

A NOTE ON E. CARTAN'S METHOD OF EQUIVALENCE AND LOCAL INVARIANTS FOR ISOMETRIC EMBEDDINGS OF RIEMANNIAN MANIFOLDS

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ABSTRACT. By using the method of equivalence of E. Cartan we calculate the local scalar invariants for Riemannian 2-manifolds. We define also a notion of local invariants for submanifolds in \mathbb{R}^{n+d} , $n \geq 2$, $d \geq 1$, in terms of the symmetry of the local isometric embedding equations of Riemannian n -manifolds into \mathbb{R}^{n+d} . We show that the local invariants obtained by the Cartan's method are the intrinsic expressions of the local invariants in our sense in the cases of surfaces in \mathbb{R}^3 .

0. Introduction

Let M be a smooth (C^∞) manifold with a certain geometric structure. E. Cartan's local equivalence problem is finding a complete system of invariants on a principal fibre bundle over M , so that there exists a structure preserving local diffeomorphism f of M onto a manifold \tilde{M} with a structure of the same type if and only if the invariants of M and \tilde{M} agree (see [2], [7], [13], [14]). By a geometric structure we mean a G -structure (see [5]). Then the problem can be described as follows : Given a set of n linearly independent differential 1-forms $\theta^i(x, dx)$, $i = 1, \dots, n$, in the coordinates $x = (x^1, \dots, x^n)$, another such set $\tilde{\theta}^i(y, dy)$ in the coordinates $y = (y^1, \dots, y^n)$, and given a Lie group $G \subset GL(n; \mathbb{R})$, the problem is determining whether there exists a mapping f

$$y^i = f^i(x^1, \dots, x^n), \quad i = 1, \dots, n,$$

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which satisfies

$$f^*\tilde{\theta} = a\theta, \quad a \in G,$$

where $\tilde{\theta} = (\tilde{\theta}^1, \dots, \tilde{\theta}^n)^t$, $\theta = (\theta^1, \dots, \theta^n)^t$, and a is a G -valued function of M . When M and \tilde{M} are Riemannian manifolds, θ and $\tilde{\theta}$ are orthonormal coframes over M and \tilde{M} respectively, and $G = SO(n; \mathbb{R})$, and the equivalence problem in this case is constructing a canonical complete system of invariants so that there exists a Riemannian isometry $f: M \rightarrow \tilde{M}$ if and only if those invariants of M and \tilde{M} agree.

In §1, we explain how to get the local scalar invariants by the method of equivalence of E. Cartan and then in §2, we apply this method to Riemannian 2-manifolds to get local scalar invariants.

Let $x = (x^1, \dots, x^n)$ be a coordinate system of a Riemannian manifold (M, g) and let $g_{ij}(x) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. A mapping $u = (u^1, \dots, u^{n+d})$: $M \rightarrow \mathbb{R}^{n+d}$ is a local isometric embedding if u satisfies

$$(3.1) \quad \sum_{\alpha=1}^{n+d} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} = g_{ij}(x), \quad \text{for each } i, j = 1, \dots, n.$$

For the history and the major results including the existence of solutions of (3.1) under various conditions, see [3], [8], [9], or [12].

In [11] the authors defined the local invariants for the submanifolds of Euclidean spaces (Definition 3.1) in terms of the symmetry of (3.1). This notion of invariant is valid in a wider class of systems of partial differential equations for unknown functions $u = (u^1, \dots, u^{n+d})$, $d \geq 1$, of independent variables $x = (x^1, \dots, x^n)$ where a solution $u(x)$ forms a submanifold of \mathbb{R}^{n+d} . Actual calculation of the invariants usually involves enormous symbolic calculations. As a simplest example, we show in §4 that the Gaussian curvature and the higher order invariants for surfaces $u^3 = h(u^1, u^2)$ obtained by Cartan's method are intrinsic expressions of the invariants in the sense of our definition.

Throughout this paper, all the manifolds and mappings are assumed to be C^∞ .

1. E. Cartan's method of equivalence

Let M be a C^∞ manifold of dimension n and G be a linear subgroup of $GL(n; \mathbb{R})$. A G -structure on M is reduction of coframe bundle of M

to a subbundle with the structure group G . For instance, a Riemannian structure on M is a $SO(n)$ -structure and the subbundle in this case is the orthonormal coframe bundle of M .

Now let M and \tilde{M} be manifolds of dimension n with G -structure. The equivalence problem is deciding whether there exists a structure preserving mapping $f : M \rightarrow \tilde{M}$. Locally, this is a question of existence of solutions for an overdetermined system of first order partial differential equations in cases where G is a sufficiently small group.

E. Cartan's method to this problem is as follows: We fix coframes $\theta = (\theta^1, \dots, \theta^n)^t$ of M and $\tilde{\theta} = (\tilde{\theta}^1, \dots, \tilde{\theta}^n)^t$ of \tilde{M} adapted to the G -structure, where θ and $\tilde{\theta}$ are defined over an open set U of M and an open set \tilde{U} of \tilde{M} , respectively. Then the question is whether there exists a mapping $f : M \rightarrow \tilde{M}$ that satisfies

$$(1.1) \quad f^* \tilde{\theta}^\alpha = a_\beta^\alpha \theta^\beta,$$

where $a := [a_\beta^\alpha(x)]_{n \times n}$ is a G -valued function of M . In terms of local coordinates, (1.1) is a system of first order partial differential equations for $f = (f^1, \dots, f^n)$ and system of algebraic equations for $a_\beta^\alpha(x)$. Thus we consider the product $U \times G$ and define a tautological 1-form $\Theta = g\theta$ on $U \times G$, namely

$$(1.2) \quad \Theta_{(x,g)} = g\theta_x, \quad \forall x \in U, \quad \forall g \in G,$$

where θ_x is a column vector $(\theta_x^1, \dots, \theta_x^n)^t$. G acts on $U \times G$ on the left by the action defined by

$$h(x, g) = (x, hg), \quad \forall x \in U, \quad \forall g, h \in G.$$

PROPOSITION 1.1. *A diffeomorphism $f : U \rightarrow \tilde{U}$ satisfies (1.1) if and only if there exists a diffeomorphism $F : U \times G \rightarrow \tilde{U} \times G$ satisfying*

$$\text{i) } F^* \tilde{\Theta} = \Theta$$

ii) *the following diagram commutes:*

$$\begin{array}{ccc} U \times G & \xrightarrow{F} & \tilde{U} \times G \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ U & \xrightarrow{f} & \tilde{U} \end{array}$$

$$\text{iii) } F(x, gh) = gF(x, h), \quad \text{for each } x \in U, \text{ and } g, h \in G.$$

