J. Korean Math. Soc. **34** (1997), No. 3, pp. 731–770

DIRICHLET FORMS, DIRICHLET OPERATORS, AND LOG-SOBOLEV INEQUALITIES FOR GIBBS MEASURES OF CLASSICAL UNBOUNDED SPIN SYSTEM

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ABSTRACT. We study Dirichlet forms and related subjects for the Gibbs measures of classical unbounded spin systems interacting via potentials which are superstable and regular. For any Gibbs measure μ, we construct a Dirichlet form and the associated diffusion process on $L^2(\Omega, d\mu)$, where $\Omega = (\mathbb{R}^d)^{\mathbb{Z}^\nu}$. Under appropriate conditions on the potential we show that the Dirichlet operator associated to a Gibbs measure μ is essentially self-adjoint on the space of smooth bounded cylinder functions. Under the condition of uniform log-concavity, the Gibbs measure exists uniquely and there exists a mass gap in the lower end of the spectrum of the Dirichlet operator. We also show that under the condition of uniform log-concavity, the unique Gibbs measure satisfies the log-Sobolev inequality. We utilize the general scheme of the previous works on the theory in infinite dimensional spaces developed by e.g., Albeverio, Antonjuk, Høegh-Krohn, Kondratiev, Röckner, and Kusuoka, etc, and also use the equilibrium condition and the regularity of Gibbs measures extensively.

1. Introduction

In this paper we study Dirichlet forms and the associated diffusion processes for the Gibbs measures of classical unbounded spin systems interacting via potentials which are superstable and regular in the sense of Ruelle [38-2]. For any Gibbs measure μ on $\Omega = (\mathbb{R}^d)^{\mathbb{Z}^\nu}$, we construct a Dirichlet form and the associated diffusion process on $L^2(\Omega, d\mu)$.

Received June 7, 1997.

1991 Mathematics Subject Classification: 47D07, 47N55, 60J60.

Key words and phrases: classical unbounded spin systems; equilibrium condition; Gibbs measures; regularity of measures; Dirichlet forms; diffusion processes; Dirichlet operators; log-Sobolev inequality.

Under appropriate conditions on the potential we then show that the Dirichlet operator associated to the Gibbs measure μ is essentially self-adjoint on the space of smooth bounded cylinder functions. We will give sufficient conditions to the potential so that the corresponding Gibbs measure satisfies the uniform log-concavity (R_μ-positivity) condition. Under the condition, we then show that the Gibbs measure exists uniquely and there exists a mass gap in the lower end of the spectrum of the Dirichlet operator. Furthermore, we will show that the unique Gibbs measure satisfies the log-Sobolev inequality. In this study we utilize the general scheme on the theory of Dirichlet forms in the infinite dimensional state spaces [3-2, 6-3, 10-2, 12-2, 15, 27, 29-2, 35] together with the equilibrium condition and the regularity of Gibbs measures [28, 33].

In [8], the essential self-adjointness of Dirichlet operators and the log-Sobolev inequality have been proved for polynomially bounded one-body and finite range two-body potentials. Thus, the results in this paper can be considered as an extension of those in [8] to more general class of potentials. See Assumption 2.1, Assumption 2.9, and Remark 2.10 in Section 2.

Dirichlet forms and the associated diffusion processes have been intensively investigated in connection with their important applications to mathematical physics and to the theory of random processes (see [22, 25, 37, 42] and references therein). The theory of Dirichlet forms on finite dimensional spaces is a well-known modern tool in the potential theory [22] and quantum mechanics [5]. There have been many efforts to extend the general theory to the case where the state spaces are of infinite dimensional, hence non-locally compact topological spaces [1-2, 3-2, 25, 6-3, 37]. In all cases the forms are given first on some minimal domains of smooth functions with compact support or cylinder ones. Most of results then touch upon the problems of the closability of the forms and the construction of corresponding diffusion processes. The uniqueness problem of determining whether a given closable form possessing the contraction property has a unique closed extension has also been discussed in recent years ([4, 6-3, 26, 30, 35] and references therein). Clearly the essential self-adjointness of the associated Dirichlet operator implies the uniqueness. In this direction, various conditions for the essential self-adjointness of Dirichlet operators have been

obtained [6-3, 26, 50].

In applications, the important property of a Dirichlet operator is when the logarithmic Sobolev inequality for the Gibbs measure appeares. The log-Sobolev inequality was first proven by Gross [24] for Gaussian measures on $\mathbb{R}^n$, and then extended in many directions [14-2, 19-2, 31, 32, 46, 47-3, 51]. The log-Sobolev inequality leads to the hypercontractivity for the semigroup generated by the Dirichlet operator and has a wide range of applications [19]. For Gibbs measures of bounded spin systems the log-Sobolev inequality was established by Stroock and Zegarlinski [46, 47-3]. In [8] and [51], the log-Sobolev inequality for Gibbs measures of unbounded spin systems with finite range pair potentials has been obtained. In [35], the essential self-adjointness of Dirichlet operators and the log-Sobolev inequality for Gibbs measures on loop spaces has been obtained. The log-Sobolev inequality for Gibbs measures on loop spaces has been also independently obtained in [10] and moreover in [10-2], the log-Sobolev inequality has been applied to show the uniqueness of Gibbs measures for quantum unbounded spin systems. In [30], we have proved the essential self-adjointness of Dirichlet operators and also proved that under the condition of uniform log-concavity (cf. Definition 2.12), the unique Gibbs measure for quantum unbounded spin systems satisfies the log-Sobolev inequality.

Let us describe briefly the results and the basic ideas in this paper. We deal with classical unbounded spin systems interacting via superstable and regular potentials [17, 28, 34, 39]. Let $\mathcal{C}$ be the family of finite subsets of the ν-dimensional lattice space $\mathbb{Z}^\nu$. Let $\Omega = (\mathbb{R}^d)^{\mathbb{Z}^\nu}$ and for each $i \in \mathbb{Z}^\nu$, let $\pi_i : \Omega \to \mathbb{R}^d$ be the projection $\pi_i(x) = x_i$, $x = (x_i)_{i\in\mathbb{Z}^\nu} \in \Omega$. For $y \in \mathbb{R}^d$, denote by $|y|$ the Euclidean norm on $\mathbb{R}^d$. We topologize Ω by the countable seminorms, $\{\rho_i\}_{i\in\mathbb{Z}^\nu}$, $\rho_i(x) = |\pi_i(x)|$. For each $\Lambda \subset \mathbb{Z}^\nu$ we have a local σ-algebra $\mathcal{F}_\Lambda$ of Borel sets for which ρ_i, $i \in \Lambda$, is continuous. Let $\mathcal{F} = \mathcal{F}_{\mathbb{Z}^\nu}$. For given interaction Φ, we denote by $\mathcal{G}^\Phi(\Omega)$ the set of corresponding Gibbs measures on $(\Omega, \mathcal{F})$. See Section 2.1 for the details. Then $\mathcal{G}^\Phi(\Omega)$ is non-empty, convex, compact in the local convergence topology, and a Choquet simplex (Theorem 2.5). Furthermore, each $\mu \in \mathcal{G}^\Phi(\Omega)$ is regular and satisfies the equilibrium condition. See Definition 2.3 and Definition 2.4.

Let Ω_{fin} be the subspace of Ω consisting of elements which have only

finite numbers of non-zero components: if $x = (x_i)_{i\in\mathbb{Z}^\nu} \in \Omega_{\text{fin}}$, there exists $\Delta \in \mathcal{C}$ such that $x_i = 0$ unless $i \in \Delta$. We shall introduce three inner products $(\cdot,\cdot)_0$, $(\cdot,\cdot)_-$, and $(\cdot,\cdot)_+$ on Ω_{fin}, and denote by $\mathcal{H}_0$, $\mathcal{H}_-$, and $\mathcal{H}_+$ the completions of Ω_{fin} with respect to the norms $|\cdot|_0$, $|\cdot|_-$, and $|\cdot|_+$ induced by $(\cdot,\cdot)_0$, $(\cdot,\cdot)_-$, and $(\cdot,\cdot)_+$, respectively. These inner products are introduced so that

$$\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_- \tag{1.1}$$

is a rigging of $\mathcal{H}_0$ by $\mathcal{H}_+$ and $\mathcal{H}_-$ and such that the embeddings are everywhere dense and belong to the Hilbert-Schmidt class. The duality between $\mathcal{H}_+$ and $\mathcal{H}_-$ given by $(\cdot,\cdot)_0$ will be denoted by $<\cdot,\cdot>$. We shall also introduce the subspace $\Omega_{\log}$ of $\mathcal{H}_-$ defined by

$$\Omega_{\log} = \{x \in \mathcal{H}_- : \exists N \in \mathbb{N} \text{ s.t. } |x_i|^2 \le N \log(|i| + 1),\ \forall i \neq 0\}. \tag{1.2}$$

By the regularity of Gibbs measures, it turns out that (Remark 2.6 (b))

$$\mu(\Omega_{\log}) = 1 \quad \text{for any } \mu \in \mathcal{G}^\Phi(\Omega). \tag{1.3}$$

Denote by $C_b^k(\mathcal{H}_-, \mathbb{R})$, $k \in \mathbb{N} \cup \{0, \infty\}$, the set of mappings from $\mathcal{H}_-$ into $\mathbb{R}$ that are k-times continuously Fréchet differentiable [15] and have bounded derivatives. Let $\mathcal{F}C_b^\infty$ be the set of smooth bounded cylinder functions on $\mathcal{H}_-$: that is, if $u \in \mathcal{F}C_b^\infty$, there exist $\Lambda \in \mathcal{C}$ and $f \in C_b^\infty((\mathbb{R}^d)^\Lambda)$ such that $u(x) = f(x_\Lambda)$. For given $\mu \in \mathcal{G}^\Phi(\Omega)$, we shall consider a form on $L^2(\mathcal{H}_-, d\mu)$ of the following type: for an orthonormal basis $\{k_n\}_{n=1}^\infty \subset \Omega_{\text{fin}}$ for $\mathcal{H}_0$,

$$D(\mathcal{E}_\mu) = \mathcal{F}C_b^\infty$$

$$\mathcal{E}_\mu(u, v) = \frac{1}{2}\sum_{n=1}^\infty \int_{\mathcal{H}_-} (\nabla^n u, \nabla^n v)_0 \, d\mu,$$

where $\nabla^n u$ is the directional derivative of u in the direction of k_n. Let β be the logarithmic derivative of the Gibbs measure μ given as in (2.19). For any $x \in \Omega_{\log}$, $|\beta(x)|_-$ is finite. Let H_μ be the Dirichlet operator defined by

$$H_\mu u(x) = -\frac{1}{2}\Delta u(x) - \frac{1}{2} < \beta(x), \nabla u(x) >, \quad u \in \mathcal{F}C_b^\infty.$$

See Section 2.2 for the notations. Using the regularity of a given Gibbs measure μ, it can be shown that $|\beta|_- \in L^2(\mathcal{H}_-, d\mu)$ (Lemma 3.1). From this fact and the equilibrium condition for Gibbs measure μ, it follows that H_μ is a well-defined symmetric operator and satisfies the relation (Proposition 3.3)

$$\mathcal{E}_\mu(u, v) = (u, H_\mu v)_{L^2}, \quad u, v \in \mathcal{F}C_b^\infty.$$

Since $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$ is associated with a symmetric operator $(H_\mu, \mathcal{F}C_b^\infty)$, it is closable. By use of the method of [12], the closure becomes a *Dirichlet form*.

Following Albeverio and Röckner [13], we will construct the associated diffusion process. For the purpose, we need to show that several conditions must be satisfied. Among them the hardest part is to give a sequence $K_n \subset \mathcal{H}_-$, $n \in \mathbb{N}$, K_n compact, such that $\lim_{n\to\infty} \mathrm{Cap}(H_- \setminus K_n) = 0$. We will use the method of [27] together with the equilibrium condition and the regularity of Gibbs measures to show that such a sequence actually exists.

Next we consider the problem of the essential self-adjointness of the Dirichlet operator H_μ for a given Gibbs measure μ. In [6-2], Albeverio, Kondratiev, and Röckner gave an *approximate criterium* of essential self-adjointness of Dirichlet operators on $C_b^2(\mathcal{H}_-)$. We impose further conditions on the potentials (Assumption 2.9) and check that under the conditions the approximate criterium of [6-2] are satisfied in our case. In [8], the same authors mentioned above gave a slightly improved approximation criterion (the basic idea, however, flows the same line) for the essential self-adjointness.

