q}$ *is characterized by state and adjoint equations and inequality:*

$$\begin{cases} y(\overline{q})'' + A_2(t,\overline{q})y(\overline{q})' + A_1(t)y(\overline{q}) = f \quad in \quad (0,T), \\ y(0,\overline{q}) = y_0 \in V, \\ y'(0,\overline{q}) = y_1 \in H, \end{cases}$$

$$\begin{cases} \eta''(\overline{q}) - A_2(t,\overline{q})\eta'(\overline{q}) + (A_1(t) - A_2'(t,\overline{q}))\eta(\overline{q}) = 0 \quad in \quad (0,T), \\ \eta(T,\overline{q}) = 0, \\ \eta'(T,\overline{q}) = \mathcal{C}_2^*\Lambda_{\mathcal{M}}(\mathcal{C}_2 y(T,\overline{q}) - z_d), \end{cases}$$

$$\int_0^T \langle \eta(t,\overline{q}), \delta A_2(t,\overline{q}; q-\overline{q})y'(t,\overline{q})\rangle_{V,V'}dt \geq 0, \quad \forall q \in Q_T.$$

Proof. We prove the optimal control only. Multiplying (3.26) by z, which is a weak solution to (3.12), integrating it by parts after integrating it on $[0,T]$, we obtain

$$\begin{aligned} 0 &= (z(T), \eta'(T,\overline{q}))_H + \int_0^T \langle \eta(t,\overline{q}), z'' + A_2(t,\overline{q})z' + A_1(t)z\rangle_{V,V'}dt \\ &= (z(T), \eta'(T,\overline{q}))_H + \int_0^T \langle \eta(t,\overline{q}), -\delta A_2(t,\overline{q}; q-\overline{q})y'(\overline{q})\rangle_{V,V'}dt. \end{aligned}$$

Hence from (3.26) and (3.25) we conclude that

$$\begin{aligned} &(z(T), \mathcal{C}_2^*\Lambda_{\mathcal{M}}(\mathcal{C}_2 y(T,\overline{q}) - z_d))_H \\ &\quad = \int_0^T \langle \eta(t,\overline{q}), \delta A_2(t,\overline{q}; q-\overline{q})y'(t,\overline{q})\rangle_{V,V'}dt \geq 0. \end{aligned}$$

□

4. Applications to partial differential equations

Let Ω be an open bounded subset of $\mathbf{R}^n$ with the smooth boundary Γ, and let $\Omega_T = (0,T) \times \Omega$ and $\sum = (0,T) \times \Gamma$. We will give some examples for the case where the operator $A_1(t)$ has the second (or the fourth) differential order and $A_2(t)$ has the second (or o-th) differential order in the variable $x \in \Omega$.

EXAMPLE 4.1. One of the simplest example in the linear damped second order equation is the following damped wave equation

$$
(4.1) \qquad \begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \kappa\Delta\dfrac{\partial y}{\partial t} - \Delta y = f \ \text{ in } \ \Omega_T, \\ y = 0 \ \text{ on } \ \Sigma, \\ y(0) = y_0, \ \ \dfrac{\partial y}{\partial t}(0) = y_1 \ \text{ in } \ \Omega. \end{cases}
$$

Let us take $V = V_2 = H_0^1(\Omega), H = L^2(\Omega)$ and $y_0 \in V, y_1 \in H$, and let $f \in L^2(\Omega_T)$. Define a bilinear form σ as

$$
(4.2) \qquad \sigma(q;\phi,\psi) = \int_\Omega q(x)\nabla\phi(x)\cdot\nabla\psi(x)dx, \quad \phi,\psi \in H_0^1(\Omega),
$$

and take $a_1(q;\phi,\psi) = \sigma(1;\phi,\psi)$ and $a_2(q;\phi,\psi) = \sigma(\kappa;\phi,\psi)$, where $\kappa > 0$ is a positive constant. It is easy to check for a_i satisfying all conditions to be verified. Therefore there is a unique weak solution y to (4.1), whose solution satisfies

$$
y, \nabla y, \frac{\partial y}{\partial t}, \nabla\frac{\partial y}{\partial t} \in L^2(\Omega_T).
$$

Take $\mathcal{M} = L^2(\Omega_T)$ and $\mathcal{C} = I$, the identity observation, and consider the cost functional defined as

$$
(4.3) \qquad J(\kappa) = \int_{\Omega_T} (y(t,x,\kappa) - z_d(t,x))^2 dx, \quad \kappa > 0.
$$