Proof. Suppose that f satisfies $f^*\tilde{\theta} = g_0\theta$, where g_0 is a G -valued function on M . Define $F : U \times G \rightarrow \tilde{U} \times G$ by $F(x, g) = (f(x), gg_0^{-1}(x))$. Then F satisfies ii) and iii). Moreover,

$$F^*\tilde{\Theta} = F^*(\tilde{g}\tilde{\theta}) = (gg_0^{-1})f^*\tilde{\theta} = (gg_0^{-1})g_0\theta = g\theta = \Theta.$$

Conversely, suppose that $F : U \times G \rightarrow \tilde{U} \times G$ satisfies i) - iii). Define $f : U \rightarrow \tilde{U}$ and $g_0 : U \rightarrow G$ by $F(x, e) = (f(x), g_0(x)^{-1})$, where e is the identity of G . Then $F(x, g) = gF(x, e) = (f(x), gg_0^{-1})$, and i) implies that

$$g\theta = F^*(\tilde{g}\tilde{\theta}) = (gg_0^{-1})f^*\tilde{\theta}$$

therefore, $f^*\tilde{\theta} = g_0\theta$. □

Now apply d to (1.2). We get

$$d\Theta = dg \wedge \theta + g d\theta;$$

substituting $\theta = g^{-1}\Theta$, we obtain

$$(1.3) \quad d\Theta = dgg^{-1} \wedge \Theta + g d\theta.$$

We need the following

HYPOTHESIS. *There exists unique 1-forms ω_j^i , $i, j = 1, \dots, n$, such that*

$$(1.4) \quad d\theta^i = -\omega_j^i \wedge \theta^j$$

and

$$[\omega_j^i(x)] \in \mathcal{G}, \text{ for each } x \in U,$$

where \mathcal{G} is the Lie algebra of G .

This Lie algebra valued 1-form $\omega = [\omega_j^i]$ is called a torsion-free connection (see [5]). Substitute $d\theta = -\omega \wedge \theta$ and $\theta = g^{-1}\Theta$ in (1.3), to get

$$d\Theta = dgg^{-1} \wedge \Theta - g\omega \wedge g^{-1}\Theta = (dgg^{-1} - g\omega g^{-1}) \wedge \Theta.$$

Let

$$(1.5) \quad \Omega = -(dgg^{-1} - g\omega g^{-1}),$$

then Ω is a \mathcal{G} -valued 1-form on $U \times G$ and we have

$$(1.6) \quad d\Theta = -\Omega \wedge \Theta.$$

Now it is easy to show

PROPOSITION 1.2. *Let Θ^i and Ω_j^i , $i, j = 1, \dots, n$, be the 1-forms defined by (1.2) and (1.5) on $U \times G$. Then Θ^i, Ω_j^i spans the cotangent space at each point of $U \times G$. Furthermore, if $\tilde{\Theta}^i, \tilde{\Omega}_j^i$ are the corresponding 1-forms on $\tilde{U} \times G$ and*

$$F : U \times G \rightarrow \tilde{U} \times G$$

is the mapping as in Proposition 1.1, then

$$(1.7) \quad F^* \tilde{\Omega}_j^i = \Omega_j^i.$$

The set $\{\Theta^i, \Omega_j^i\}$ is called a complete set of invariants for the equivalence problem. Ω is called a torsion-free connection form on $U \times G$. Note that ω is a 1-form on the base manifold U and that the restriction of Ω on each fibre is the Maurer-Cartan form of G . Now apply d to (1.6), to get

$$\begin{aligned} 0 &= -d\Omega \wedge \Theta + \Omega \wedge d\Theta \\ &\quad \text{substitute (1.6) for } d\Theta \\ &= -(d\Omega + \Omega \wedge \Omega) \wedge \Theta. \end{aligned}$$

The curvature 2-form on $U \times G$ is defined by

$$(1.8) \quad \mathfrak{R} = d\Omega + \Omega \wedge \Omega.$$

Then \mathfrak{R} is a \mathcal{G} -valued 2-form such that

$$(1.9) \quad \mathfrak{R} \wedge \Theta = 0.$$

Similarly, from (1.4) we get

$$0 = -(d\omega + \omega \wedge \omega) \wedge \theta.$$

The curvature 2-form on U is defined as

$$(1.10) \quad R = d\omega + \omega \wedge \omega.$$

Then we have

$$R \wedge \theta = 0.$$

Now apply d to (1.5). Then computation shows that

$$(1.11) \quad d\Omega = -\Omega \wedge \Omega + gRg^{-1}.$$

By (1.8) and (1.11) we have

$$(1.12) \quad \mathfrak{R} = gRg^{-1}.$$

For the problem of existence of f satisfying (1.1), or equivalently, the existence of F as in Proposition 1.1, we consider $V := U \times G$, $\tilde{V} := \tilde{U} \times G$, and 1-forms ψ^j on $V \times \tilde{V}$ defined by

$$\psi^j = \Theta^j - \tilde{\Theta}^j, \quad j = 1, \dots, n.$$

Then the problem is finding an integral manifold of the Pfaffian system

$$\psi^j = 0, \quad j = 1, \dots, n.$$

We are concerned in this paper with finding local geometric invariants. Let $F : U \times G \rightarrow \tilde{U} \times G$ be as in Proposition 1.1. Then by (1.7) and (1.8) we have

$$(1.13) \quad F^*\tilde{\mathfrak{R}} = \mathfrak{R},$$

where $\tilde{\mathfrak{R}} = d\tilde{\Omega} + \tilde{\Omega} \wedge \tilde{\Omega}$.