Finally we discuss the existence of a mass gap of the Dirichlet operator and the log-Sobolev inequality for the Gibbs measure. As in [15, 8], we introduce the notion of uniform log-concavity (R_μ-positivity) of a Gibbs measure μ. We will give sufficient conditions to the potential for the uniform log-concavity (Theorem 2.14). It turns out that under the condition, the Gibbs measure exists uniquely. We will then show that under the both condition of essential self-adjointness of the Dirichlet operator H_μ and of uniform log-concavity, the unique Gibbs measure satisfies the log-Sobolev inequality. See [14] and [8] for the related results.

We organize this paper as follows: In Section 2.1, we introduce notations, definitions and basic assumptions on the potentials, and then

describe specific properties of Gibbs measures from [28, 33]. In Section 2.2, we introduce Dirichlet forms and Dirichlet operators for Gibbs measures, and give the main results in this paper. In Section 3, we construct Dirichlet forms and the associated diffusion processes employing the methods in [12-2] and [27]. In Section 4, we show the essential self-adjointness of Dirichlet operators under Assumption 2.1 and Assumption 2.9. Main ingredients shall be the approximate criterium of essential self-adjointness of Dirichlet operators [6] and the regularity of Gibbs measures. In Section 5, we will prove the uniform log-concavity of the Gibbs measure under proper conditions on the potential (Theorem 2.14). We also prove the log-Sobolev inequality for the unique Gibbs measure under the uniform log-concavity condition (Theorem 2.16).

We remark here that the original version of this manuscript has been already appeared two years ago. In the mean time, some further results have been obtained by many authors, see e.g., [8, 51, 30, 35]. In order to accommodate those results we have revised the original version of this paper.

2. Notations, preliminaries, and main results

2.1. Classical unbounded spin systems; Gibbs measures

We consider the classical unbounded spin systems interacting via potentials which are superstable and regular. The systems were studied in detail in [28] (see also [17, 34, 39]). As a preparation, we briefly describe the systems we consider and collect basic results which will be used in the sequel.

Let $\mathbb{Z}^\nu$ be the ν-dimensional lattice space. Denote by $\mathcal{C}$ the class of finite subsets of $\mathbb{Z}^\nu$. At each site $i \in \mathbb{Z}^\nu$ we associate an identical copy of $\mathbb{R}^d$. For $x = (x^1, x^2, \cdots, x^d) \in \mathbb{R}^d$ and $i = (i_1, i_2, \cdots, i_\nu) \in \mathbb{Z}^\nu$ we write

$$|x| = \left(\sum_{l=1}^{d} (x^l)^2\right)^{1/2}, \quad |i| = \max_{1\le k\le \nu} |i_k|. \tag{2.1}$$

For each $\Lambda \in \mathcal{C}$, we write

$$x_\Lambda = \{x_i : i \in \Lambda\},\ dx_\Lambda = \prod_{i\in\Lambda} dx_i, \tag{2.2}$$

where dx_i is the Lebesgue measure on $\mathbb{R}^d$. In this paper, we only consider one-body and two-body potentials, and introduce the local potential

$$V(x_\Lambda) = \sum_{i\in\Lambda} \Phi_{\{i\}}(x_i) + \sum_{\substack{\{i,j\}:\\ i,j\in\Lambda}} \Phi_{\{i,j\}}(x_i, x_j), \tag{2.3}$$

where for any $i, j \in \mathbb{Z}^\nu$, $\Phi_{\{i\}}$ and $\Phi_{\{i,j\}}$ are one-body and two-body interaction potentials which are measurable real valued functions on $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^d$, respectively. Throughout this paper we impose the following conditions on the interaction:

ASSUMPTION 2.1. *The interaction $\Phi = (\Phi_\Delta)_{\substack{\Delta\subset\mathbb{Z}^\nu\\ |\Delta|\le 2}}$ satisfies the following conditions:*

(a) *There exist a differentiable function $P(x)$ on $\mathbb{R}^d$ and positive constants a and b such that for each $i \in \mathbb{Z}^\nu$, $\Phi_{\{i\}}(x_i) = P(x_i)$ and for some $\gamma \ge 2$*

$$P(x) \ge a|x|^\gamma - b.$$

(If $\gamma = 2$, then the constant a is assumed to be sufficiently large.) Moreover, for any $\alpha > 0$ there exists $M(\alpha)$ such that the bound

$$\left|\frac{\partial}{\partial x^l} P(x)\right| \le M(\alpha)\exp(\alpha|x|^2), \quad l = 1, 2, \cdots, d,$$

holds.

(b) *For each $r \in \mathbb{N}$, there exists differentiable symmetric function $U(\cdot,\cdot;r) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $\Phi_{\{i,j\}}(x_i, x_j) = U(x_i, x_j; |i-j|) = U(x_j, x_i; |i-j|)$ for any $i, j \in \mathbb{Z}^\nu$. Moreover, there exists a decreasing function Ψ on $\mathbb{N}$ such that $\Psi(r) \le Kr^{-\nu-\varepsilon}$ for some constants K and $\varepsilon > 0$ and such that the bounds*

$$|U(x, y; |i-j|)| \le \Psi(|i-j|)\frac{1}{2}(|x|^2 + |y|^2),$$

$$\left|\frac{\partial}{\partial x^l} U(x, y; |i-j|)\right| \le \Psi(|i-j|)(|x| + |y|), \quad l = 1, 2, \cdots, d$$

hold.

REMARK 2.2. By the above conditions the interaction is invariant under $\mathbb{Z}^\nu$-actions and moreover, superstable and regular, i.e., there exists $A > 0$ and $c \in \mathbb{R}$ such that for any $x_\Lambda \in (\mathbb{R}^d)^\Lambda$,

$$V(x_\Lambda) \geq \sum_{i\in\Lambda} \left(A|x_i|^2 - c\right), \tag{2.4}$$

and if Λ_1, Λ_2 are disjoint finite subsets of $\mathbb{Z}^\nu$ and if we write

$$V(x_{\Lambda_1\cup\Lambda_2}) = V(x_{\Lambda_1}) + V(x_{\Lambda_2}) + W(x_{\Lambda_1}, x_{\Lambda_2}),$$

then the bound

$$|W(x_{\Lambda_1}, x_{\Lambda_2})| \leq \sum_{i\in\Lambda_1}\sum_{j\in\Lambda_2} \Psi(|i-j|)\frac{1}{2}\left(|x_i|^2 + |x_j|^2\right) \tag{2.5}$$

holds. Furthermore, by Assumption 2.1 (a) we may take A in (2.4) to be sufficiently large so that we may assume that

$$\sum_{i\in\mathbb{Z}^\nu} \Psi(|i|) < A,$$

which was needed to show the regularity of Gibbs measures in Theorem 2.5 [28, 33, 39].

Let $\Omega = (\mathbb{R}^d)^{\mathbb{Z}^\nu}$ and for each $i \in \mathbb{Z}^\nu$, let $\pi_i : \Omega \to \mathbb{R}^d$ be the projection $\pi_i(x) = x_i$, $x = (x_i)_{i\in\mathbb{Z}^\nu} \in \Omega$. We topologize Ω by the countable seminorms, $\{\rho_i\}_{i\in\mathbb{Z}^\nu}$: $\rho_i(x) = |\pi_i(x)|$. Notice that this topology is equivalent to the metric topology given by the metric

$$d(x,y) = \sum_{i\in\mathbb{Z}^\nu} 2^{-|i|}\frac{|x_i - y_i|}{1+|x_i - y_i|}, \quad x, y \in \Omega, \tag{2.6}$$

where $x = (x_i)_{i\in\mathbb{Z}^\nu}$ and $y = (y_i)_{i\in\mathbb{Z}^\nu}$. For each subset $\Lambda \subset \mathbb{Z}^\nu$, we have a local σ-algebra $\mathcal{F}_\Lambda$, which is the minimal σ-algebra of Borel sets for which ρ_i, $i \in \Lambda$, is continuous. We simply write $\mathcal{F}$ for $\mathcal{F}_{\mathbb{Z}^\nu}$. By $\mathcal{P}(\Omega, \mathcal{F})$ we mean the set of probability measures on $(\Omega, \mathcal{F})$.

Before introducing Gibbs measures on Ω, we give the notion of regular measures on Ω:

DEFINITION 2.3. A Borel probability measure μ on $(\Omega, \mathcal{F})$ is said to be *regular* if there exist A^* and $\delta > 0$ so that the projection μ_Λ of μ on any $(\Omega, \mathcal{F}_\Lambda)$, being understood as a measure on $((\mathbb{R}^d)^\Lambda, \mathcal{B}((\mathbb{R}^d)^\Lambda))$, satisfies

$$g(x_\Lambda|\mu) \leq \exp\left[-\sum_{i\in\Lambda}(A^*|x_i|^2 - \delta)\right],$$

where $g(x_\Lambda|\mu)$ is such that $\mu_\Lambda(dx_\Lambda) = g(x_\Lambda|\mu)dx_\Lambda$.

Let us define

$$\mathfrak{S} = \bigcup_{N\in\mathbb{N}} \mathfrak{S}_N, \tag{2.7}$$

$$\mathfrak{S}_N = \{x \in \Omega : \forall l, \sum_{|i|\leq l} |x_i|^2 \leq N^2(2l+1)^\nu\}.$$

This definition is invariant under linear translations of $\mathbb{Z}^\nu$. It can be shown that each regular measure on $(\Omega, \mathcal{F})$ has its support on $\mathfrak{S}$ [38]. We say that a measure μ is *tempered* if $\mu(\mathfrak{S}) = 1$.

For $x \in \Omega$ and $\Lambda \in \mathcal{C}$, we write

$$\begin{aligned} W(x_\Lambda, x_{\Lambda^c}) &= \sum_{i\in\Lambda,\, j\in\Lambda^c} \Phi_{\{i,j\}}(x_i, x_j) \\ &= \sum_{i\in\Lambda,\, j\in\Lambda^c} U(x_i, x_j; |i-j|). \end{aligned} \tag{2.8}$$

The partition function in a finite $\Lambda \subset \mathbb{Z}^\nu$ for the interaction Φ with boundary condition $y \in \mathfrak{S}$ is defined by

$$Z_\Lambda^\Phi(y) \equiv \int dx_\Lambda \exp[-V(x_\Lambda) - W(x_\Lambda, y_{\Lambda^c})]. \tag{2.9}$$

Notice that the partition function is well defined from the assumptions on Φ [38, 28]. The *Gibbs specification* $\gamma^\Phi = (\gamma_\Lambda^\Phi)_{\Lambda\in\mathcal{C}}$ with respect to $\mathfrak{S}$ is defined by [23, 28, 33]

$$\gamma_\Lambda^\Phi(A|y) = \begin{cases} Z_\Lambda^\Phi(y)^{-1} \int dx_\Lambda \exp[-V(x_\Lambda) - W(x_\Lambda, y_{\Lambda^c})] \\ \qquad \times 1_A(x_\Lambda y_{\Lambda^c}), & \text{if } y \in \mathfrak{S} \\ 0, \qquad \text{if } y \notin \mathfrak{S}, \end{cases} \tag{2.10}$$

where $A \in \mathcal{F}$ and 1_A is the indicator function of A and $x_\Lambda y_{\Lambda^c}$ is the configuration defined by x_Λ on Λ and y_{Λ^c} on Λ^c, respectively. It is easy to check that the Gibbs specification satisfies the consistent condition [23]: for $\Delta \subset \Lambda$, $y \in \mathfrak{S}$,

$$\begin{aligned}\gamma_\Lambda^\Phi \gamma_\Delta^\Phi(A|y) &\equiv \int_{\mathfrak{S}} \gamma_\Lambda^\Phi(d\tilde{x}|y)\gamma_\Delta^\Phi(A|\tilde{x}) \\ &= \gamma_\Lambda^\Phi(A|y).\end{aligned}$$

We now give a definition of Gibbs measures on $(\Omega, \mathcal{F})$:

DEFINITION 2.4. A Gibbs measure μ for the potential Φ is a tempered Borel probability measure on $(\Omega, \mathcal{F})$ satisfying the equilibrium conditions (equations)

$$\mu(A) = \int \mu(dx)\gamma_\Lambda^\Phi(A|x), \quad A \in \mathcal{F}.$$

We denote by $\mathcal{G}^\Phi(\Omega)$ the family of all Gibbs measures.

We summarize the results from Theorems 4.3 – 4.5 of [28] and Theorem 2.7 of [33]:

THEOREM 2.5. *Let the hypotheses of Assumption 2.1 hold. Then any Gibbs measure is regular. Furthermore, $\mathcal{G}^\Phi(\Omega)$ is non-empty, convex, compact in the local convergence topology, and a Choquet simplex.*

REMARK 2.6. (a) The above is the classical version of Theorem 2.7 of [33]. The existence and the regularity of Gibbs measures were shown in [28]. Direct applications of the methods used in the proof of Theorem 2.7 of [33] give the rest of the theorem.
(b) Denote by $\Omega_{\log}$ the subset of Ω defined by
(2.11)

$$\Omega_{\log} \equiv \{x = (x_i)_{i\in\mathbb{Z}^\nu} \in \Omega : \exists N \text{ s.t. } |x_i|^2 \le N \log(|i| + 1),\ \forall i \neq 0\}.$$

Due to the regularity of Gibbs measures, one can show for any Gibbs measure μ that $\mu(\Omega_{\log}) = 1$. In fact, this follows from Lemma 3.1 of [28] as a corollary.

