J. Korean Math. Soc. **34** (1997), No. 3, pp. 719–730

A GENERALIZED MINIMAX INEQUALITY RELATED TO ADMISSIBLE MULTIMAPS AND ITS APPLICATIONS

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ABSTRACT. From a minimax inequality related to admissible multimaps, we deduce generalized versions of lopsided saddle point theorems, fixed point theorems, existence of maximizable linear functionals, the Walras excess demand theorem, and the Gale-Nikaido-Debreu theorem.

0. Introduction

In the frame of the KKM theory, the author [15, Theorem 11] obtained a far-reaching generalization of the Ky Fan minimax inequality related to the admissible classes of multimaps. It was first used to establish the fixed point theory of admissible maps in [14], and then applied to the Karamardian type variational inequalities and generalized complementarity problems in [16]. On the other hand, in [17], we introduced the "better" admissible class $\mathfrak{B}$ of multimaps and obtained a basic coincidence theorems for $\mathfrak{B}$ as well as a matching theorem and a KKM theorem.

In the present paper, we obtain a new minimax or equilibrium theorem (Theorem 1) involving a map in $\mathfrak{B}$ and apply it to some other problems.

In section 2, we obtain a refined version (Theorem 3) of a lopsided saddle point theorem in [14], which is applied to obtain a Browder type fixed point theorem (Theorem 4) in section 3. Moreover, in section

Received June 7, 1997.

1991 Mathematics Subject Classification: Primary 49J35, 47H10, 49K35, 52A07.

Key words and phrases: Convex space, polytope, l.s.c., u.s.c., quasiconcave, quasiconvex, multimap (map), acyclic, admissible map, minimax inequality, fixed point, lopsided saddle point, weakly inward (outward) sets, Walras excess demand theorem, the Gale-Nikaido-Debreu theorem.

Supported in part by the SNU-Daewoo Research Fund in 1996.

4, we deduce a generalization (Theorem 5) of the existence theorem of maximizable linear functionals due to Simons. Finally, section 4 deals with generalized forms of the Walras excess demand theorem (Theorems 6 and 7) and the Gale-Nikaido-Debreu theorem (Theorem 8).

All results in this paper can be stated with respect to admissible class $\mathfrak{A}_c^\kappa$ of multimaps, which include continuous (single-valued) functions, Kakutani maps, Aronszajn maps, acyclic maps, Górniewicz maps, Dzedzej maps, approximable maps, and others which frequently appear in nonlinear analysis and algebraic topology.

1. Preliminaries

A *convex space* is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*; see Lassonde [11].

Recall that an extended real-valued function $f : X \to \overline{\mathbf{R}}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (*l.s.c.* [resp. *u.s.c.*]) whenever $\{x \in X \mid fx > r\}$ [resp. $\{x \in X \mid fx < r\}$] is open for each $r \in \overline{\mathbf{R}}$. If X is a convex space, then f is *quasiconcave* [resp. *quasiconvex*] whenever $\{x \in X \mid fx > r\}$ [resp. $\{x \in X \mid fx < r\}$] is convex for each $r \in \overline{\mathbf{R}}$.

For topological spaces X and Y, a *multimap* or a *map* $T : X \multimap Y$ is a function from X into the set of nonempty subsets of Y. A map $T : X \multimap Y$ is *upper semicontinuous* (*u.s.c.*) if, for each open subset G of Y, the set $\{x \in X \mid Tx \subset G\}$ is open in X; *lower semicontinuous* (*l.s.c.*) if, for each closed subset F of Y, the set $\{x \in X \mid Tx \subset F\}$ is closed in X; *continuous* if it is u.s.c. and l.s.c.; and *compact* if the range $T(X) = \{y \in Y \mid y \in Tx$ for some $x \in X\}$ is contained in a compact subset of Y. As usual, the set $\{(x, y) \mid y \in Tx\}$ is called either the *graph* of F or, simply, F.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

An *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ is one such that, for each T and each compact subset K of X, there exists a map $\Gamma \in \mathfrak{A}_c(K, Y)$ satisfying $\Gamma x \subset Tx$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite composites of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of $\mathfrak{A}$ are continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ in topological vector spaces, admissible maps of Górniewicz, permissible maps of Dzedzej, and many others. Note that $\mathbb{V}_c^\sigma$ due to Park, Singh and Watson, and $\mathbb{K}_c^\sigma$ due to Lassonde are examples of $\mathfrak{A}_c^\kappa$. For details, see [14-16, 19].