Thus \mathfrak{R} is an invariant 2-form on $U \times G$. Substituting (1.12) and the same expression with tilde in (1.13), we obtain

$$F^*(\tilde{g}\tilde{R}\tilde{g}^{-1}) = gRg^{-1}.$$

Since $\tilde{g} = gg_0^{-1}$, this implies that

$$(1.14) \quad f^*\tilde{R} = g_0Rg_0^{-1}.$$

To get the scalar invariants, we shall express the invariant 2-form \mathfrak{R} in terms of the invariant 1-form Θ and Ω . Set

$$\mathfrak{R}_j^i = a_{j,\lambda\eta}^{i,\mu\nu} \Omega_\mu^\lambda \wedge \Omega_\nu^\eta + b_{j,\lambda\mu}^i \Theta^\lambda \wedge \Theta^\mu + c_{j,\lambda\mu}^{i,\nu} \Theta^\lambda \wedge \Omega_\nu^\mu,$$

(summation convention in $\lambda, \mu, \eta, \nu = 1, \dots, n$).

Then $a_{j,\lambda\eta}^{i,\mu\nu}$ are all zero by (1.9), $b_{j,\lambda\mu}^i$ and $c_{j,\lambda\mu}^{i,\nu}$ are uniquely determined if we require $b_{j,\lambda\mu}^i = -b_{j,\mu\lambda}^i$. Thus we get for each $i, j = 1, \dots, n$,

$$(1.15) \quad \mathfrak{R}_j^i = b_{j,\lambda\mu}^i(x, g) \Theta^\lambda \wedge \Theta^\mu + c_{j,\lambda\mu}^{i,\nu}(x, g) \Theta^\lambda \wedge \Omega_\nu^\mu, \\ (\text{summation convention in } \lambda, \mu, \nu = 1, \dots, n).$$

These functions $b_{j,\lambda\mu}^i(x, g)$ and $c_{j,\lambda\mu}^{i,\nu}(x, g)$ are all scalar invariants in the following sense :

$$\tilde{a}(F(x, g)) = \tilde{a}(f(x), gg_0^{-1}) = a(x, g), \quad \forall x \in U, \quad \forall g \in G.$$

By differentiating the scalar invariants repeatedly, we get further scalar invariants, for instance, if a is a scalar invariant we set

$$da = d_\lambda \Theta^\lambda + e_\lambda^\mu \Omega_\mu^\lambda, \\ (\text{summation convention in } \lambda, \mu = 1, \dots, n),$$

then $d_\lambda(x, g)$ and $e_\lambda^\mu(x, g)$ are scalar invariants. Since $\mathfrak{R} = (\mathfrak{R}_j^i)$ and $\Omega = (\Omega_j^i)$ are \mathcal{G} -valued 2-form and 1-form, respectively, they satisfy a system of linear equations

$$(1.16) \quad c_i^j \mathfrak{R}_j^i = 0, \\ c_i^j \Omega_j^i = 0 \quad (\text{summation convention in } i, j = 1, \dots, n),$$

where c_i^j are the structural constants of \mathcal{G} . (1.15) and (1.16) give linear equations in $b_{j,\lambda\mu}^i(x, g)$ and $c_{j,\lambda\mu}^{i,\nu}(x, g)$. If we can eliminate the variables g using these linear equations and get an invariant expressed in variables x only, this is a local geometric invariant of the base manifold M . We shall see an example of such cases in the next section.

2. Local invariants for Riemannian 2-Manifolds

In this section we apply the method in §1 to get the local invariants of a Riemannian 2-manifold.

Let M be a Riemannian 2-manifold with a local coordinate system (x, y) and metric $g_{ij}(x, y)$, $i, j = 1, 2$. Using the Gram-Schmidt orthonormalization process, we choose orthonormal coframes θ^1 and θ^2 on M as follows :

$$(2.1) \quad \begin{cases} \theta^1 = \Gamma_1^1 dx + \Gamma_2^1 dy, \\ \theta^2 = \Gamma_2^2 dy, \end{cases}$$

where

$$\Gamma_1^1 = \sqrt{g_{11}}, \quad \Gamma_2^1 = \frac{g_{12}}{\sqrt{g_{11}}},$$

and

$$\Gamma_2^2 = \frac{\sqrt{g_{11}g_{22} - (g_{12})^2}}{\sqrt{g_{11}}}.$$

Taking the exterior differentiation d of (2.1), we have

$$(2.2) \quad \begin{aligned} d\theta^1 &= d\Gamma_1^1 \wedge dx + d\Gamma_2^1 \wedge dy = \{(\Gamma_2^1)_x - (\Gamma_1^1)_y\} dx \wedge dy, \\ d\theta^2 &= d\Gamma_2^2 \wedge dy = (\Gamma_2^2)_x dx \wedge dy. \end{aligned}$$

Take the wedge product of θ^1 and θ^2 in (2.1), to get

$$(2.3) \quad dx \wedge dy = \frac{1}{\sqrt{g_{11}g_{22} - (g_{12})^2}} \theta^1 \wedge \theta^2.$$