2.2 Main results

We introduce Dirichlet form and Dirichlet operator associated to a Gibbs measure $\mu \in \mathcal{G}^{\Phi}(\Omega)$ and then list main results in this paper. We begin with some notations. As in Introduction let Ω_{fin} be the subset of Ω consisting of elements which have only finite members of non-zero components, i.e.,

$$\Omega_{\text{fin}} := \{x = (x_i)_{i\in\mathbb{Z}^\nu} \in \Omega : \exists\Delta \in \mathcal{C} \text{ s.t. } x_i = 0 \text{ if } i \notin \Delta\}. \tag{2.12}$$

Let us introduce inner products in Ω_{fin} as follows: for a given (fixed) real number $\sigma > 0$ and for $x = (x_i),\ y = (y_i) \in \Omega_{\text{fin}}$,

$$\begin{aligned} (x,y)_0 &\equiv \sum_{i\in\mathbb{Z}^\nu} (x_i, y_i), \\ (x,y)_- &\equiv \sum_{i\in\mathbb{Z}^\nu} \mathrm{e}^{-\sigma|i|}(x_i, y_i), \\ (x,y)_+ &\equiv \sum_{i\in\mathbb{Z}^\nu} \mathrm{e}^{\sigma|i|}(x_i, y_i), \end{aligned} \tag{2.13}$$

where for each $i \in \mathbb{Z}^\nu$, (x_i, y_i) is the Euclidean inner product in $\mathbb{R}^d$. The constant $\sigma > 0$ will be fixed according to Assumption 2.9 (d). Denote by $|\cdot|_0$, $|\cdot|_-$, and $|\cdot|_+$ the norms induced by $(\cdot,\cdot)_0$, $(\cdot,\cdot)_-$, and $(\cdot,\cdot)_+$, respectively. Let $\mathcal{H}_0$, $\mathcal{H}_-$, and $\mathcal{H}_+$ be the separable real Hilbert spaces obtained by completions of Ω_{fin} with respect to $|\cdot|_0$, $|\cdot|_-$,and $|\cdot|_+$, respectively. Notice that the embeddings

$$\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_- \tag{2.14}$$

are everywhere dense and belong to the Hilbert-Schmidt class. Thus, the above is a rigging of $\mathcal{H}_0$ by $\mathcal{H}_+$ and $\mathcal{H}_-$. The duality between $\mathcal{H}_+$ and $\mathcal{H}_-$ given by the inner product in $\mathcal{H}_0$ will be denoted by $< \cdot,\cdot >$. Let $\Omega_{\log}$ be the subset of Ω defined as in (2.11). Since $\Omega_{\log} \subset \mathcal{H}_-$, $\mu(\mathcal{H}_-) = 1$.

We now focus on $L^2(\mathcal{H}_-, d\mu)$ for a given $\mu \in \mathcal{G}^{\Phi}(\Omega)$. For $n = 0, 1, 2, \cdots$, we denote by $C_b^n(\mathbb{R}^m)$ the n-times continuously differentiable functions with bounded l-th derivatives, $l = 0, 1, \cdots, n$. Let $\mathcal{F}C_b^n$ be the family of cylinder functions of the C_b^n type:

$$\mathcal{F}C_b^n = \{u : \mathcal{H}_- \to \mathbb{R} : \exists\Lambda \in \mathcal{C},\ f \in C_b^n((\mathbb{R}^d)^\Lambda) \text{ s.t. } u(x) = f(x_\Lambda)\}.$$

Put $\mathcal{F}C_b^\infty = \cap_n \mathcal{F}C_b^n$. It turns out that $\mathcal{F}C_b^\infty$ is dense in $L^p(\mathcal{H}_-, d\mu)$, $p \geq 1$, for any Gibbs measure μ. See Lemma 3.4.

Following [3-2, 14-2, 6-3, 12-2, 26] we introduce a Dirichlet form for a Gibbs measure $\mu \in \mathcal{G}^\Phi(\Omega)$. Define for $u \in \mathcal{F}C_b^\infty$ and $k \in \mathcal{H}_-$ the Gâteaux-type derivative of u in the direction k:

$$\frac{\partial u(x)}{\partial k} = \frac{d}{ds} u(x + sk)|_{s=0}, \quad x \in \mathcal{H}_-.$$

Define for $k \in \mathcal{H}_-$

$$\mathcal{E}_k(u, v) = \frac{1}{2} \int_{\mathcal{H}_-} \frac{\partial u(x)}{\partial k} \frac{\partial v(x)}{\partial k} d\mu(x), \quad u, v \in \mathcal{F}C_b^\infty.$$

Observe that if $k = (k_i) \in \mathcal{H}_-$ and $u(x) = f(x_\Lambda)$,

$$\frac{\partial u(x)}{\partial k} = \sum_{i \in \Lambda} (\nabla^i f(x_\Lambda), k_i), \tag{2.15}$$

where ∇^i is the gradient operator with respect to $x_i = (x_i^1, \cdots, x_i^d)$ variable. Let $\{k_n\}$ be an orthonormal basis of $\mathcal{H}_0$ such that $k_n \in \Omega_{\text{fin}} \subset \mathcal{H}_+$ for each $n \in \mathbb{N}$. Define a form on $L^2(\mathcal{H}_-, d\mu)$ by

$$\begin{aligned} D(\mathcal{E}_\mu) &= \mathcal{F}C_b^\infty \\ \mathcal{E}_\mu(u, v) &= \sum_{n=1}^\infty \mathcal{E}_{k_n}(u, v). \end{aligned} \tag{2.16}$$

Obviously, $\mathcal{E}_\mu(u, v)$ is independent of the choice of orthonormal basis [37].

Let us consider a coordinate free version of the form (2.16) [37]. Observe that by (2.15), for $u \in \mathcal{F}C_b^\infty$ and $x \in \mathcal{H}_-$ fixed, $k \to \frac{\partial u(x)}{\partial k}$ is a continuous linear functional on $\mathcal{H}_-$. Define $\nabla u(x) \in \mathcal{H}_+$ by

$$< \nabla u(x), k >= \frac{\partial u(x)}{\partial k}, \quad k \in \mathcal{H}_-. \tag{2.17}$$

It follows from (2.15) – (2.17) that

$$\begin{aligned} D(\mathcal{E}_\mu) &= \mathcal{F}C_b^\infty \\ \mathcal{E}_\mu(u, v) &= \frac{1}{2} \int_{\mathcal{H}_-} (\nabla u(x), \nabla u(x))_{\mathcal{H}_0} d\mu. \end{aligned} \tag{2.18}$$

The form $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$ is a densely defined positive definite symmetric bilinear form on $L^2(\mathcal{H}_-, d\mu)$.

We review briefly Dirichlet forms and the associated processes [22] on a separable real Hilbert space $(\mathfrak{H}, (\cdot,\cdot)) \equiv L^2(\Omega, d\mu)$. A pair $(\mathcal{E}, D(\mathcal{E}))$ is a form on $\mathfrak{H}$ if $D(\mathcal{E})$ is a linear subspace of $\mathfrak{H}$ and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is non-negative symmetric bilinear form. Given a form $(\mathcal{E}, D(\mathcal{E}))$ and $\alpha > 0$, we set $\mathcal{E}_\alpha \equiv \mathcal{E} + \alpha(\cdot,\cdot)$, $D(\mathcal{E}_\alpha) = D(\mathcal{E})$. $(\mathcal{E}, D(\mathcal{E}))$ is said to be *closed* if the pre-Hilbert space $(\mathcal{E}_1, D(\mathcal{E}_1))$ is complete and *closable* if it has a closed extension, i.e., there exists a closed form $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ on $\mathfrak{H}$ such that $D(\mathcal{E}) \subset D(\tilde{\mathcal{E}})$ and $\mathcal{E} = \tilde{\mathcal{E}}$ on $D(\mathcal{E})$. A form $(\mathcal{E}, D(\mathcal{E}))$ on a Hilbert space $L^2(\Omega, d\mu)$ is said to be *Markovian* if for each $\varepsilon > 0$ there exists a real function $\phi_\varepsilon(t)$, $t \in \mathbb{R}$, such that $\phi_\varepsilon(t) = t$ for $0 < t < 1$, $-\varepsilon \le \phi_\varepsilon(t) \le 1 + \varepsilon$ for any $t \in \mathbb{R}$, $0 \le \phi_\varepsilon(t) - \phi_\varepsilon(s) \le t - s$ whenever $s < t$, and for any $u \in D(\mathcal{E})$, it holds that $\phi_\varepsilon(u) \in D(\mathcal{E})$ and $\mathcal{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \le \mathcal{E}(u, u)$. A *Dirichlet form* on a real Hilbert space $L^2(\Omega, d\mu)$ is defined to be a closed Markovian form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\Omega, d\mu)$.

We return to the form defined in (2.16) ((2.18)). We have the following result:

THEOREM 2.7. *Let μ be a Gibbs measure with interaction Φ. Under Assumption 2.1, the form $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$ defined by (2.16) is closable on $L^2(\mathcal{H}_-, d\mu)$ and its closure is a symmetric Dirichlet form.*

Next, we consider the associated diffusion process. Let $(\mathcal{E}, D(\mathcal{E}))$ be an arbitrary Dirichlet form on $L^2(\Omega, d\mu)$. Here we do not restrict Ω and μ to any special case. Define

$$D(L) = \{u \in D(\mathcal{E}) : v \mapsto \mathcal{E}(u, v) \text{ is continuous w.r.t. } (\cdot,\cdot)_{L^2}^{1/2} \text{ on } D(\mathcal{E})\},$$

and let $(L, D(L))$ be the linear operator defined by $(-Lu, v) = \mathcal{E}(u, v)$. Then, L is the generator of a strongly continuous Markovian semigroup $(T_t)_{t\ge 0}$, i.e., $T_t = e^{tL}$, $t \ge 0$, and for all $u \in L^2(\Omega, d\mu)$, $0 \le u \le 1$ implies $0 \le T_t u \le 1$, $t \ge 0$. See [22, 37] for details. A Markov process $(\tilde{\Omega}, \tilde{\mathcal{F}}, (X_t)_{t\ge 0}, P_x)$ with state space Ω is said to be *associated* with $(\mathcal{E}, D(\mathcal{E}))$ if for any $u : \Omega \to \mathbb{R}$, $\mathcal{F}$-measurable, bounded, and $t \ge 0$,

$$(T_t u)(x) = \int_{\tilde{\Omega}} u(X_t)\, dP_x \quad \text{for } \mu - \text{a.e. } x \in \Omega.$$

A Markov process associated with $(\mathcal{E}, D(\mathcal{E}))$ is called a *diffusion process* if it is a Hunt process having continuous sample paths P_x-almost surely for each $x \in \Omega$ [22].

THEOREM 2.8. *Let the hypotheses of Assumption 2.1 be satisfied and $\mu \in \mathcal{G}^{\Phi}(\Omega)$. Then there exists a diffusion process with state space $\mathcal{H}_-$ associated with the closure of $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$.*

In fact, under Assumption 2.1 and Assumption 2.9 listed below, the diffusion process associated to the Dirichlet form for a Gibbs measure is unique by Theorem 2.11. The proofs of Theorem 2.7 and Theorem 2.8 will be produced in the next section.