More recently, the approximable maps $\mathbb{A}^\kappa(X, Y)$ is due to Ben-El-Mechaiekh and Idzik [2], where X and Y are subsets of topological vector spaces. They noted that if X is a closed subset of a locally convex Hausdorff topological vector space, then any u.s.c. compact map $T : X \multimap X$ with closed values belongs to $\mathbb{A}^\kappa$ whenever the functional values are all (1) convex, (2) contractible, (3) decomposable, or (4) ∞-proximally connected.

Let X be a convex space and Y a Hausdorff space. In [17], we introduced a new "better" admissible class $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ such that, for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P)$ has a fixed point.

Moreover, we defined one more class of multimaps as follows:

$F \in \mathfrak{B}^\kappa(X, Y) \iff$ for any compact convex subset K of X, there is a closed map $\Gamma \in \mathfrak{B}(K, Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

Our new classes contain the admissible class $\mathfrak{A}_c^\kappa$ properly.

The following KKM theorem is due to the author [17, Theorem 3]:

THEOREM 0. *Let X be a convex space, Y a Hausdorff space, $F \in \mathfrak{B}(X, Y)$ a compact map, and $S : X \multimap Y$ a map. Suppose that*

(0.1) *for each $x \in X$, Sx is closed; and*
(0.2) *for each $N \in \langle X \rangle$, $F(\mathrm{co}\, N) \subset S(N)$.*

Then $\overline{F(X)} \cap \bigcap\{Sx : x \in X\} \neq \emptyset$.

Here, $\langle X \rangle$ denotes the set of all nonempty finite subsets of X.

2. Basic minimax inequalities

We begin with the following minimax inequality equivalent to Theorem 0:

THEOREM 1. *Let X be a convex space, Y a Hausdorff space, $F \in \mathfrak{B}(X, Y)$ a compact map, $f, g : X \times Y \to \overline{\mathbf{R}}$ extended real-valued functions, and $\gamma \in \overline{\mathbf{R}}$. Suppose that*

(1.1) *for each $x \in X$, $\{y \in Y \mid f(x, y) \leq \gamma\}$ is closed;*
(1.2) *for each $N \in \langle X \rangle$ and $y \in F(\mathrm{co}\, N)$, $\min\{g(x, y) \mid x \in N\} \leq \gamma$; and*
(1.3) *$f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$.*

Then (a) *there exists a $\hat{y} \in \overline{F(X)}$ such that*

$$f(x, \hat{y}) \leq \gamma \quad \text{for all} \quad x \in X;$$

and (b) *if $\gamma = \sup\{g(x, y) \mid (x, y) \in F\}$, then we have the minimax inequality:*

$$\min_{y \in \overline{F(X)}} \sup_{x \in X} f(x, y) \leq \sup_{(x,y) \in F} g(x, y).$$

Proof of Theorem 1 using Theorem 0. For each $x \in X$, let $Sx = \{y \in Y \mid f(x, y) \leq \gamma\}$. Then Sx is closed by (1.1). We show that (0.2) holds. Suppose that there exists an $N \in \langle X \rangle$ such that $F(\mathrm{co}\, N) \not\subset S(N)$. Choose a $y \in F(\mathrm{co}\, N)$ such that $y \notin S(N)$, whence $g(x, y) \geq f(x, y) > \gamma$ for all $x \in N$. Then $\min_{x \in N} g(x, y) > \gamma$, which contradicts (1.2). Therefore, all of the requirements of Theorem 0 are satisfied, and hence there exists a $\hat{y} \in \overline{F(X)}$ such that $\hat{y} \in Sx$ for all $x \in X$. This completes the proof of (a). Note that (b) clearly follows from (a). □