By substituting (2.3) for $dx \wedge dy$ in (2.2), we have

$$(2.4) \quad \begin{aligned} d\theta^1 &= \frac{\{(\Gamma_2^1)_x - (\Gamma_1^1)_y\}}{G} \theta^1 \wedge \theta^2, \\ d\theta^2 &= \frac{(\Gamma_2^2)_x}{G} \theta^1 \wedge \theta^2, \end{aligned}$$

where $G = \sqrt{g_{11}g_{22} - (g_{12})^2}$. Define ω_2^1 and ω_1^2 by

$$(2.5) \quad \omega_2^1 = -\omega_1^2 = -\frac{(\Gamma_2^1)_x - (\Gamma_1^1)_y}{G} \theta^1 - \frac{(\Gamma_2^2)_x}{G} \theta^2.$$

Then (2.4) becomes

$$(2.6) \quad \begin{aligned} d\theta^1 &= -\omega_2^1 \wedge \theta^2, \\ d\theta^2 &= -\omega_1^2 \wedge \theta^1. \end{aligned}$$

ω_2^1 is the unique 1-form which satisfies (1.4).

In order to apply the method in §1, we consider differential 1-forms Θ^1 and Θ^2 on $M \times SO(2)$ defined by

$$(2.7) \quad \begin{cases} \Theta^1 = \cos \tau \theta^1 - \sin \tau \theta^2, \\ \Theta^2 = \sin \tau \theta^1 + \cos \tau \theta^2, \end{cases}$$

where τ is a parameter of $SO(2)$ (cf. (1.2)). Apply d to (2.7), to get

$$(2.8) \quad \begin{aligned} d\Theta^1 &= -\sin \tau d\tau \wedge \theta^1 - \cos \tau d\tau \wedge \theta^2 + \cos \tau d\theta^1 - \sin \tau d\theta^2, \\ d\Theta^2 &= \cos \tau d\tau \wedge \theta^1 - \sin \tau d\tau \wedge \theta^2 + \sin \tau d\theta^1 + \cos \tau d\theta^2. \end{aligned}$$

Substitute (2.6) for $d\theta^1$ and $d\theta^2$ in (2.8), and define 1-forms

$$\begin{aligned} \Omega_2^1 &= +d\tau + \omega_2^1, \\ \Omega_1^2 &= -d\tau + \omega_1^2, \end{aligned}$$

on $M \times SO(2)$ (cf. (1.5)), then (2.8) becomes

$$(2.9) \quad \begin{aligned} d\Theta^1 &= -\Omega_2^1 \wedge \Theta^2, \\ d\Theta^2 &= -\Omega_1^2 \wedge \Theta^1. \end{aligned}$$

Since $\omega_2^1 = -\omega_1^2$, we have

$$(2.10) \quad \Omega_2^1 = -\Omega_1^2.$$

Applying d to (2.9) and substituting (2.9) for $d\Theta^1$ and $d\Theta^2$, we get

$$(2.11) \quad \begin{cases} d\Omega_2^1 \wedge \Theta^2 = 0, \\ d\Omega_1^2 \wedge \Theta^1 = 0. \end{cases}$$

Then, by Cartan's Lemma,

$$(2.12) \quad d\Omega_2^1 = d\omega_2^1 := K\Theta^1 \wedge \Theta^2,$$

for some function K on $M \times SO(2)$. Applying d to (2.12) and substituting (2.9) for $d\Theta^1$ and $d\Theta^2$, we get

$$0 = dK \wedge \Theta^1 \wedge \Theta^2,$$

so $\partial K / \partial \tau = 0$, that is, K is independent of the parameter τ . From (2.7) it follows that $\Theta^1 \wedge \Theta^2 = \theta^1 \wedge \theta^2$. In (2.12) substitute (2.5) for ω_2^1 and $\theta^1 \wedge \theta^2$ for $\Theta^1 \wedge \Theta^2$, then by (2.3), (2.4), and (2.5) we have

$$(2.13) \quad 4K(x, y)$$

$$\begin{aligned} &= (g_{11}g_{22} - (g_{12})^2)^{-2} [2((g_{12})^2 - g_{11}g_{22})(g_{11,yy} - 2g_{12,xy} + g_{22,xx}) \\ &+ g_{11}(g_{11,y}g_{22,y} - 2g_{12,x}g_{22,y} + (g_{22,x})^2) \\ &+ g_{12}(g_{11,x}g_{22,y} - g_{11,y}g_{22,x} - 2g_{11,y}g_{12,y} + 4g_{12,x}g_{12,y} - 2g_{12,x}g_{22,x}) \\ &+ g_{22}(g_{11,x}g_{22,x} - 2g_{11,x}g_{12,y} + (g_{11,y})^2)]. \end{aligned}$$

Notice that K is the classical Gaussian curvature.

To find the further invariants of M apply d to $K(x, y)$, to get

$$\begin{aligned} (2.14) \quad dK &= K_x dx + K_y dy, \\ &= K_x \left(\frac{1}{\sqrt{g_{11}}} \theta^1 - \frac{g_{12}}{G\sqrt{g_{11}}} \theta^2 \right) + K_y \left(\frac{\sqrt{g_{11}}}{G} \theta^2 \right), \text{ by (2.1)} \\ &= \frac{K_x}{\sqrt{g_{11}}} \theta^1 + \frac{g_{11}K_y - g_{12}K_x}{G\sqrt{g_{11}}} \theta^2. \end{aligned}$$

From (2.7) we have

$$(2.15) \quad \begin{cases} \theta^1 = \cos \tau \Theta^1 + \sin \tau \Theta^2 \\ \theta^2 = -\sin \tau \Theta^1 + \cos \tau \Theta^2. \end{cases}$$