We introduce the Dirichlet operator associated to a Gibbs measure. Let $P(x)$ and $U(x, y; |i-j|)$ be the one-body and two-body potentials given in Assumption 2.1. For $x \in \Omega_{\log}$, let $\beta(x)$ be an element of $\mathcal{H}_-$ given by

$$\begin{aligned} (2.19) \qquad & \beta(x) = (\beta_i(x))_{i \in \mathbb{Z}^\nu}, \quad x \in \Omega_{\log}, \\ & \beta_i(x) = -\nabla^i P(x_i) - \sum_{\substack{j \in \mathbb{Z}^\nu: \\ j \neq i}} \nabla^i U(x_i, x_j; |i-j|), \end{aligned}$$

where ∇^i is the gradient operator with respect to $x_i = (x_i^1, \cdots, x_i^d)$. In Section 3, we will show that $\beta(x) \in \mathcal{H}_-$ if $x \in \Omega_{\log}$. Denote by $C^k(\mathcal{H}_-, \mathcal{B})$ the set of mappings from $\mathcal{H}_-$ into a Banach space $\mathcal{B}$ that are k-times continuously differentiable in the sense of Fréchet. See, e.g., [14-2]. Define $C_b^k(\mathcal{H}_-, \mathcal{B})$ as the subset of $C^k(\mathcal{H}_-, \mathcal{B})$ which is characterized by the boundedness in usual operator norms of the derivatives

$$f^{(l)} : \mathcal{H}_- \to \mathcal{L}(\mathcal{H}_-, \mathcal{L}(\mathcal{H}_-, \cdots, \mathcal{L}(\mathcal{H}_-, \mathcal{B}) \cdots)), \quad l = 0, 1, 2, \cdots, k.$$

For $f : \mathcal{H}_- \to \mathbb{R}$, identify $f'(\cdot) \in \mathcal{L}(\mathcal{H}_-, \mathbb{R})$ with the vector $\widehat{f'}(\cdot) \in \mathcal{H}_+$ and $f''(\cdot)$ with the operator $\widehat{f''}(\cdot) \in \mathcal{L}(\mathcal{H}_-, \mathcal{H}_+)$ by the formulae

$$\begin{aligned} (2.20) \qquad & f'(x)h = < \widehat{f'}(x), h >, \\ & (f''(x)h)g = < \widehat{f''}(x)h, g >, \quad h, g \in \mathcal{H}_-, \ x \in \mathcal{H}_-. \end{aligned}$$

For the function $f \in C_b^2 \equiv C_b^2(\mathcal{H}_-, \mathbb{R})$ we use the symbol $\nabla f = f'$ and $\Delta f = \mathrm{Tr}_{\mathcal{H}_0}(f'')$. Notice that ∇f is consistent with that of (2.17). We

introduce in the space C_b^2 the norm

$$\|f\|_{C_b^2} \equiv \sup_{x\in\mathcal{H}_-} \{|f(x)| + |f'(x)|_+ + \|f''(x)\|_{\mathcal{L}(\mathcal{H}_-,\mathcal{H}_+)}\}.$$

Define a differential operator H_μ on the domain $D(H_\mu) = \mathcal{F}C_b^\infty$ by the formula

(2.21)

$$H_\mu u(x) = -\frac{1}{2}\Delta u(x) - \frac{1}{2} < \beta(x), \nabla u(x) >, \quad u \in \mathcal{F}C_b^\infty,\ x \in \mathcal{H}_-.$$

In the next section, we will show that H_μ is a well-defined symmetric operator in $L^2(\mathcal{H}_-, d\mu)$ and that the relation

$$\mathcal{E}_\mu(u,v) = (u, H_\mu v)_{L^2}, \quad u, v \in \mathcal{F}C_b^\infty, \tag{2.22}$$

holds. Since $\mathcal{E}_\mu$ is associated to the symmetric operator H_μ, $\mathcal{E}_\mu$ is closable. We call the operator H_μ in $L^2(\Omega, d\mu)$ with $D(H_\mu) = \mathcal{F}C_b^\infty$ the *Dirichlet operator* associated to μ.

In order to ensure the essential self-adjointness of H_μ, we need to impose additional conditions to the interaction.

ASSUMPTION 2.9. *Let $P(x)$ and $U(x,y;|i-j|)$ be the one-body and two-body interactions introduced in Assumption 2.1. We assume further that the following properties hold : $P(x)$ and $U(x,y;|i-j|)$, $i,j \in \mathbb{Z}^\nu$, are three times continuously differentiable functions satisfying the following conditions:*

(a) *For any positive real number $\alpha > 0$, there exists positive constant $M(\alpha)$ such that the bound*

$$\sum_{l,k=1}^{d} \left| \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^k} P(x) \right| \le M(\alpha)\exp(\alpha|x|^2)$$

holds.

(b) *There exists $M \in \mathbb{R}$ such that*

$$Hess.\ P(x) \ge M\mathbf{1}, \quad x \in \mathbb{R}^d,$$

where Hess.$P(x)$ is the Hessian of $P(x)$, i.e., the $d \times d$ matrix whose l-k elements are given by $(\frac{\partial}{\partial x^l}\frac{\partial}{\partial x^k}P(x))$, $l,k = 1,2,\cdots,d$.

(c) *In the case of $d \geq 2$, there exist a function $Q : \mathbb{R} \to \mathbb{R}$ and an element $b \in \mathbb{R}^d$ such that $P(x) = Q(|x|) + b \cdot x$, $x \in \mathbb{R}^d$.*

(d) *The function Ψ of Assumption 2.1* (b) *is exponentially decreasing: there exist $K > 0$ and $\sigma > 0$ such that*

$$\Psi(r) \leq Ke^{-2\sigma r}, \quad r \in \mathbb{N}.$$

Furthermore, the bounds

$$\left|\frac{\partial}{\partial x^l}\frac{\partial}{\partial x^k}U(x,y;|i-j|)\right| + \left|\frac{\partial}{\partial x^l}\frac{\partial}{\partial y^k}U(x,y;|i-j|)\right| \leq \Psi(|i-j|), \quad l,k = 1,2,\cdots,d$$

hold and for any $y \in \mathbb{R}^d$ and $r \in \mathbb{N}$ the third order partial derivatives of $U(x,y;r)$ with respect to x and y variables assume to be bounded by $\Psi(r)$.

REMARK 2.10. We impose the strong regularity condition (Assumption 2.9 (d)) to show the essential self-adjointness of H_μ. If $P(x)$ is polynomially bounded, one can replace $e^{-\sigma|i|}$ by $(|i|+1)^{-\sigma}$, $\sigma > \nu$, in the definition of the norm $|x|_-$. Then, the bound $\Psi(|i|) \leq K(|i|+1)^{-2\sigma}$ is sufficient. See the proofs of Lemma 3.1 and Theorem 4.1.

Under Assumption 2.1 and Assumption 2.9, we have the following result for the Dirichlet operator H_μ:

THEOREM 2.11. *Assume that the properties in Assumption 2.1 (a) and Assumption 2.9 hold and $\mu \in \mathcal{G}^\Phi(\Omega)$. Then, the Dirichlet operator H_μ with $D(H_\mu) = \mathcal{F}C_b^\infty$ is essentially self-adjoint in $L^2(\mathcal{H}_-, d\mu)$.*

The proof of the above theorem will be given in Section 4.

Finally, we discuss the log-Sobolev inequality for a Gibbs measure μ [14-2, 6-3, 24, 46, 47-3, 30, 35, 51]. Recall the definition of $\beta(x)$ in (2.19). Notice that $\beta : \Omega_{\log} \to \mathcal{H}_-$ and so for $x \in \Omega_{\log}$, the Gâteaux derivative $\frac{\partial\beta(x)}{\partial h}$ is well defined in the direction of $h \in \Omega_{\log}$, especially in the directions of $\mathcal{H}_+$-vectors. Let us define for each $x \in \Omega_{\log}$ an operator $R_\mu(x) : \mathcal{H}_+ \to \mathcal{H}_-$ by

$$R_\mu(x)(h) = -\frac{\partial\beta(x)}{\partial h}, \quad x \in \Omega_{\log},\ h \in \mathcal{H}_+. \tag{2.23}$$

Since $\mu(\Omega_{\log}) = 1$ for any $\mu \in \mathcal{G}^{\Phi}(\Omega)$, $R_\mu(\cdot)h$, $h \in \mathcal{H}_+$, is well defined μ-a.e. In Lemma 5.1 we will show that $R_\mu(x) \in \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)$. We notice that in [15] $R_\mu(x)$ was defined to be $-\beta'(x)$, however, we cannot define $\beta'(x)$ in the Fréchet sense because β is not everywhere defined in $\mathcal{H}_-$. See Section 5 for the details.

DEFINITION 2.12. We say that a Gibbs measure μ is *uniformly log-concave (R_μ-positive)* if and only if there exists $\lambda > 0$ such that for any $y \in \mathcal{H}_+$ and $x \in \Omega_{\log}$, the bound

$$< R_\mu(x)y, y > \geq \lambda |y|_0^2$$

holds for μ-a.a. $x \in \mathcal{H}_-$.

Let us fix a dense linear subset $\mathcal{K} \subset \mathcal{H}_+$. We say that a measure $\mu \in \mathcal{P}(\mathcal{H}_-)$ is $\mathcal{K}$-ergodic if and only if the only measurable subset of $\mathcal{H}_-$ which are $\mathcal{K}$-invariant has μ-measure zero or one. We recall that a μ-measurable set $A \subset \mathcal{H}_-$ is $\mathcal{K}$-invariant if $\forall h \in \mathcal{K}$, $\mu((A \setminus A_h) \cup (A_h \setminus A)) = 0$, where $A_h = A + h = \{x + h : x \in A\}$.

We define the space $W_2^1(\mu)$ and $W_2^2(\mu)$ as the closures of C_b^2 in the norms

$$\|u\|^2_{W_2^1(\mu)} = \int_{\mathcal{H}_-} (|u|^2 + |\nabla u|_0^2)\, d\mu,$$

$$\|u\|^2_{W_2^2(\mu)} = \|u\|^2_{W_2^1(\mu)} + \int_{\mathcal{H}_-} \mathrm{Tr}_{\mathcal{H}_0}(u'' \cdot u'')\, d\mu, \tag{2.24}$$

respectively.

As in [8], we denote by $\mathcal{P}_{\mathrm{sa}}(\mathcal{H}_-) \subset \mathcal{P}(\mathcal{H}_-)$ the set of all probability measures in $\mathcal{H}_-$ which is characterized by the following two conditions:

(a) For any $\mu \in \mathcal{P}_{\mathrm{sa}}(\mathcal{H}_-)$ there exists the square integrable logarithmic derivative β of μ and therefore the Dirichlet operator H_μ is well defined on $C_b^2(\mathcal{H}_-)$ by the formula (2.21).

(b) H_μ is essentially self-adjoint in $L^2(\mathcal{H}_-, d\mu)$ with a core $C_b^2(\mathcal{H}_-)$.

If $\mu \in \mathcal{P}_{\mathrm{sa}}(\mathcal{H}_-)$, we will use the same notation H_μ for the closure of H_μ for simplicity. The following theorem was proven in [8].

THEOREM 2.13 ([8, THEOREM 2]). *Suppose $\mu \in \mathcal{P}_{\mathrm{sa}}(\mathcal{H}_-)$ is uniformly log-concave with a constant $\lambda > 0$. Then,*

(a) $D(H_\mu) \subset W_2^2(\mu)$.

(b) *If the measure μ is $\mathcal{K}$-ergodic, then the point $0 \in \mathbb{R}$ is a simple eigenvalue of H_μ.*

(c) *If the measure μ is $\mathcal{K}$-ergodic, then there is a gap in the lower end of the spectrum of H_μ; moreover, $H_\mu \geq \lambda/4$ on the orthogonal complement of constants in $L^2(\Omega, d\mu)$.*

For the uniform log-concavity of Gibbs measures we have the folloing result.

THEOREM 2.14. *Suppose that the hypotheses in Assumption 2.1 and Assumption 2.9 hold. In addition, suppose that Assumption 2.9 (d) hold with a positive constant $M > 0$, i.e., $\exists\, M > 0$ such that*

$$Hess.P(x) \geq M\,\mathbf{1}, \quad x \in \mathbb{R}^d.$$

Furthermore, suppose that

$$M' := 2d \sum_{i \in \mathbb{Z}^\nu : i \neq 0} \Psi(|i|) < M.$$

Then, the Gibbs measure exists uniquely and the unique Gibbs measure is uniformly log-concave with a concavity constant $\lambda = M - M' > 0$.

The proof of the above Theorem will be given in Section 5. Let us take $\mathcal{K} \subset \mathcal{H}_+$ in Theorem 2.13 to be the special one Ω_{fin} defined in (2.12). We say that a Dirichlet form $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is irreducible if for any $u \in D(\mathcal{E}_\mu)$ with $\mathcal{E}_\mu(u,u) = 0$ it follows that u is constant μ-a.e. [9]. In [9], it was shown that the irreducibility of $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is equivalent with the extremality of μ on the set of measures that have the same logarithmic derivatives. Moreover, by Theorem 3.4 ond Theorem 3.7 of [9], the irreducibility in turn is equivalent with (space) ergodicity of μ under the condition (3.2) of [9]. For Gibbs measures, by the equilibrium condition, the condition (3.2) of [9] holds true for any $k \in \Omega_{\text{fin}}$ through equation (3.3) of [9]. On the other hand, since the Gibbs measure exists uniquely under the condition of uniform log-concavity, the unique Gibbs measure is automatically an extremal one. Thus, we state as a corollary the following result.