Proof of Theorem 0 using Theorem 1. Define $f = g : X \times Y \to \overline{\mathbf{R}}$ by

$$f(x, y) = \begin{cases} 0 & \text{if } y \in Sx \\ 1 & \text{otherwise} \end{cases}$$

for $(x, y) \in X \times Y$, and let $\gamma = 0$ Then (0.1) clearly implies (1.1). We show that (0.2) implies (1.2). In fact, suppose that there exist an

$N \in \langle X \rangle$ and a $y \in F(\operatorname{co} N)$ such that $\min\{f(x,y) \mid x \in N\} > 0$. Then $y \notin Sx$ for all $x \in N$; that is, $F(\operatorname{co} N) \not\subset S(N)$, which contradicts (0.2). Therefore, all of the requirements of Theorem 1 are satisfied, and hence there exists a $\hat{y} \in \overline{F(X)}$ such that $f(x,\hat{y}) = 0$ for all $x \in X$; that is, $\hat{y} \in \bigcap\{Sx : x \in X\}$. This completes our proof. □

Theorem 1 is a far-reaching generalization of the celebrated minimax inequality of Ky Fan [7] and includes a large number of particular known forms; for example, [22, Theorem 2.11] and [4, Corollary 3.5]. Note that if T is single-valued, then the Hausdorffness assumption on Y is not necessary; see [15].

A particular form of Theorem 1 is applied in [18] to obtain existence theorems of solutions of variational inequalities.

From Theorem 1, we deduce the following result on lopsided saddle points:

THEOREM 2. *Let X be a compact convex space, Y a Hausdorff space, and $T \in \mathfrak{B}(X,Y)$ a closed compact map. Let $\phi : X \times Y \to \mathbf{R}$ be a continuous function such that for each $y \in Y$, $x \mapsto \phi(x,y)$ is quasiconvex on X. Then there exists an $(x_0,y_0) \in T$ such that*

$$\phi(x_0,y_0) \le \phi(x,y_0) \quad \textit{for all} \quad x \in X.$$

Proof. Define $f : X \times Y \to \mathbf{R}$ by

$$f(x,y) = \min_{z \in X} \phi(z,y) - \phi(x,y)$$

for $(x,y) \in X \times Y$. Then f is continuous on $X \times Y$ [1, p.70] and satisfies (1.1)-(1.3) of Theorem 1 with $f = g$. Therefore, by Theorem 1, there exists a $\hat{y} \in \overline{T(X)}$ such that

$$\sup_{x \in X} f(x,\hat{y}) \le \sup_{(x,y) \in T} f(x,y).$$

Since $x \mapsto \phi(x,\hat{y})$ is continuous on the compact set X, there exists an $\hat{x} \in X$ such that $\phi(\hat{x},\hat{y}) = \min_{z \in X} \phi(z,\hat{y})$ or $f(\hat{x},\hat{y}) = 0$. Hence, we have

$$0 \le \sup_{(x,y) \in T} f(x,y).$$

Since the graph of T is closed and hence compact in $X \times Y$, the supremum in the above inequality is attained on some $(x_0, y_0) \in T$. This completes our proof. $\square$

Note that, in Theorem 2, a closed compact map $T \in \mathfrak{B}(X, Y)$ can be replaced by a compact map $T \in \mathfrak{B}^\kappa(X, Y)$, a map $T \in \mathfrak{A}_c^\kappa(X, Y)$ or a map $T \in \mathfrak{A}_c(X, Y)$. In the following, we use mainly the class $\mathfrak{A}_c^\kappa$ for the simplicity.

EXAMPLES. 1. Ky Fan [7, Corollary 1]: $X = Y$ and $T = 1_X$, where the Hausdorffness assumption is superfluous. Moreover, for a normed vector space $E = Y$, $X \subset E$, $T : X \hookrightarrow E$, and $\phi(x, y) = \|x - y\|$, Theorem 2 reduces to Fan [6, Theorem 2], which is usually called a best approximation theorem.