Substitute (2.15) for θ^1 and θ^2 in (2.14), to get

$$(2.16) \quad dK = \alpha(x, y, \tau) \Theta^1 + \beta(x, y, \tau) \Theta^2,$$

where

$$(2.17) \quad \begin{aligned} \alpha(x, y, \tau) &= \frac{K_x}{\sqrt{g_{11}}} \cos \tau - \frac{g_{11}K_y - g_{12}K_x}{G\sqrt{g_{11}}} \sin \tau, \\ \beta(x, y, \tau) &= \frac{K_x}{\sqrt{g_{11}}} \sin \tau + \frac{g_{11}K_y - g_{12}K_x}{G\sqrt{g_{11}}} \cos \tau. \end{aligned}$$

Therefore, α and β are scalar invariants on $M \times SO(2)$. Notice that

$$(2.18) \quad \alpha^2 + \beta^2 = \frac{g_{22}(K_x)^2 - 2g_{12}K_xK_y + g_{11}(K_y)^2}{G^2}.$$

is a scalar invariant which does not depend on the fibre coordinate τ and thus a third order invariant of a Riemannian 2-manifold.

An invariant of next order can be found by differentiating (2.18): Let H be the right hand side of (2.18), then the same procedure gives a scalar invariant

$$\frac{g_{22}(H_x)^2 - 2g_{12}H_xH_y + g_{11}(H_y)^2}{G^2}.$$

We can continue the same procedure to get invariants of higher order.

3. Invariants for submanifolds in \mathbb{R}^{n+d}

In this section we define the notion of local invariant for n -dimensional submanifolds in \mathbb{R}^{n+d} , $n \geq 2$, $d \geq 1$, in terms of the symmetry of the local isometric embedding equations. We review some background in the theory of jets and symmetry of differential equations and then explain how we are led to Definition 3.1, which was first introduced in [11].

First, we fix notations and definitions of jet theoretic notions. Our standard reference is [15]. Let $X = \{(x^1, \dots, x^n)\}$ be an open subset of \mathbb{R}^n and $U = \{(u^1, \dots, u^q)\}$ be an open subset of \mathbb{R}^q . Let $U^{(m)}$ be an open subset of a Euclidean space whose coordinates represent all the partial derivatives of smooth maps $u(x) = (u^1(x), \dots, u^q(x))$ from X to U of all orders 0 to m . A multi-index of order r is an unordered r -tuple of integers $J = (j_1, \dots, j_r)$ with $1 \leq j_s \leq n$. The order of

multi-index J is denoted by $|J|$. A typical point in $U^{(m)}$ is denoted by $u^{(m)}$, so that

$$u^{(m)} = (u_j^\alpha), \quad 1 \leq \alpha \leq q, \quad 0 \leq |J| \leq m.$$

Then $U^{(m)}$ is an open subset of the Euclidean space of dimension $q \cdot \binom{n+m}{m}$. The product space $X \times U^{(m)}$ is called the m -th order jet space and is denoted by $J^m(X, U)$.

Let A be a commutative algebra of smooth functions $\mathbf{a}(x, u^{(m)})$ depending on x, u , and derivatives of u up to some finite, but unspecified order m . An element of A is called a differential function and denoted by $\mathbf{a}[u]$. If m is the highest order of the partial derivatives that are in the arguments, $\mathbf{a}[u]$ is called a differential function of order m . The subset $A^{(m)}$ of A consisting of the differential functions of order less than or equal to m forms a subalgebra. Now consider a system of m -th order differential equations

$$\Delta^\nu(x, u^{(m)}) = 0, \quad 1 \leq \nu \leq l,$$

for unknown functions $u = (u^1, \dots, u^q)$ of n variables $x = (x^1, \dots, x^n)$ $\in X$. Let I be the set of all differential functions of the form

$$\sum_{|J| \geq 0} \sum_{\nu=1}^l P_\nu^J[u](D_J \Delta^\nu), \quad P_\nu^J[u] \in A,$$

where $D_J = D_{(j_1, \dots, j_r)} = D_{j_1} \circ \dots \circ D_{j_r}$ is a composition of total differential operators. Then we see that I is an ideal of A and that I is closed under total differentiation.

Now let M be an open subset of \mathbb{R}^n with the standard coordinates $x = (x^1, \dots, x^n)$ and let g be a Riemannian metric on M . A C^1 mapping $u = (u^1, \dots, u^{n+d})$ of M into a Euclidean space \mathbb{R}^{n+d} is a local isometric embedding if and only if u satisfies

$$(3.1) \quad \sum_{\alpha=1}^{n+d} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} = g_{ij}(x), \quad 1 \leq i, j \leq n,$$

where $g_{ij}(x) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. For each $i, j = 1, \dots, n$, let

$$(3.2) \quad \Delta^{ij} = \sum_{\alpha=1}^{n+d} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} - g_{ij}(x),$$

then Δ^{ij} is differential functions of order one defined on $J^1(M, \mathbb{R}^{n+d})$.

Now we denote by script letters the jet theoretic notions associated with (3.1). So, for instance, \mathcal{A} the algebra of differential functions in the arguments

$$(x^1, \dots, x^n, u^1, \dots, u^{n+d}, u_i^\alpha, u_{ij}^\alpha, \dots),$$

where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$, $u_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j}$, \dots and $\mathcal{A}^{(m)}$ is the subalgebra of \mathcal{A} consisting of the differential functions of order less than or equal to m . And also we denote by \mathcal{I} an ideal of \mathcal{A} consisting of all the differential functions of the form

$$\sum_J \sum_{i,j=1}^n P_{ij}^J[u](D_J \Delta^{ij}), \quad P_{ij}^J[u] \in \mathcal{A}.$$

Now consider an n -dimensional submanifold $S \subset \mathbb{R}^{n+d} = \{(u^1, \dots, u^{n+d})\}$ given by

$$(3.3) \quad u^{n+r} = h^r(u^1, \dots, u^n), \quad r = 1, \dots, d.$$

Let $J^m(M, \mathbb{R}^{n+d})$ be the m -th jet space of embeddings $u : M \rightarrow \mathbb{R}^{n+d}$ and let $J^m(\mathbb{R}^n, \mathbb{R}^d)$ be the m -th jet space of (3.3). Let Ω^m be the open subset of $J^m(M, \mathbb{R}^{n+d})$ on which a submatrix $[\frac{\partial u^k}{\partial x^i}]_{i=1, \dots, n}^{k=1, \dots, n}$ of the Jacobian $[\frac{\partial u^\alpha}{\partial x^i}]_{i=1, \dots, n}^{\alpha=1, \dots, n+d}$ is nonsingular.