COROLLARY 2.15. *Suppose that the hypotheses in Theorem 2.14 hold and let $\mu \in \mathcal{G}^{\Phi}(\Omega)$ be the unique Gibbs measure. Then, the conclusions* (a), (b), *and* (c) *of Theorem 2.13 hold for the Dirichlet operator H_μ.*

From now on, we discuss the log-Sobolev inequality for Gibbs measures. Let us recall that a probability measure μ satisfies a log-Sobolev inequality if and only if there exists some constant $c_\mu > 0$ such that for all $f \in W_2^1$ the following inequality holds [24]:

$$\begin{aligned} &\int_{\mathcal{H}_-} |f(x)|^2 \log |f(x)| d\mu(x) \\ &\leq c_\mu \int_{\mathcal{H}_-} |\nabla f(x)|_0^2 d\mu(x) + \|f\|^2_{L^2(\mu)} \log \|f\|_{L^2(\mu)}. \end{aligned} \tag{2.25}$$

The coefficient c_μ is called a *Sobolev coefficient.*

We have the following result for the log-Sobolev inequality.

THEOREM 2.16. *Suppose that the hypotheses in Theorem 2.14 hold. Then, the unique Gibbs measure $\mu \in \mathcal{G}^{\Phi}(\Omega)$ satisfies the log-Sobolev inequality with a Sobolev coefficient $c_\mu = \lambda^{-1}$, where $\lambda = M - M'$.*

The proof of the theorem will be given in Section 5. An important consequence of the log-Sobolev inequality is that the semi-group $\{T_t\}_{t\geq 0}$ in $L^2(\mu)$ defined by

$$T_t := \exp(-tH_\mu), \quad t \geq 0. \tag{2.26}$$

is hypercontractive [24]. From Theorem 2.16 and Rothaus-Simon mass gap theorem [36, Sim], we have that $0 \in \mathbb{R}$ is a simple eigenvalue for H_μ and $H_\mu \geq (2c_\mu)^{-1}$ on the orthogonal complement to the constants in $L^2(\mu)$. By the spectral theorem, this implies the L^2-ergodicity of the semigroup T_t, $t \geq 0$:

$$\|T_t f - E_\mu f\|_{L^2(\mu)} \leq \exp\left(-\frac{t}{2c_\mu}\|f - E_\mu f\|_{L^2(\mu)}\right) \quad \forall\, f \in L^2(\mu), \forall\, t \geq 0, \tag{2.27}$$

where $E_\mu f = \int_{\mathcal{H}_-} f(x) d\mu(x)$.

Before closing this section, it may be worth to give a typical example for which all the results in this paper are valid.

EXAMPLE 2.17. Let $P(x)$ and $U(x, y; r)$ be the one-body and two-body potentials given as follows:

$$P(x) = \sum_{l=1}^{n} a_{2l}|x|^{2l}, \quad x \in \mathbb{R}^d, \tag{2.28}$$

where $a_{2n} > 0$, $a_2 > 0$, and $a_{2l} \geq 0$ $(l = 2, \cdots, n-1)$.

$$U(x, y; |i-j|) = f(|i-j|)(x, y), \quad x, y \in \mathbb{R}^d, \tag{2.29}$$

where $f : \mathbb{N} \to \mathbb{R}$. Assume that there exist constants $K > 0$ and $\sigma > 0$ such that $|f(|i-j|)| \leq K \exp(-2\sigma|i-j|)$. Then, all the conditions in Assumption 2.1 and Assumption 2.9 are satisfied. Furthermore, if the strict inequality

$$2dK \sum_{\substack{j \in \mathbb{Z}^\nu : \\ j \neq 0}} \exp(-2\sigma|j|) < a_2$$

holds, then the uniform log-concavity (R_μ-positivity) holds.

3. Dirichlet forms and associated diffusion processes

In this section we produce the proofs of Theorem 2.7 and Theorem 2.8. We first establish the relation (2.22) by using the equilibrium condition (Definition 2.4) and then show that for any Gibbs measure μ the Dirichlet operator H_μ is a well-defined symmetric operator. Then, the proof of Theorem 2.7 follows from the relation (2.22). For the proof of Theorem 2.8 we use the well-known method developed by Albeverio and Röckner [13].

Recall the definition of $\Omega_{\log}$ in (2.11) and Ω_{fin} in (2.12). The following inclusions

$$\Omega_{\text{fin}} \subset \mathcal{H}_+ \subset \mathcal{H}_0 \subset \Omega_{\log} \subset \mathcal{H}_- \subset \Omega$$

hold and for any Gibbs measure μ, $\mu(\Omega_{\log}) = 1$. See Remark 2.6 (b). In the rest of this section we assume that the properties in Assumption 2.1 hold and a Gibbs measure $\mu \in \mathcal{G}^\Phi(\Omega)$ is given. We begin with the following result:

LEMMA 3.1. *Let β be defined as in (2.19). Then $|\beta(x)|_-$ is finite for any $x \in \Omega_{\log}$ and $|\beta|_- \in L^2(\mathcal{H}_-, d\mu)$.*

Proof. Notice that

$$|\beta(x)|_-^2 = \sum_{i \in \mathbb{Z}^\nu} \mathrm{e}^{-\sigma|i|} |\beta_i(x)|^2. \tag{3.1}$$

Using the Schwarz inequality and Assumption 2.1 (b) one obtains that for $x = (x_i)_{i \in \mathbb{Z}^\nu} \in \Omega_{\log}$

$$|\beta_i(x)|^2 \le 2|\nabla^i P(x_i)|^2 + M \sum_{\substack{j \in \mathbb{Z}^\nu: \\ j \ne i}} \Psi(|i-j|)(|x_i|^2 + |x_j|^2) \tag{3.2}$$

for some constant $M > 0$. It follows from Assumption 2.1 (a) and the definition of $\Omega_{\log}$ that $|\beta(x)|_- < \infty$ for any $x \in \Omega_{\log}$. Since $\mu(\Omega_{\log}) = 1$, $|\beta|_- : \mathcal{H}_- \to \mathbb{R} \cup \{-\infty, \infty\}$ is finite μ-a.e. Due to the regularity of μ and Assumption 2.1 (a), there exist positive constants c, M_1, and M_2 such that the bounds

$$\begin{aligned} \int_{\mathcal{H}_-} |\nabla^i P(x_i)| d\mu(x) &\le c \int_{\mathbb{R}^d} |\nabla P(x)|^2 \exp(-A^*|x|^2)\, dx \\ &\le M_1 \end{aligned}$$

and

$$\begin{aligned} \int_{\mathcal{H}_-} (|x_i|^2 + |x_j|^2)\, d\mu(x) &\le 2c \int_{\mathbb{R}^d} |x|^2 \exp(-A^*|x|^2)\, dx \\ &\le M_2 \end{aligned}$$

hold. The lemma follows from (3.1) and the above bounds. □

Define for $k \in \mathcal{H}_-$, $\tau_k : \mathcal{H}_- \to \mathcal{H}_-$ by $\tau_k(x) = x + k$, and let $\tau_k(\mu)$ be the image measure of μ under τ_k. If $\tau_k(\mu)$ is absolutely continuous with respect to μ, we set

$$a_k(x) = \frac{d\tau_k(\mu)}{d\mu}(x).$$

For $k \in \Omega_{\mathrm{fin}}$ we show that the logarithmic derivative β_k of μ in the direction k exists.

PROPOSITION 3.2. (a) *For any $k \in \Omega_{\text{fin}}$ and $s \in \mathbb{R}$, $\tau_{sk}(\mu)$ is absolutely continuous with respect to μ. Furthermore,*

$$\beta_k \equiv \frac{d}{ds} a_{-sk}\Big|_{s=0}$$

is an element in $L^2(\mathcal{H}_-, d\mu)$.

(b) *If $(\frac{\partial}{\partial k})^*$ denotes the adjoint of $\frac{\partial}{\partial k}$, then $C_b^2(\mathcal{H}_-, \mathbb{R}) \subset D((\frac{\partial}{\partial k})^*)$ for any $k \in \Omega_{\text{fin}}$ and for any $k \in \Omega_{\text{fin}}$ and $u \in C_b^2$, the relation*

$$\left(\frac{\partial}{\partial k}\right)^* u = -\frac{\partial}{\partial k} u - \beta_k u$$

holds.

Proof. We use the equilibrium condition and the regularity of μ extensively for the proof. Let $k = (k_i)_{i \in \mathbb{Z}^\nu} \in \Omega_{\text{fin}}$ be such that there exists finite subset $\Delta \in \mathcal{C}$ so that $k_i = 0$ if $i \notin \Delta$. Using the equilibrium condition (Definition 2.4) for μ we obtain that for any $A \in \mathcal{B}(\mathcal{H}_-)$

$$\begin{aligned}
\tau_{sk}(\mu)(A) &= \int d\mu(\widetilde{x}) \Big\{ Z_\Delta^{-1}(\widetilde{x}) \int dx_\Delta \exp[-V(x_\Delta) - W(x_\Delta, \widetilde{x}_{\Delta^c})] \\
&\quad \times 1_{\tau_{-sk}(A)}(x_\Delta \widetilde{x}_{\Delta^c}) \Big\} \\
&= \int d\mu(\widetilde{x}) \Big\{ Z_\Delta^{-1}(\widetilde{x}) \int dx_\Delta \exp[-V(x_\Delta) - W(x_\Delta, \widetilde{x}_{\Delta^c})] \\
&\quad \times 1_A(\tau_{sk}(x_\Delta) \widetilde{x}_{\Delta^c}) \Big\},
\end{aligned}$$

where 1_A is the indicator function of A. We use the change of variables $\tau_{sk}(x_\Delta) \to x_\Delta$ and the equilibrium condition once again to get

(3.3)

$$\begin{aligned}
\tau_{sk}(\mu)(A) = \int_A d\mu(x) \exp\Big\{ &- \sum_{i \in \Delta} [P(x_i - sk_i) - P(x_i)] \\
&- \sum_{\{i,j\} \subset \Delta} [U(x_i - sk_i, x_j - sk_j; |i-j|) - U(x_i, x_j; |i-j|)] \\
&- \sum_{i \in \Delta, j \in \Delta^c} [U(x_i - sk_i, x_j; |i-j|) - U(x_i, x_j; |i-j|)] \Big\}.
\end{aligned}$$

The above relation implies that for any $s \in \mathbb{R}$, $\tau_{sk}(\mu)$ is absolutely continuous w.r.t. μ with the Radon-Nikodym derivative a_{sk} the factor $\exp\{\cdots\}$ in (3.3). From this fact it follows that

$$(3.4)\quad \beta_k(x) = -\sum_{i\in\mathbb{Z}^\nu}(\nabla^i P(x_i), k_i) - \sum_{i\in\mathbb{Z}^\nu}\sum_{\substack{j\in\mathbb{Z}^\nu:\\ j\neq i}}(\nabla^i U(x_i, x_j; |i-j|), k_i)$$

where for each $i \in \mathbb{Z}^\nu$, (y_i, k_i) denotes the inner product in $\mathbb{R}^d$. With the notation in (2.19) and (2.20) we write

$$(3.5)\qquad \beta_k(x) =< \beta(x), k > .$$

Since $|\beta_k(x)| \le |\beta(x)|_-|k|_+$, $\beta_k \in L^2(\mathcal{H}_-, d\mu)$ by Lemma 3.1.

(b) The part (b) of the lemma follows from the part (a) and Proposition 4.5 of [12]. See also [15, 26]. In fact we can show the part (b) directly by using the equilibrium condition and an integration by parts formula as follows: Let $\{e_i^l : i \in \mathbb{Z}^\nu,\ l = 1, 2, \cdots, d\}$ be the standard basis for $\mathcal{H}_0$. Then for any $u, v \in C_b^2$ and e_i^l one has

$$\begin{aligned}
&\int u(x)\frac{\partial v}{\partial e_i^l}(x)\,d\mu(x)\\
&\quad= \int u(x_i x_{\{i\}^c})\frac{\partial v}{\partial x_i^l}(x_i x_{\{i\}^c})\,d\mu(x)\\
&\quad= \int d\mu(\widetilde{x})\Big\{Z_{\{i\}}^{-1}(\widetilde{x})\int dx_i \exp[-P(x_i) - W(x_{\{i\}}, \widetilde{x}_{\{i\}^c})]\\
&\qquad\qquad \times u(x_i\widetilde{x}_{\{i\}^c})\frac{\partial v}{\partial x_i^l}(x_i\widetilde{x}_{\{i\}^c})\Big\}\\
&\quad= \int d\mu(x)\Big[-\frac{\partial u}{\partial x_i^l}(x_i x_{\{i\}^c}) - \beta_i^l(x)u(x)\Big]v(x),
\end{aligned}$$

where $\beta_i^l(x) =< \beta(x), e_i^l >$. Here we have used the integration by parts formula and the equilibrium condition to obtain the third equality. Since any $k \in \Omega_{\text{fin}}$ is a finite linear combination of elements in the standard basis, we proved the part (b) of the lemma completely. □

We now establish the relation (2.22). In fact we have the following result:

PROPOSITION 3.3. *The Dirichlet operator*

$$H_\mu u(x) = -\frac{1}{2}\Delta u(x) - \frac{1}{2} < \beta(x), \nabla u(x) >, \quad u \in C_b^2,$$

is a positive-definite symmetric operator in L^2. *Furthermore,* H_μ *is associated with the form* $\mathcal{E}_\mu(u,v)$ *of (2.16) by the equality* $\mathcal{E}_\mu(u,v) = (u, H_\mu v)_{L^2}$.