2. Ha [10, Theorem 2]: Y is a nonempty compact convex subset of a Hausdorff topological vector space and $T \in \mathbb{K}(X, Y)$. Ha applied his theorem to obtain fixed point theorems for Kakutani maps.

3. Park [13, Theorem 2]: This is the case $T \in \mathbb{V}(X, Y)$ and applied to obtain fixed point theorems for acyclic maps. In [14], Theorem 2 is applied to obtain results for admissible maps.

4. Recently, in our work [16], Theorem 2 is applied to generalized equilibrium problems and generalized complementarity problems.

For a subset X of a topological vector space E and $x \in E$, the *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined by

$$I_X(x) = \{x + r(u - x) \in E \mid u \in X,\ r > 0\},$$
$$O_X(x) = \{x - r(u - x) \in E \mid u \in X,\ r > 0\},$$

resp. Their closures are called *weakly inward* and *outward sets*, and denoted $\overline{I}_X(x)$ and $\overline{O}_X(x)$, resp.

The following slightly weaker form (with "quasiconvex" replaced by "convex") of Theorem 2 is useful in various problems:

THEOREM 3. *Let X be a compact convex subset of a topological vector space E, Y a Hausdorff space, and $T \in \mathfrak{A}_c^\kappa(X, Y)$. Let $\phi : E \times Y \to \mathbf{R}$ be a continuous function such that for each $y \in Y$, $x \mapsto \phi(x, y)$ is convex on E. Then there exists an $(x_0, y_0) \in T$ such that*

$$\phi(x_0, y_0) \le \phi(x, y_0) \quad \textit{for all} \quad x \in \overline{I}_X(x_0).$$

Proof. From Theorem 2, there exists an $(x_0, y_0) \in T$ such that

$$\phi(x_0, y_0) \leq \phi(x, y_0) \quad \text{for all} \quad x \in X.$$

For $x \in I_X(x_0)\backslash X$, there exist $u \in X$ and $r > 1$ such that $x = x_0 + r(u - x_0)$. Suppose that $\phi(x, y_0) < \phi(x_0, y_0)$. Since

$$\frac{1}{r}x + (1 - \frac{1}{r})x_0 = u \in X$$

and $\phi(\cdot, y_0)$ is convex, we have

$$\phi(u, y_0) \leq \frac{1}{r}\phi(x, y_0) + (1 - \frac{1}{r})\phi(x_0, y_0) < \phi(x_0, y_0),$$

which is a contradiction. Therefore, $\phi(x_0, y_0) \leq \phi(x, y_0)$ holds for $x \in I_X(x_0)$; that is, for all $x \in \overline{I}_X(x_0)$. □

EXAMPLES. 1. Fan [5, p.118]: $X = Y$ and $T = 1_X$. This result was shown to imply Tychonoff's fixed point theorem.

2. Park [12, Theorem 1]: $Y = E$ and $T : X \multimap E$. This was applied to obtain fixed point theorems for weakly inward maps.

3. Fixed point theorems

From Theorem 3, we immediately deduce the following:

THEOREM 4. *Let X be a compact convex subset of a Hausdorff topological vector space E, $T \in \mathfrak{A}_c^\kappa(X, E)$, and $\phi : E \times E \to \mathbf{R}$ be a continuous function such that for each $y \in E$, $x \mapsto \phi(x, y)$ is convex on E. Suppose that for any $x \in X$ satisfying $x \notin Tx$, there exists a $z \in \overline{I}_X(x)$ such that*

$$\phi(z, y) < \phi(x, y) \quad \textit{for all} \quad y \in Tx.$$

Then T has a fixed point $x_0 \in X$, that is, $x_0 \in Tx_0$.

Proof. From Theorem 3 with $Y = E$, there exists an $(x_0, y_0) \in T$ such that

$$\phi(x_0, y_0) \leq \phi(x, y_0) \quad \text{for all} \quad x \in \overline{I}_X(x_0).$$

Suppose that $x_0 \notin Tx_0$. Then, by hypothesis, there exists a $z \in \overline{I}_X(x_0)$ such that $\phi(z, y_0) < \phi(x_0, y_0)$, which is a contradiction. This completes our proof. □

REMARK. Let a map $T' : X \multimap E$ be given by $T'x = 2x - Tx$ for $x \in X$ for $T \in \mathfrak{A}_c^\kappa(X, E)$. If $T' \in \mathfrak{A}_c^\kappa(X, E)$ and $\phi(x, y) = \psi(y - x)$ for a continuous convex function $\psi : E \to \mathbf{R}$, then the set $\overline{I}_X(x_0)$ in Theorem 4 can be replaced by $\overline{O}_X(x_0)$ in Theorem 4.