To define the invariants for submanifold S in terms of the symmetry of (3.1), we first consider a mapping π from $\Omega^m \subset J^m(M, \mathbb{R}^{n+d})$ into $J^m(\mathbb{R}^n, \mathbb{R}^d)$ defined as follows :

For $m = 1$, differentiating (3.3) by chain rule we obtain

$$(3.4) \quad \frac{\partial u^{n+r}}{\partial x^i} = \sum_{k=1}^n h_k^r \frac{\partial u^k}{\partial x^i}, \quad i = 1, \dots, n, \quad r = 1, \dots, d,$$

and we define a mapping π of $\Omega^1 \subset J^1(M, \mathbb{R}^{n+d})$ into $J^1(\mathbb{R}^n, \mathbb{R}^d)$ by

$$\begin{aligned} \pi : (x, u, u_i^\alpha : \alpha = 1, \dots, n+d, \quad i = 1, \dots, n) \\ \mapsto (u, h_k^r : k = 1, \dots, n, \quad r = 1, \dots, d). \end{aligned}$$

To define $\pi : \Omega^2 \rightarrow J^2(\mathbb{R}^n, \mathbb{R}^d)$, differentiate (3.4) to get

$$(3.5) \quad \frac{\partial^2 u^{n+r}}{\partial x^i \partial x^j} = \sum_{k,l=1}^n h_{kl}^r \frac{\partial u^k}{\partial x^i} \frac{\partial u^l}{\partial x^j} + \sum_{k=1}^n h_k^r \frac{\partial^2 u^k}{\partial x^i \partial x^j}, \quad r = 1, \dots, d.$$

Then on Ω^2 , we solve (3.4) and (3.5) for h_k^r, h_{kl}^r , in terms of $u_i^\alpha, u_{ij}^\alpha$, to define $\pi : \Omega^2 \rightarrow J^2(\mathbb{R}^n, \mathbb{R}^d)$ by

$$\begin{aligned} \pi : (x, u, u_i^\alpha, u_{ij}^\alpha : \alpha = 1, \dots, n+d, i, j = 1, \dots, n) \\ \mapsto (u, h_k^r, h_{kl}^r : k, l = 1, \dots, n, r = 1, \dots, d). \end{aligned}$$

We define $\pi : \Omega^m \rightarrow J^m(\mathbb{R}^n, \mathbb{R}^d)$ inductively for each positive integer m .

An evolutionary vector field $V_Q = \sum_{\alpha=1}^{n+d} Q^\alpha[u] \frac{\partial}{\partial u^\alpha}$ is an infinitesimal symmetry of (3.1) if for each $i, j = 1, \dots, n$,

$$prV_Q(\Delta^{ij}) = 0, \quad \text{mod } \mathcal{I},$$

where

$$prV_Q = V_Q + \sum_j (D_j Q^\alpha) \frac{\partial}{\partial u_j^\alpha}.$$

That is, the components of $Q = (Q^1[u], \dots, Q^{n+d}[u])$ of V_Q satisfies

$$(3.6) \quad \sum_{\alpha=1}^{n+d} \{(D_j Q^\alpha) u_i^\alpha + (D_i Q^\alpha) u_j^\alpha\} = 0, \quad \text{mod } \mathcal{I}, \quad \text{for each } i, j = 1, \dots, n.$$

Suppose that V_Q is an infinitesimal symmetry of (3.1) and $u = f(x)$ is a solution of (3.1). Suppose also that a function $v(x, t) : M \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+d}$ satisfies the system of evolution equations

$$(3.7) \quad \begin{cases} \frac{\partial v^\alpha(x, t)}{\partial t} = Q^\alpha(x, v^{(n)}), & \alpha = 1, \dots, n+d, \\ v(x, 0) = f(x), \end{cases}$$

where $v^{(m)} = \{(\partial/\partial x^1)^{\alpha_1} \dots (\partial/\partial x^n)^{\alpha_n} v : \alpha_1 + \dots + \alpha_n \leq m\}$. Then for each t , $v(\cdot, t)$ is a solution of (3.1) and thus $v(x, t)$ gives a one-parameter family of solutions of (3.1), i.e., a 'bending' of f .

Now consider a differential function $\mathbf{a}[h]$ of $h = (h^1, \dots, h^d)$ in (3.3) such that

$$(3.8) \quad prV_Q(\mathbf{a}[h] \circ \pi) = 0, \quad \text{mod } \mathcal{I},$$

for any infinitesimal symmetry V_Q of (3.1). Then $\mathbf{a}[h] \circ \pi$ is invariant under any bending. (3.7) has no solution in general, but we can always consider the action of V_Q on differential functions $\mathbf{a}[h] \circ \pi$, so that (3.8) defines differential functions $\mathbf{a}[h]$ which are infinitesimally invariant under isometries.