Before proving the proposition we give some comments. Since the embeddings $\mathcal{H}_+ \subset \mathcal{H}_0$ and $\mathcal{H}_0 \subset \mathcal{H}_-$ are Hilbert Schmidt operators, $\Delta u = \mathrm{Tr}_{\mathcal{H}_0} u''$ is well-defined for any $u \in C_b^2$. For $u \in C_b^2$, the function $< \beta(\cdot), \nabla u(\cdot) >$ is finite μ-a.e. and belongs to $L^2(\mathcal{H}_-, d\mu)$ by Lemma 3.1, and so H_μ is well-defined on C_b^2.

Proof of Proposition 3.3. The proposition follows from the definition of $\mathcal{E}_\mu$ in (2.16) and Proposition 3.2 together with a suitable choice of an orthonormal basis $\{k_i\} \in \Omega_{\mathrm{fin}}$. For an instance, we choose the standard basis $\{e_i^l : i \in \mathbb{Z}^\nu, l = 1, \cdots, d\}$. From (2.16) and Proposition 3.2 it follows that

$$\begin{aligned}
\mathcal{E}_\mu(u,v) &= \frac{1}{2}\sum_{i\in\mathbb{Z}^\nu}\sum_{l=1}^{d}\left(\frac{\partial u}{\partial e_i^l}, \frac{\partial v}{\partial e_i^l}\right)_{L^2} \\
&= \frac{1}{2}\sum_{i\in\mathbb{Z}^\nu}\sum_{l=1}^{d}\left(u, \left(-\frac{\partial}{\partial e_i^l} - \beta_{e_i^l}\right)\frac{\partial v}{\partial e_i^l}\right)_{L^2} \\
&= \frac{1}{2}(u, -\Delta v)_{L^2} + (u, - < \beta, \nabla v >)_{L^2} \\
&= (u, H_\mu v)_{L^2}.
\end{aligned}$$

See also the proof of Theorem 1 of [26] for the proof with arbitrary orthonormal basis. □

We now turn to the proof of Theorem 2.7.

Proof of Theorem 2.7. Since $\mathcal{F}C_b^\infty \subset C_b^2$ and $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$ is associated to the symmetric operator H_μ with the domain $\mathcal{F}C_b^\infty$, $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$ is closable by Proposition 3.3. The Markov property of $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$ follows from a well-known method [12, 37]. See the proof of Proposition 4.5 of [37]. □

We next consider a diffusion process associated to the Dirichlet form for given Gibbs measure μ. We start with the following lemma.

LEMMA 3.4. *$\mathcal{F}C_b^\infty$ is dense in $L^p(\mathcal{H}_-, d\mu)$, $1 \le p < \infty$.*

Proof. Here we present a proof which we learnt from B. Schmuland [40]. It is easy to show that any Borel set in $\mathcal{H}_-$ is also a Borel set in Ω and so one may consider μ as a Borel measure on $\mathcal{H}_-$. Since for any vector $k \in \Omega_{\text{fin}}$, $\tau_k(\mu) \ll \mu$ (Proposition 3.2 (a)) and Ω_{fin} is dense in $\mathcal{H}_-$, we can show that μ is supported in $\mathcal{H}_-$ (i.e., $\mu(U) > 0$ for any open $U \subset \mathcal{H}_-$). First, we show that there exists a countable family in $\mathcal{F}C_b^\infty$ which separates points in $\mathcal{H}_-$. Let $\{r_m \in \mathbb{R}^d : m \in \mathbb{N}\}$ be an enumeration of the family of vectors in $\mathbb{R}^d$ which have rational coordinates. For each $m, n \in \mathbb{N}$, define smooth functions a_m and $a_{m,n}$ on $\mathbb{R}^d$ by $a_m(y) = (r_m, y)$ and $a_{m,n}(y) = n \sin(a_m(y)/n)$. For each $i \in \mathbb{Z}^\nu$, let $f_{i,m,n}(x) = a_{m,n}(x_i)$, $x = (x_i)_{i \in \mathbb{Z}^\nu} \in \mathcal{H}_-$, and define $\mathcal{G}$ to be the class of those functions $f_{i,m,n}$, $i \in \mathbb{Z}^\nu$, $m, n \in \mathbb{N}$. Clearly, $\mathcal{G}$ is a countable subclass of $\mathcal{F}C_b^\infty$ and separates points in $\mathcal{H}_-$ because $\lim_{n\to\infty} n \sin(t/n) = t$, for any $t \in \mathbb{R}$. Since $\mathcal{H}_-$ is a Polish space, the σ-algebra generated by $\mathcal{G}$, and hence the σ-algebra generated by $\mathcal{F}C_b^\infty$, is the entire Borel σ-algebra $\mathcal{B}(\mathcal{H}_-)$ ([18], Corollary 8.6.8). Now, denote by $\mathcal{B}$ the family of functions $u \in \mathcal{B}_b(\mathcal{H}_-)$ for which there exists a sequence $\{u_n\}$ in $\mathcal{F}C_b^\infty$ such that $\int |u_n - u|^p d\mu \to 0$ as $n \to \infty$. Then by a Monotone Class Theorem ([41], A0.6), we see that $\mathcal{B} = \mathcal{B}_b(\mathcal{H}_-)$ and we conclude that $\mathcal{F}C_b^\infty$ is dense in $L^p(\mathcal{H}_-, d\mu)$. □

We now turn to the proof of Theorem 2.8. In [13], Albeverio and Röckner gave sufficient conditions for the construction of the associated diffusion process. We will check that all the conditions are satisfied in our case. Let $(\tilde{\mathcal{E}}_\mu, D(\tilde{\mathcal{E}}_\mu))$ be the closure of $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$, and let $\mathcal{E}_{\mu,1}(u, v) = \mathcal{E}_\mu(u, v) + \int uv\, d\mu$, $u, v \in \mathcal{F}C_b^\infty$. For any open $U \subset \mathcal{H}_-$, define the capacity [22] of U by

$$\text{Cap}(U) \equiv \inf\{\tilde{\mathcal{E}}_{\mu,1}(u, u) : u \in D(\tilde{\mathcal{E}}_\mu),\ u \ge 1 \text{ on } U\ \mu - \text{a.e.}\},$$

and for any $A \subset \mathcal{H}_-$ $\text{Cap}(A) \equiv \inf\{\text{Cap}(U) : A \subset U,\ U \text{ open}\}$.

Let us now consider the following conditions introduced in [13]:

(i) There exist $K_n \subset \mathcal{H}_-$, $n \in \mathbb{N}$, K_n compact, such that $\lim_{n\to\infty} \text{Cap}(\mathcal{H}_- \setminus K_n) = 0$.

(ii) There exists a countable set D of bounded continuous functions on $\mathcal{H}_-$ separating the points of $\mathcal{H}_-$ which is dense in $D(\widetilde{\mathcal{E}}_\mu)$ with respect to $\widetilde{\mathcal{E}}_{\mu,1}$.

(iii) $\widetilde{\mathcal{E}}_\mu(u,v) = 0$ if $u, v \in D(\widetilde{\mathcal{E}}_\mu)$, continuous, such that supp $u \cap$ supp $v = \emptyset$.

(iv) There exist $f_n : \mathcal{H}_- \to \mathbb{R}$, $n \in \mathbb{N}$, generating the topology of $\mathcal{H}_-$.

The following is a main result of [13]:

THEOREM 3.5 ([13], THEOREM 2.7). *Assume that the conditions* (i) - (iv) *listed above are satisfied. Then, there exists a diffusion process with state space* $\mathcal{H}_-$ *associated with* $(\widetilde{\mathcal{E}}_\mu, D(\widetilde{\mathcal{E}}_\mu))$.

In order to prove Theorem 2.8, we prepare as follows. Let us fix $p > \nu$ and define

$$Q(x) \equiv \left(\sum_{i \in \mathbb{Z}^\nu} (|i|+1)^{-p} |x_i|^2 \right)^{1/2}, \quad x = (x_i)_{i \in \mathbb{Z}^\nu} \in \mathcal{H}_-, \tag{3.6}$$

and

$$\Omega_0 \equiv \{x \in \mathcal{H}_- : Q(x) < \infty\}. \tag{3.7}$$

Since $\Omega_{\log} \subset \Omega_0$, $\mu(\Omega_0) = 1$.

Proof of Theorem 2.8. By Theorem 3.5, we only need to check the conditions (i) - (iv) listed above. The condition (ii) is satisfied by Proposition 2.6 of [13]. (iii) is obvious and (iv) is satisfied since $\mathcal{H}_-$ is a complete metric space. Thus, it only remains to check the condition (i).

Notice that the validity of the condition (i) follows from Proposition 3.2 of [37]. Here, for concreteness, we construct a sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact sets in $\mathcal{H}_-$ such that $\lim_{n \to \infty} \mathrm{Cap}(\mathcal{H}_- \setminus K_n) = 0$ directly. Put $F_r \equiv \{x \in \Omega_0 : Q(x) \le r\}$, where $Q(x)$ and Ω_0 have been defined in (3.6) and (3.7), respectively. Notice that Ω_0 is a reflexive Banach space (a Hilbert space) and the embedding $\Omega_0 \hookrightarrow \mathcal{H}_-$ is a Hilbert-Schmidt operator. Thus, F_r is weakly compact and the embedding image of F_r in $\mathcal{H}_-$ is compact. Let K_n, $n \in \mathbb{N}$, be the compact image of F_n in $\mathcal{H}_-$.

Following [27], we introduce a smooth increasing function ϕ on $\mathbb{R}$ such that $\phi(t) = 0$ if $t \le n^2$, $\phi(t) = 1$ if $t \ge (n+1)^2$, and $|\phi'(t)| \le 2/(n+1)$, $t \in \mathbb{R}$. Define $u(x) = \phi(Q(x)^2)$, $x \in \mathcal{H}_-$, with the convention that $\phi(\infty) = 1$. By the regularity of Gibbs measures, it can be checked that $Q \in L^2(\mu)$. Thus, it follows that $u \in D(\widetilde{\mathcal{E}}_\mu)$ (Lemma 3.1, Chapter IV, of [37]).

We note that $\mathcal{H}_- \setminus K_{n+1}$ is open and $u(x) \ge 1$ on $\mathcal{H}_- \setminus K_{n+1}$. Thus,

$$\mathrm{Cap}(\mathcal{H}_- \setminus K_{n+1}) \le \widetilde{\mathcal{E}}_\mu(u,u) + \int_{\mathcal{H}_-} u(x)^2 \, d\mu(x). \tag{3.8}$$

Let $\{e_i^l : i \in \mathbb{Z}^\nu,\ l = 1, 2, \cdots, d\}$ be the standard basis of $\mathcal{H}_0$. Then, with the notation $\nabla_l^i = \frac{\partial}{\partial e_i^l}$,

$$\begin{aligned}
\widetilde{\mathcal{E}}_\mu(u,u) &= \sum_{i \in \mathbb{Z}^\nu} \sum_{l=1}^{d} \int |\nabla_l^i u(x)|^2 \, d\mu(x) \\
&= 4 \sum_{i \in \mathbb{Z}^\nu} \int (\phi'(Q(x)^2))^2 (|i|+1)^{-2p} |x_i|^2 \, d\mu(x) \\
&\le 4 \int (\phi'(Q(x)^2))^2 Q(x)^2 \, d\mu(x). \qquad (3.9)
\end{aligned}$$

Since $\phi'(t) \le 2/(n+1)$ and $\phi'(Q(x)^2) = 0$ on K_n and on $\mathcal{H}_- \setminus K_{n+1}$, the right hand side of (3.9) is bounded by

$$\int_{K_{n+1} \setminus K_n} \left(\frac{4}{n+1}\right)^2 (n+1)^2 \, d\mu(x) \to 0, \quad \text{as } n \to \infty. \tag{3.10}$$

Here, we have used the fact that $\mu(\Omega_0) = 1$. On the other hand,

$$\int u(x)^2 \, d\mu(x) \le \int_{\mathcal{H}_- \setminus K_n} d\mu(x) \to 0, \quad \text{as } n \to \infty. \tag{3.11}$$

Combining (3.8) – (3.11), we conclude that

$$\mathrm{Cap}(\mathcal{H}_- \setminus K_n) \to 0, \quad \text{as } n \to \infty.$$

This completes the proof of Theorem 2.8. □

4. Essential self-adjointness of the dirichlet operator

Let H_μ be the Dirichlet operator with $D(H_\mu) = C_b^2$. We first show that H_μ is essentially self-adjoint and then show that $\mathcal{F}C_b^2$ is also a core for the self-adjoint extension of H_μ. The proof of essentially self-adjointness will be relied on the basic criterium of essential self-adjointness given in [6]. We state the main result in this section.