In fact, by Theorem 4, there exists an $(x_0, y_1) \in \Gamma'$ satisfying $\psi(x_0 - y_1) \leq \psi(x' - y_1)$ for all $x' \in \overline{I}_X(x_0)$. For $x \in O_X(x_0)$, let $x' = 2x_0 - x$ and $y_1 = 2x_0 - y_0$ where $y_0 \in Tx_0$. Then we have

$$\begin{aligned} \phi(x_0, y_0) = \psi(y_0 - x_0) &= \psi(x_0 - y_1) \\ &\leq \psi(x' - y_1) = \psi(y_0 - x) = \phi(x, y_0) \end{aligned}$$

for all $x \in O_X(x_0)$; and hence, for all $x \in \overline{O}_X(x_0)$.

We followed the method of Browder [3].

We give examples of Theorem 4 as follows:

EXAMPLES. 1. Browder [3, Theorems 1 and 2]: E is a locally convex topological vector space, $T = f \in \mathbb{C}(X, E)$, and $\phi(x, y) = p(x - y)$, where $p : E \to \mathbf{R}$ is a continuous convex function. He also obtained variants of his theorems employing the concept of subgradient $\partial h : E \multimap E^*$ of a convex real function h on E.

2. Park [14, Corollary 3.1]: E is a metrizable topological vector space where the metric d on E has been chosen so that balls are convex, and $\phi = d$. This extends earlier works of Brouwer, Schauder, Cellina, Fan, and Rassias for $T = f \in \mathbb{C}(X, E)$, and of Kakutani and Bohnenblust-Karlin for $T \in \mathbb{K}(X, E)$; see [14].

4. Existence of maximizable linear functionals

Let Φ be either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. Let E be a topological vector space over Φ, E^* its topological dual, and $\langle\ ,\ \rangle$ the pairing between E^* and E.

Let X be a compact subset of E. In this section, we assume that E^* is equipped with any topology such that the pairing $\langle\ ,\ \rangle : E^* \times X \to \Phi$ is continuous. For example, the topology of E^* can be the uniform convergence topology on the bounded sets or on the compact sets in E.

From Theorem 3, we have the following:

THEOREM 5. *Let X be a compact convex subset of a topological vector space E, and $T \in \mathfrak{A}_c^\kappa(X, E^*)$. If E^* is Hausdorff, then there exists an $(x_0, f_0) \in T$ such that*

$$\operatorname{Re}\langle f_0, x_0\rangle = \max_{x \in \overline{I}_X(x_0)} \operatorname{Re}\langle f_0, x\rangle.$$

Proof. Let $\phi : E \times E^* \to \mathbf{R}$ be defined by

$$\phi(x, f) = -\operatorname{Re}\langle f, x\rangle \quad \text{for} \quad (x, f) \in E \times E^*.$$

Since ϕ is continuous and $\phi(\cdot, f)$ is convex on E by Theorem 3, there exists an $(x_0, f_0) \in T$ such that

$$\phi(x_0, f_0) = \min_{x \in X} \phi(x, f_0).$$

This is equivalent to the conclusion. □

EXAMPLES. 1. If $T = t \in \mathbb{C}_c^\kappa(X, E^*)$ in Theorem 5, we do not need to assume the Hausdorffness of E^*. This case includes the following:

2. Simons [20, Theorem 4.5]: X is a subset of a real Hausdorff topological vector space and $T : X \multimap E^*$ has nonempty convex values and open fibers.