Now consider a differential function $\mathbf{a}[h]$ such that $\mathbf{a}[h] \circ \pi$ is equal to some function $k(x)$ of M after eliminating the elements of \mathcal{I} , that is,

$$(3.9) \quad \mathbf{a}[h] \circ \pi = k(x) + b,$$

for some $b \in \mathcal{I}$. Then $\mathbf{a}[h]$ determines $k(x)$, which is independent of the choice of isometric embedding u . Note that if $\mathbf{a}[h]$ satisfies (3.9), then it satisfies (3.8) also and therefore, it is invariant under infinitesimal symmetries. Thus we are led to define the invariants for submanifolds as follows :

DEFINITION 3.1. *A differential function $\mathbf{a}[h]$ of $h = (h^1, \dots, h^d)$ in (3.3) defined on $J^m(\mathbb{R}^n, \mathbb{R}^d)$ is an invariant of order m if $\mathbf{a}[h] \circ \pi = k(x)$, mod \mathcal{I} , for some function k that depends only on x .*

We will call $k(x)$ the intrinsic expression of the invariant $\mathbf{a}[h]$.

4. Local invariants for surfaces in \mathbb{R}^3

This section is a test case for Definition 3.1. We shall show that the local invariants calculated in §2 by the Cartan's method are intrinsic expressions of some invariants in the sense of Definition 3.1. For the symbolic calculations we used Mathematica®.

Now let S be a surface in \mathbb{R}^3 given by $u^3 = h(u^1, u^2)$. Recall that the principal curvatures $\lambda_i, i = 1, 2$, for S are the eigenvalues of the second fundamental form (see p. 149 of [1])

$$\frac{1}{\sqrt{D}} \begin{pmatrix} 1 + (h_1)^2 & h_1 h_2 \\ h_1 h_2 & 1 + (h_2)^2 \end{pmatrix}^{-1} \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix},$$

where $D = 1 + (h_1)^2 + (h_2)^2$. They are given by

$$(4.1) \quad \frac{(1 + (h_2)^2)h_{11} - 2h_1h_2h_{12} + (1 + (h_1)^2)h_{22} \pm \sqrt{A}}{2(1 + (h_1)^2 + (h_2)^2)^{3/2}},$$

where

$$A = [(1 + (h_2)^2)h_{11} - 2h_1h_2h_{12} + (1 + (h_1)^2)h_{22}]^2 - 4(1 + (h_1)^2 + (h_2)^2)(h_{11}h_{22} - (h_{12})^2).$$

Note that λ_1 and λ_2 are invariants under the infinitesimal rigid motion in \mathbb{R}^3 .

Let

$$(4.2) \quad \Lambda[h] := \lambda_1\lambda_2 = \frac{h_{11}h_{22} - (h_{12})^2}{(1 + (h_1)^2 + (h_2)^2)^2}.$$

Then we have

THEOREM 4.1. $\Lambda[h]$ is an invariant of order two in the sense of Definition (3.1). In fact,

$$(4.3) \quad \begin{aligned} & 4 \Lambda[h] \circ \pi \\ &= (g_{11}g_{22} - (g_{12})^2)^{-2} [2((g_{12})^2 - g_{11}g_{22})(g_{11,y y} - 2g_{12,x y} + g_{22,x x}) \\ &+ g_{11}(g_{11,y}g_{22,y} - 2g_{12,x}g_{22,y} + (g_{22,x})^2) \\ &+ g_{12}(g_{11,x}g_{22,y} - g_{11,y}g_{22,x} - 2g_{11,y}g_{12,y} + 4g_{12,x}g_{12,y} - 2g_{12,x}g_{22,x}) \\ &+ g_{22}(g_{11,x}g_{22,x} - 2g_{11,x}g_{12,y} + (g_{11,y})^2)], \quad \text{mod } \mathcal{I}. \end{aligned}$$

Observe that the right hand side of (4.3) is the Gaussian curvature K as in (2.13) and that Theorem 4.1 implies that the product of the two principal curvatures, that is, the determinant of the Gauss map of a surface embedded in \mathbb{R}^3 is, in fact, independent of embeddings, which is the ‘Theorema Egregium’ of Gauss (cf. [10]).

Proof of Theorem 4.1. First, we express the left hand side of (4.3) as a differential function on $\Omega^2 \subset J^2(\mathbb{R}^2, \mathbb{R}^3)$: Since (3.4) and (3.5) in this case are

$$(4.4) \quad \begin{cases} u_x^3 = h_1 u_x^1 + h_2 u_x^2, \\ u_y^3 = h_1 u_y^1 + h_2 u_y^2, \end{cases}$$

and

$$(4.5) \quad \begin{cases} u_{xx}^3 = h_{11}(u_x^1)^2 + 2h_{12}u_x^1u_x^2 + h_{22}(u_x^2)^2 + h_1u_{xx}^1 + h_2u_{xx}^2, \\ u_{xy}^3 = h_{11}u_x^1u_y^1 + h_{12}(u_x^1u_y^2 + u_x^2u_y^1) + h_{22}u_x^2u_y^2 + h_1u_{xy}^1 + h_2u_{xy}^2, \\ u_{yy}^3 = h_{11}(u_y^1)^2 + 2h_{12}u_y^1u_y^2 + h_{22}(u_y^2)^2 + h_1u_{yy}^1 + h_2u_{yy}^2. \end{cases}$$

By solving (4.4) and (4.5) for h_i and h_{ij} , $i, j = 1, 2$, on Ω^2 (recall that the matrix of coefficients in (4.4) is nonsingular on Ω^2), and substituting in (4.2), we get

$$(4.6) \quad 4\Lambda[h] \circ \pi = 4[(J_3)^2(u_{xx}^3u_{yy}^3 - (u_{xy}^3)^2) + J_2J_3(u_{yy}^2u_{xx}^3 - 2u_{xy}^2u_{xy}^3 + u_{xx}^2u_{yy}^3) + (J_2)^2(u_{xx}^2u_{yy}^2 - (u_{xy}^2)^2) + J_1J_3(u_{yy}^1u_{xx}^3 - 2u_{xy}^1u_{xy}^3 + u_{xx}^1u_{yy}^3) + J_1J_2(u_{yy}^1u_{xx}^2 - 2u_{xy}^1u_{xy}^2 + u_{xx}^1u_{yy}^2) + (J_1)^2(u_{xx}^1u_{yy}^1 - (u_{xy}^1)^2)] / ((J_1)^2 + (J_2)^2 + (J_3)^2)^2,$$

where

$$J_1 = u_x^2u_y^3 - u_y^2u_x^3, \quad J_2 = u_x^1u_y^3 - u_y^1u_x^3, \quad J_3 = u_x^1u_y^2 - u_y^1u_x^2.$$