THEOREM 4.1. *Let $\mu \in \mathcal{G}^\Phi(\Omega)$. Under Assumption 2.1 and Assumption 2.9, the Dirichlet operator H_μ with $D(H_\mu) = C_b^2$ is essentially self-adjoint in $L^2(\mathcal{H}_-, d\mu)$.*

The basic criterium of essential self-adjointness of [6] in the form applicable to our case is the following:

PROPOSITION 4.2 ([6, THEOREM 1]). *Let H_μ be the Dirichlet operator with $D(H_\mu) = C_b^2$. Suppose $|\beta|_- \in L^2$ and that there exists a sequence $\{b_n : n \in \mathbb{N}\}$, $b_n : \mathcal{H}_- \to \mathcal{H}_-$, $n \in \mathbb{N}$, such that*

(a) for any $n \in \mathbb{N}$, $b_n \in C_b^2(\mathcal{H}_-, \mathcal{H}_-)$,

(b) $|b_n - \beta|_- \to 0$ in L^2 as $n \to \infty$,

(c) there exists a constant $c \in \mathbb{R}$ such that for any $x \in \mathcal{H}_-$, $h \in \mathcal{H}_-$, $n \in \mathbb{N}$,

$$(b_n'(x)h, h)_- \leq c|h|_-^2.$$

Then H_μ is essentially self-adjoint in $L^2(\mathcal{H}_-, d\mu)$.

Due to Lemma 3.1, it is sufficient to construct a sequence $\{b_n, n \in \mathbb{N}\}$ satisfying the conditions (a) – (c) of Proposition 4.2.

Proof of Theorem 4.1. We will construct a sequence $\{b_n, n \in \mathbb{N}\}$ which satisfies the properties in Proposition 4.2. Let $g : \mathbb{R} \to \mathbb{R}$ be a C^∞-function satisfying the following conditions:

(a) g is an odd function: $g(-t) = -g(t)$,

(b) $g(t) = t$ for $t \in (-1, 1)$,

(c) g is monotonic increasing: $0 \leq g'(t) \leq 1$,

(d) $g(t) \to 2$ as $t \to \infty$.

For $n \in \mathbb{N}$ we set

$$g_n(t) = ng(t/n). \tag{4.1}$$

Then for each $n \in \mathbb{N}$, g_n is a monotonic increasing function such that $g_n(t) = t$ for $t \in (-n, n)$ and $|g_n(t)| \leq 2n$ for any $t \in \mathbb{R}$.

For $x = (x^1, \cdots, x^d) \in \mathbb{R}^d$, we define $g_n(x) := (g_n(x^1), \cdots, g_n(x^d)) \in \mathbb{R}^d$. Recall the definition $\beta(x)$ in (2.19) and the one-body interaction $P(x)$ in Assumption 2.9 (c) for $d \geq 2$. For any $n \in \mathbb{N}$ and $x = (x_i)_{i\in\mathbb{Z}^\nu} \in \mathcal{H}_-$ we define $b_n : \mathcal{H}_- \to \mathcal{H}_-$ by

$$b_n(x) = (b_{n,i}(x))_{i\in\mathbb{Z}^\nu},$$

$$b_{n,i}(x) = b_{n,i}^{(1)}(x) + b_{n,i}^{(2)}(x), \tag{4.2}$$

where

$$b_{n,i}^{(1)} = \begin{cases} -P'(g_n(x_i)), & d = 1 \\ -Q'(g_n(|x_i|))\frac{x_i}{|x_i|} - b, & d \geq 2 \end{cases} \tag{4.3}$$

and

$$b_{n,i}^{(2)}(x) = -\sum_{\substack{j\in\mathbb{Z}^\nu: \\ j\neq i}} \nabla^i U(g_n(x_i), g_n(x_j); |i-j|). \tag{4.4}$$

We prove that b_n satisfies the desired properties in Proposition 4.2 through the following steps. We will consider the case $d \geq 2$ only. The case $d = 1$ can be dealt with similary.

Step 1. The fact that $b_n \in C_b^2(\mathcal{H}_-, \mathcal{H}_-)$ follows directly from the definition of b_n, $n \in \mathbb{N}$.

Step 2. Let us show that $|b_n - \beta|_- \to 0$ in $L^2(\mu)$ as $n \to \infty$. From (2.19) and (4.2) – (4.4) we see that

$$\begin{aligned}
&\frac{1}{2}|b_n(x) - \beta(x)|_-^2 \\
&\leq \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|}\left|Q'(|x_i|)\frac{x_i}{|x_i|} - Q'(g_n(|x_i|))\frac{x_i}{|x_i|}\right|^2 \\
&\quad + \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|}\Bigg|\sum_{\substack{j\in\mathbb{Z}^\nu: \\ j\neq i}} \nabla^i U(x_i - x_j; |i-j|) \\
&\quad - \sum_{\substack{j\in\mathbb{Z}^\nu: \\ j\neq i}} \nabla^i U(g_n(x_i) - g_n(x_j); |i-j|)\Bigg|^2 \\
&\equiv I_1(x;n) + I_2(x;n).
\end{aligned}$$

By the regularity of μ and Assumption 2.1(a) we have for some $0 < \alpha < A^*$ that

$$\begin{aligned}
&\int_{\mathcal{H}_-} I_1(x;n)d\mu(x) \\
&\le \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|} \int_{\mathbb{R}^d} |Q'(|x_i|) - Q'(g_n(|x_i|))|^2 e^{-A^*|x_i|^2+\delta} dx_i \\
&\le \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|} M_1 \int_{B_n(0)^c} e^{-(A^*-\alpha)|x_i|^2} dx_i \\
&\to 0, \qquad \text{as } n \to \infty,
\end{aligned} \tag{4.5}$$

where $B_n(0)$ is the ball in $\mathbb{R}^d$ with radius and centered at 0. We consider $I_2(x;n)$. Denote by B_n the subset of $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$B_n = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \ : \ |x| \ge n \text{ or } |y| \ge n\}.$$

Using the property of g_n in (4.1), Assumption 2.1 (b) and the Schwarz inequality, we see that there exists a constant M_2 such that

$$\begin{aligned}
&I_2(x_i;n) \\
&\le \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|} \Big(\sum_{\substack{j\in\mathbb{Z}^\nu: \\ j\ne i}} e^{-\sigma|i-j|} \Big) \sum_{\substack{j\in\mathbb{Z}^\nu \\ j\ne i}} e^{+\sigma|i-j|} \\
&|\nabla^i U(x_i - x_j; i-j) - \nabla^i U(g_n(x_i) - g_n(x_j); i-j)|^2 \\
&\le M_2 \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|} \sum_{\substack{j\in\mathbb{Z}^\nu: \\ j\ne i}} e^{+\sigma|i-j|} \Psi(|i-j|)(|x_i|^2 + |x_j|^2) 1_{B_n}(x_i, x_j).
\end{aligned}$$

Let us put

$$K_n = \int_{\mathbb{R}^{2d}} (|x|^2 + |y|^2) 1_{B_n}(x,y) \exp[-A^*(|x|^2 + |y|^2)] dx dy.$$

It is obvious that $K_n \to 0$ as $n \to \infty$. From the regularity of Gibbs measure μ and Assumption 2.9 (d) it follows that for some constant $M_2' > 0$,

$$\int I_2(x;n) d\mu(x) \le M_2' K_n \to 0 \qquad \text{as } n \to \infty. \tag{4.6}$$

Combining (4.5) – (4.6) we prove that $|b_n - \beta|_- \to 0$ in $L^2(\mu)$ as $n \to \infty$.

Step 3. We prove that the property (c) in Proposition 4.2 holds. Let $h = (h_i)_{i\in\mathbb{Z}^\nu} \in \mathcal{H}_-$. From (4.2) – (4.4), it follows that

$$
\begin{aligned}
(h, b_n'(x)h)_- &= \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|}(h_i, (b_n'(x)h)_i),\\
(b_n'(x)h)_i &= -\Big[Q''(g_n(|x_i|))g_n'(|x_i|)\vec{x}_i\vec{x}_i/|x_i|^2\\
&\qquad + Q'(g_n(|x_i|))\Big(\mathbf{1} - \frac{\vec{x}_i\vec{x}_i}{|x_i|^2}\Big)/|x_i|\Big]h_i\\
&\quad - \nabla^i\Big\{\sum_{\substack{i\in\mathbb{Z}^\nu:\\ j\neq i}}\Big[(\nabla^i U(g_n(x_i), g_n(x_j); |i-j|), h_i)\\
&\qquad + (\nabla^j U(g_n(x_i), g_n(x_j); |i-j|), h_j)\Big]\Big\}\\
&\equiv g_i^{(1)}(x;h) + g_i^{(2)}(x;h). \qquad (4.7)
\end{aligned}
$$

In the above, we have used the notation : for $x \in \mathbb{R}^d$, $\vec{x}\vec{x}$ is the $d \times d$ matrix whose $l-k$ element is given by $(\vec{x}\vec{x})_{lk} = x^l x^k$, $l, k = 1, \cdots, d$. First we consider $(h_i, g_i^{(2)}(x;h))$. Due to Assumption 2.9 we obtain that

$$
|(h_i, g_i^{(2)}(x;h)| \le d \sum_{\substack{j\in\mathbb{Z}^\nu:\\ j\neq i}} \Psi(|i-j|)|h_i|(|h_i| + |h_j|) \qquad (4.8)
$$

Notice that by Assumption 2.9 (d),

$$
e^{-\sigma|i|/2}\Psi(|i-j|)e^{\sigma|j|/2} \le Ke^{-c|i-j|}. \qquad (4.9)
$$

By (4.8) and (4.9) and the Schwarz inequality we see that

$$
\begin{aligned}
&|\sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|}(h_i, g_i^{(2)}(x;h)|\\
&\le c_1 \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|}|h_i|^2 = c_1|h|_-^2 \qquad (4.10)
\end{aligned}
$$

for some constant c_1.

In order to control $\sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|}(h_i, g_i^{(1)}(x,h))$ we use the following trick. Choose an $\epsilon > 0$ and let $M_\epsilon := M - \epsilon$, where M is the constant given in Assumption 2.9 (b). We rewrite $Q(|x|)$ as

$$Q(|x|) = \widehat{Q}(|x|) + \frac{1}{2}M_\epsilon |x|^2.$$

Then $g_i^{(1)}(x;h)$ is devided into two parts:

$$\begin{aligned} g_i^{(1)}(x;h) = &-\Big[\widehat{Q}''(|x_i|)g_n'(|x|)\vec{x}_i\vec{x}_i/|x_i|^2 \\ &+ \widehat{Q}'(g_n(|x_i|))\Big(\mathbf{1} - \vec{x}_i\vec{x}_i/|x_i|^2\Big)/|x_i|\Big]h_i \\ &- \Big[g_n'(|x_i|)\frac{\vec{x}_i\vec{x}_i}{|x_i|^2} + \Big(\mathbf{1} - \frac{\vec{x}_i\vec{x}_i}{|x_i|^2}\Big)\frac{g_n(|x_i|)}{|x_i|}\Big]h_i \\ \equiv\ & g_i^{(1,1)}(x;h) + g_i^{(1,2)}(x;h). \end{aligned} \tag{4.11}$$

Since $g_n'(|x_i|) \geq 0$ and $0 \leq \frac{g_n(|x_i|)}{|x_i|} \leq 1$, it is obvious that

$$(h_i, g_i^{(1,2)}(x;h)) \geq 0. \tag{4.12}$$

On the other hand since $Hess.P(x) \geq M\mathbf{1}$ by Assumption 2.9 (b), we see that

$$Hess.\widehat{Q}(|x|) = Hess.P(x) - M_\epsilon\mathbf{1} \geq \epsilon\mathbf{1}.$$

Therefore, $\exists\, R > 0$ such that

$$\widehat{Q}(|x|) \geq 0 \ \text{ and } \ \widehat{Q}''(|x|) \geq 0 \text{ if } |x| \geq R. \tag{4.13}$$

Thus from (4.11),

$$(h_i, g^{(1,1)}(x;h)) \leq 0 \ \text{ if } \ |x| \geq R \tag{4.14}$$

Finally, since $Hess.\widehat{Q}(x)$ is a real symmetric matrix, its eigenvalues are real. Let $m(x)$ be the minimum of the eigenvalues of $\widehat{Q}(x)$ and put $m = \inf\{m(x) : |x| \leq R\}$. Then, we see that

$$\begin{aligned} (h_i, g^{(1,1)}(x;h)) &= -(h_i, Hess.\widehat{Q}(x)h_i) \\ &\leq -m|h_i|^2, \text{ if } |x| \leq R. \end{aligned} \tag{4.15}$$

From (4.10), (4.12), and (4.14) – (4.15), we conclude that

$$(h, b_n'(x)h)_- \leq c|h|_-^2, \quad c = \min\{c_1, c_1 - m\}.$$

The proof of Theorem 4.1 is now completed. □

Finally, we present the proof of Theorem 2.11 by using the method employed in the proof of Corollary 2 of [6]. See [6] for the details.