If we take Q the set of all positive constants and give a L^∞−norm topology on Q, then Q is compact in $L^\infty(\Omega)$. Hence there is at least one optimal control subject to (4.1) and (4.3), say it $\overline{k}$. Denote by $y(\overline{\kappa})$ the solution of the state equation (4.1). Then by Theorem 3.1 adjoint state equation and necessary condition on $\overline{\kappa}$ are given as follows:

$$
(4.4) \qquad \begin{cases} \dfrac{\partial^2 \eta(\overline{\kappa})}{\partial t^2} + \overline{\kappa}\Delta\dfrac{\partial \eta(\overline{\kappa})}{\partial t} - \Delta\eta(\overline{\kappa}) = y(\overline{\kappa}) - z_d \ \text{ in } \ \Omega_T, \\ \eta(\overline{\kappa}) = 0 \ \text{ on } \ \Sigma, \\ \eta(T,\overline{\kappa}) = \dfrac{\partial}{\partial t}\eta(T,\overline{\kappa}) = 0 \ \text{ in } \ \Omega \end{cases}
$$

and

$$
\int_0^T \int_\Omega \nabla\frac{\partial y(\overline{\kappa})}{\partial t}\cdot\nabla\eta(\overline{\kappa})dxdt \le 0.
$$

EXAMPLE 4.2. As one of the realistic system, we can give the structurally damped plate equation of

$$(4.5)\qquad \begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \nabla \cdot \left(\alpha(x)\nabla \dfrac{\partial y}{\partial t}\right) + \Delta^2 y = f \text{ in } \Omega_T, \\ y = \Delta y = 0 \text{ on } \Sigma, \\ y(0) = y_0, \quad \dfrac{\partial y}{\partial t}(0) = y_1 \text{ in } \Omega. \end{cases}$$

We take a parameter set $Q = \{\alpha \in L^\infty(\Omega) : a \leq \alpha(x) \leq b < \infty \text{ a.e. } x \in \Omega\}$ for some positive constants a and b. Let us take $V = H_0^2(\Omega), V_2 = H_0^1(\Omega), H = L^2(\Omega)$. Define a bilinear form a_1 as

$$(4.6)\qquad a_1(\phi, \psi) = \int_\Omega \Delta\phi(x)\Delta\psi(x)dx, \quad \phi, \psi \in H_0^2(\Omega),$$

and take $a_2(q; \phi, \psi) = \sigma(\alpha; \phi, \psi)$, which is given Example 4.1. For $y_0 \in V, y_1 \in H$ and $f \in L^2(\Omega_T)$, there exists a unique weak solution y to (4.5) such that

$$y, \Delta y, \frac{\partial y}{\partial t}, \Delta \frac{\partial y}{\partial t} \in L^2(\Omega_T).$$

Take $\mathcal{M} = L^2(\Omega)$ and $\mathcal{C} = I$, the identity observation, and consider the cost functional defined as

$$(4.7)\qquad J(\alpha) = \int_\Omega (y(T, x, \alpha) - z_d(x))^2 dx, \quad \alpha \in Q,$$

where $z_d \in L^2(\Omega)$. Let $\overline{\alpha}$ be the optimal control subject to (4.5) and (4.7) and denote $y(\overline{\alpha})$ the solution of the state equation (4.5). Then by Theorem 3.2 adjoint state equation and necessary condition on $\overline{\alpha}$ are given as follows:

$$(4.8)\qquad \begin{cases} \dfrac{\partial^2 \eta(\overline{\alpha})}{\partial t^2} + \nabla \cdot \left(\alpha(x)\nabla \dfrac{\partial \eta(\overline{\alpha})}{\partial t}\right) + \Delta^2 \eta(\overline{\alpha}) = 0 \text{ in } \Omega_T, \\ \eta(\overline{\alpha}) = \Delta\eta(\overline{\alpha}) = 0 \text{ on } \Sigma, \\ \eta(T, \overline{\alpha}) = 0 \text{ in } \Omega, \\ \dfrac{\partial}{\partial t}\eta(T, \overline{\alpha}) = y(T, \overline{\alpha}) - z_d \text{ in } \Omega \end{cases}$$

and

$$\int_0^T \int_\Omega (\alpha(x) - \overline{\alpha}(x))\nabla \frac{\partial y(\overline{\alpha})}{\partial t} \cdot \nabla\eta(\overline{\alpha}) dx dt \geq 0, \quad \forall \alpha \in Q.$$

References

[1] N. U. Ahmed, *Optimization and identification of systems governed by evolution equations on Banach space*, Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, **184** (1988).

[2] N. U. Ahmed, *Necessary conditions of optimality for a class of second-order hyperbolic systems with spatially dependent controls in the coefficients*, J. Optim. Theory and Appl., **38** (1982), 423-446.

[3] H. T. Banks, K. Ito and Y. Wang, *Well Posedness for damped second order systems with unbounded input operators*, Center for Research in Scientific Computation, North Carolina State Univ., 1993.

[4] H. T. Banks and K. Kunisch, *Estimations Techniques for distributed parameter systems*, Birkhäuser, 1989.

[5] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer-Verlag, **5**, Evolution Problems, 1992.

[6] Junhong Ha, *Optimal control problems for hyperbolic distributed parameter systems with damping terms*, Doctoral Thesis, Kobe Univ., Japan, 1996.

Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: jypark@hyowon.pusan.ac.kr