On the other hand, let $p_{ij}[u]$, $i, j = 1, 2$, be the left hand side of (3.1) with $n = 2, d = 1$, namely,

$$(4.7) \quad \begin{cases} p_{11}[u] = (u_x^1)^2 + (u_x^2)^2 + (u_x^3)^2, \\ p_{12}[u] = p_{21}[u] = u_x^1u_y^1 + u_x^2u_y^2 + u_x^3u_y^3, \\ p_{22}[u] = (u_y^1)^2 + (u_y^2)^2 + (u_y^3)^2. \end{cases}$$

Since $p_{ij}[u] = g_{ij}(x, y)$, mod \mathcal{I} , and therefore $p_{ij,x}[u] = \frac{\partial g_{ij}}{\partial x}$, mod \mathcal{I} , and so forth, where $p_{ij,x}$ is the total derivative of p_{ij} with respect to x , (2.13) implies that

$$(4.8) \quad 4K(x, y) = (p_{11}p_{22} - (p_{12})^2)^{-2} [2((p_{12})^2 - p_{11}p_{22})(p_{11,yy} - 2p_{12,xy} + p_{22,xx})$$

$$\begin{aligned}
& + p_{11}(p_{11,y}p_{22,y} - 2p_{12,x}p_{22,y} + (p_{22,x})^2) \\
& + p_{12}(p_{11,x}p_{22,y} - p_{11,y}p_{22,x} - 2p_{11,y}p_{12,y} + 4p_{12,x}p_{12,y} - 2p_{12,x}p_{22,x}) \\
& + p_{22}(p_{11,x}p_{22,x} - 2p_{11,x}p_{12,y} + (p_{11,y})^2)], \quad \text{mod } \mathcal{I}.
\end{aligned}$$

Substitute in (4.8) the right hand side of (4.7) for p_{ij} , and their total derivatives for $p_{ij,x}$, $p_{ij,y}$, $p_{ij,xx}$, $p_{ij,xy}$, $p_{ij,yy}$, respectively. Then computations using Mathematica® show that the right hand side of (4.8) is equal to the right hand side of (4.6), thus we have

$$(4.9) \quad \Lambda[h] \circ \pi = K(x, y), \quad \text{mod } \mathcal{I}. \quad \square$$

REMARK. We have found an algorithm of subtracting differential functions of \mathcal{I} from the right hand side of (4.6) to get the right hand side of (2.13), which also proves (4.9) (see, [16]).

Now we show that the right hand side of (2.18) is an intrinsic expression of an invariant of order three in the sense of Definition 3.1. Apply total differential operators D_x and D_y , respectively, to (4.9) to get

$$(4.10) \quad \begin{cases} (\Lambda[h]_1 \circ \pi)u_x^1 + (\Lambda[h]_2 \circ \pi)u_x^2 = K_x, & \text{mod } \mathcal{I}, \\ (\Lambda[h]_1 \circ \pi)u_y^1 + (\Lambda[h]_2 \circ \pi)u_y^2 = K_y, & \text{mod } \mathcal{I}, \end{cases}$$

where

$$\Lambda[h]_i = \frac{\partial}{\partial u^i} \left\{ \frac{h_{11}h_{22} - (h_{12})^2}{(1 + (h_1)^2 + (h_2)^2)^2} \right\}, \quad i = 1, 2.$$

In (2.18), substitute (4.10) for K_x and K_y , respectively, and substitute (4.7) for g_{ij} , then (2.18) becomes a differential function of $J^3(M, \mathbb{R}^3)$. Now we substitute (4.4) for u_x^3 and u_y^3 , and from the resulting expression we eliminate the elements of \mathcal{I} then all terms involving the derivatives of u^α , $\alpha = 1, 2, 3$, cancel out and we get an invariant of order three. In fact, if we define a differential function $\Lambda'[h]$ on $\Omega^3 \subset J^3(\mathbb{R}^2, \mathbb{R})$ by

$$(4.11) \quad \Lambda'[h] := \frac{(1 + (h_2)^2)(Q_1)^2 - 2h_1h_2Q_1Q_2 + (1 + (h_1)^2)(Q_2)^2}{1 + (h_1)^2 + (h_2)^2},$$

where

$$Q_1 = 4 \frac{h_1 h_{11} (h_{12})^2 + h_2 (h_{12})^3 - h_1 (h_{11})^2 h_{22} - h_2 h_{11} h_{12} h_{22}}{D^3} + \frac{h_{11} h_{122} - 2 h_{12} h_{112} - h_{22} h_{111}}{D^2},$$

$$Q_2 = 4 \frac{h_2 (h_{12})^2 h_{22} + h_1 (h_{12})^3 - h_2 h_{11} (h_{22})^2 - h_1 h_{11} h_{12} h_{22}}{D^3} + \frac{h_{22} h_{112} - 2 h_{12} h_{122} + h_{11} h_{222}}{D^2},$$

and

$$D = (1 + (h_1)^2 + (h_2)^2),$$

then by direct calculations as in the proof of Theorem 4.1, we can prove

THEOREM 4.2. $\Lambda'[h] \circ \pi = H(x, y), \mod \mathcal{I}$, where $H(x, y)$ is the right hand side of (2.18).

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