Proof of Theorem 2.11. For each finite subset $\Delta \in \mathcal{C}$ of $\mathbb{Z}^\nu$, let P_Δ be the orthogonal projection from $\mathcal{H}_0$ to $(\mathbb{R}^d)^\Delta \subset \mathcal{H}_+$. Here, we have identified $(\mathbb{R}^d)^\Delta$ as the subspace $\{x \in \mathcal{H}_- : x_i = 0, \forall i \notin \Delta\}$ of $\mathcal{H}_-$. Then P_Δ extends by continuity to continuous projection from $\mathcal{H}_-$ into $\mathcal{H}_+$ with range $P_\Delta \mathcal{H}_- = (\mathbb{R}^d)^\Delta$. If $f \in C_b^2(\mathcal{H}_-)$, then the function $f_\Delta(x) \equiv f(P_\Delta x)$ is cylindrical. The relations

$$f_\Delta'(x) = P_\Delta f'(P_\Delta x), \quad f_\Delta''(x) = P_\Delta f''(P_\Delta x) P_\Delta, \quad x \in \mathcal{H}_-,$$

show that $f_\Delta \in \mathcal{F}C_b^2(\mathcal{H}_-)$.

Let P_{Δ_n}, $n \in \mathbb{N}$, be a sequence of projections such that $\cup_n P_{\Delta_n} \mathcal{H}_-$ is dense in $\mathcal{H}_-$. It is not hard to show that for the sequence $\{f_{\Delta_n} : n \in \mathbb{N}\} \subset \mathcal{F}C_b^2$ we have convergence $H_\mu f_{\Delta_n} \to H_\mu f$ in L^2 as $n \to \infty$. By a standard approximation for functions from $C_b^2(\mathbb{R}^n)$ by elements of $C_b^\infty(\mathbb{R}^n)$, for any $f \in C_b^2(\mathcal{H}_-)$, we can choose a sequence $f_j \in \mathcal{F}C_b^\infty(\mathcal{H}_-)$ such that $H_\mu f_j \to H_\mu f$ in L^2 as $j \to \infty$. This completes the proof of Theorem 2.11. □

5. Log-Sobolev inequality for Gibbs measures

As stated in Introduction, the Dirichlet operator plays an important role when the Gibbs measure satisfies the log-Sobolev inequality, which implies the strong convergence of the semi-group generated by the Dirichlet operator. The log-Sobolev inequality was firstly proved by Gross [24] for the case of Gaussian measures on $\mathbb{R}^d$. The Gross inequality leads to hypercontractivity for the semigroup generated by Dirichlet operators and to a wide range of applications [19]. In statistical mechanics, the log-Sobolev inequality for Gibbs measures for bounded spin systems have been established under the Dobrushin-Schlosman uniqueness criterion [46, 47-3]. The log-Sobolev inequality

for unbounded spin systems of finite range interactions has been also proven in [8] and [51]. In this section we produce the proof of the uniform log-concavity (Theorem 2.14) and the log-Sobolev inequality for the Gibbs measure (Theorem 2.16). We begin with the following result:

LEMMA 5.1. *Suppose that the hypotheses of Assumption 2.1 and Assumption 2.9 hold. Let R_μ be defined as in* (2.23). *Then, $R_\mu(x) \in \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)$ for any $x \in \Omega_{\log}$.*

Proof. For $x \in \Omega_{\log}$ and $h \in \mathcal{H}_+$ we have

$$\begin{aligned}(R_\mu(x)h)_i &= \nabla^i(\nabla^i P(x_i), h_i) \\ &\quad + \sum_{\substack{j\in\mathbb{Z}^\nu: \\ j\neq i}} \nabla^i \left[(\nabla^i U(x_i, x_j; |i-j|), h_i) + (\nabla^j U(x_i, x_j; |i-j|), h_j)\right].\end{aligned}$$

Using the conditions in Assumption 2.9 (a) and (d), we see that for some small $\alpha > 0$

$$\begin{aligned}|(R_\mu(x)h)_i|^2 \leq & 2d^2 M(\alpha)^2 e^{2\alpha|x_i|^2}|h_i|^2 \\ & + 2d^2\Big(\sum_{\substack{j\in\mathbb{Z}^\nu: \\ j\neq i}} \Psi(|i-j|)(|h_i| + |h_j|)\Big)^2.\end{aligned}$$

Since $x \in \Omega_{\log}$, there exists an $N \in \mathbb{N}$ such that $|x_i| \leq N\log(|i|+1)$, $i \in \mathbb{Z}^\nu$. Substituting this into the above we see after some calculation that

$$\begin{aligned}|(R_\mu(x)h)|_-^2 &= \sum_{i\in\mathbb{Z}^\nu} e^{-\sigma|i|}|(R_\mu(x)h)_i|^2 \\ &\leq K|h|_+^2\end{aligned}$$

for some constant K. This proves that $R_\mu(x) \in \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)$ (actually it holds that $R_\mu(x) \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_-)$ for each $x \in \Omega_{\log}$). □

We prove the uniform log-concavity of the Gibbs measure, i.e., Theorem 2.14.

Proof of Theorem 2.14. The uniqueness of the Gibbs measure under the hypotheses of Theorem 2.14 can be proved by the method similar to that used in [10-2]. The basic idea is the Dobrushin's criterion of uniqueness of Gibbs measures [21], in which one considers the interdependence between particles via the Wasserstein distance between the distributions of single site specifications. In [10-2], for unbounded quantum spin systems, the Wasserstein distance was controlled by the log-Sobolev inequality for the single site specifications. The method used in [10-2] can be directly applied to the systems that we are dealing with in this paper. For the details, we refer the reader to [10-2].

Thus, it remains only to show the uniform log-concavity of the Gibbs measure. Suppose that $\mu \in \mathcal{G}^{\Phi}(\Omega)$ is the unique Gibbs measure and $y = (y_i)_{i\in\mathbb{Z}^\nu} \in \mathcal{H}_+$, $x = (x_i)_{i\in\mathbb{Z}^\nu} \in \Omega_{\log}$. Then, it follows that

$$\begin{aligned}
< y, R_\mu(x)(y) > &= \sum_{i\in\mathbb{Z}^\nu} (y_i, (R_\mu(x)y)_i) \\
&= \sum_{i\in\mathbb{Z}^\nu} (y_i, \nabla^i(\nabla^i P(x_i), y_i)) \\
&+ \sum_{\substack{j\in\mathbb{Z}^\nu: \\ j\neq i}} \Big\{ (y_i, \nabla^i(\nabla^i U(x_i, x_j; |i-j|), y_i)) \\
&\qquad + (y_i, \nabla^j(\nabla^i U(x_i, x_j; |i-j|), y_j)) \Big\} \\
&\equiv < y, R_\mu^{(1)}(x)y > + < y, R_\mu^{(2)}(x)y > .
\end{aligned}$$

By the assumptions in the Theorem we see that

$$< y, R_\mu^{(1)}(x)y > = \sum_{i\in\mathbb{Z}^\nu} (y_i, Hess.P(x)y_i) \geq M|y|_0^2 \tag{5.1}$$

and

$$\begin{aligned}
&| < y, R_\mu^{(2)}(x)y > | \\
&\leq d \sum_{i\in\mathbb{Z}^\nu} |y_i| \sum_{j\neq i} \Psi(|i-j|)(|y_i|^2 + |y_j|^2) \\
&\leq M' \sum_{i\in\mathbb{Z}^\nu} |y_i|^2 =: M'|y|_0^2. \qquad (5.2)
\end{aligned}$$

Combining (5.1) and (5.2) we have proved Theorem 2.14. □

Let us now prove Theorem 2.16. For given $\Lambda \in \mathcal{C}$ and $z \in \mathfrak{S}$, let $\mu_\Lambda^{(z)}$ be the local Gibbs measure given by [28]

$$d\mu_\Lambda^{(z)} := \frac{1}{Z_\Lambda^{(z)}} \exp[-V(x_\Lambda) - W(x_\Lambda, z_{\Lambda^c})] dx_\Lambda, \tag{5.3}$$

where $Z_\Lambda^{(z)}$ is the normalization constant. It was shown in [28] that for any sequence $\{\Lambda_n\}$, $\Lambda_n \uparrow \mathbb{Z}^\nu$, the sequence of local Gibbs measures $\{\mu_{\Lambda_n}^{(z)}\}$ has a limit point in $\mathcal{G}^\Phi(\Omega)$. Since $\mathcal{G}^\Phi(\Omega)$ consists of only one element we may assume that the unique Gibbs measure μ is a limit of $\mu_{\Lambda_n}^{(z)}$.

Proof of Theorem 2.16. We may assume that $\mu = \lim_{n\to\infty} \mu_{\Lambda_n}^{(z)}$ in the local convergence topology. We notice first the following fact. Let $R_{\Lambda_n}^{(z)}(x_{\Lambda_n})$ be defined by

$$R_{\Lambda_n}^{(z)}(x_{\Lambda_n}) = \nabla_{\Lambda_n} \nabla_{\Lambda_n} \left(V(x_{\Lambda_n}) + W(x_{\Lambda_n}, z_{\Lambda_n^c}) \right), \tag{5.4}$$

where ∇_{Λ_n} is the gradient operator with respect to the variables $\{x_i : i \in \Lambda_n\}$. By the exactly same method used in the proof of Theorem 2.14 we see that the uniform log-concavity

$$< R_{\Lambda_n}^{(z)}(x_{\Lambda_n}) y_{\Lambda_n}, y_{\Lambda_n} > \geq \lambda |y_{\Lambda_n}|^2, \quad \lambda = M - M', \tag{5.5}$$

holds uniformly in $n \in \mathbb{N}$. Since, under the condition stated in the theorem, the Dirichlet operator is essentially self-adjoint with a core $\mathcal{F}C_b^\infty$ by Theorem 2.11, it is enough to show the theorem only for the functions $f \in \mathcal{F}C_b^\infty$. So let us fix an $f \in \mathcal{F}C_b^\infty$ and suppose that $f(x) = f(x_\Delta)$ for some $\Delta \in \mathcal{C}$. Then, for any $\Lambda_n \supset \Delta$, the condition (5.5) implies, by the Bakry-Emery criterion [16], the following log-Sobolev inequality (see [19, Theorem 6.2.42]) :

$$\begin{aligned} &\int_{(\mathbb{R}^d)^{\Lambda_n}} |f(x_{\Lambda_n})|^2 \log |f(x_{\Lambda_n})| d\mu_{\Lambda_n}^{(z)}(x_{\Lambda_n}) \\ &\leq \frac{1}{\lambda} \int_{(\mathbb{R}^d)^{\Lambda_n}} < \nabla_{\Lambda_n} f(x_{\Lambda_n}), \nabla_{\Lambda_n} f(x_{\Lambda_n}) > d\mu_{\Lambda_n}^{(z)}(x_{\Lambda_n}) \\ &\quad + \|f\|^2_{L^2(\mu_{\Lambda_n}^{(z)})} \log \|f\|_{L^2(\mu_{\Lambda_n}^{(z)})}. \end{aligned} \tag{5.6}$$

uniformly in n and z. By letting n go to infinity, we obtain the desired log-Sobolev inequality for any $f \in \mathcal{F}C_b^\infty$ with a Sobolev coefficient $c_\mu = \frac{1}{\lambda}$. This completes the proof. □

ACKNOWLEDGEMENTS. The authors would like to thank Professor B. Schmuland for helpful discussions. This work was supported in part by KOSEF (95-0701-04-01-3) and Basic Science Research Institute Program, Korean Ministry of Education (BSRI 97-1421). H. Y. Lim would like to thank Korea Research Foundation for financial support.

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