5. Generalized Walras type theorems

From Theorem 1, we obtain the following generalization of the so-called Walras excess demand theorem:

THEOREM 6. *Let X be a convex space, Y a Hausdorff space, $T \in \mathfrak{B}(X, Y)$ a compact map, $c \in \mathbf{R}$, and $\phi, \psi : X \times Y \to \overline{\mathbf{R}}$ two extended real-valued functions such that*

(1) *$\phi(x, y) \le \psi(x, y)$ for each $(x, y) \in X \times Y$;*

(2) *for each* $x \in X$, $y \mapsto \psi(x, y)$ *is u.s.c. on* Y;
(3) *for each* $y \in Y$, $x \mapsto \phi(x, y)$ *is quasiconvex on* X; *and*
(4) $\phi(x, y) \geq c$ *for all* $(x, y) \in T$ *(Walras law).*

Then there exists a Walras equilibrium; that is, there exists a $y_0 \in Y$ *such that*

$$c \leq \phi(x, y_0) \quad \textit{for all} \quad x \in X.$$

Proof. Use Theorem 1 with $f = -\phi$ and $g = -\psi$. Then there exists a $\hat{y} \in \overline{T(X)}$ such that

$$\inf_{(x,y)\in T} \phi(x, y) \leq \inf_{x\in X} \psi(x, \hat{y}).$$

This completes our proof. □

EXAMPLE. Granas and Liu [8, Theorem (13.4)]: Y is a convex subset of a topological vector space and T belongs to a particular class of $\mathfrak{A}_c^\kappa$.

The following is a different version of Theorem 6:

THEOREM 7. *Let* X *be a compact convex space,* Y *a Hausdorff space, and* $T \in \mathfrak{A}_c^\kappa(X, Y)$. *Let* $\phi : X \times Y \to \mathbf{R}$ *be a continuous function and* $c \in \mathbf{R}$ *such that*

(1) *for each* $y \in Y$, $x \mapsto \phi(x, y)$ *is quasiconvex on* X; *and*
(2) $\phi(x, y) \geq c$ *for all* $(x, y) \in T$ *(Walras law).*

Then there exists a Walras equilibrium; that is, there exists an $(x_0, y_0) \in T$ *such that*

$$c \leq \phi(x_0, y_0) \leq \phi(x, y_0) \quad \textit{for all} \quad x \in X.$$

Proof. Immediate from Theorem 2. □

EXAMPLES. 1. Gwinner [9, Theorem 8], Zeidler [21, Theorem 77.E]: X and Y are compact convex subsets of locally convex Hausdorff topological vector spaces and $T \in \mathbb{K}(X, Y)$.

2. Granas and Liu [8, Theorem (13.5)]: Y is a convex subset of a Hausdorff topological vector space and $T : X \multimap Y$ a map with nonempty convex values and open fibers. Note that $T \in \mathbb{C}_c^\kappa(X, Y)$.

From Theorem 7, we deduce the following generalization of the Gale-Nikaido-Debreu theorem:

THEOREM 8. *Let $(E, F, \langle\ ,\ \rangle)$ be a dual system of Hausdorff topological vector spaces E and F, where the real bilinear form $\langle\ ,\ \rangle$ is continuous on compact subsets of $E \times F$. Let X be a nonempty compact convex subset of E, P the convex cone $\bigcup\{rX : r \geq 0\}$, and $P^+ = \{y \in F : \langle p, y\rangle \geq 0,\ p \in P\}$ its positive dual cone. Then for any map $T \in \mathfrak{A}_c^\kappa(X, F)$ satisfying $\langle x, y\rangle \geq 0$ for $(x, y) \in T$, there exists an $\overline{x} \in X$ such that $T\overline{x} \cap P^+ \neq \emptyset$.*

Proof. Since X is compact, we may assume that $T \in \mathfrak{A}_c(X, E)$ without loss of generality. Then $Y = \overline{T(X)}$ is compact in F. Let $\phi : X \times Y \to \mathbf{R}$ be defined by $\phi(x, y) = \langle x, y\rangle$ for $(x, y) \in X \times Y$. Then ϕ is continuous. Since ϕ satisfies the Walras law for T with $c = 0$, by Theorem 7, there exists an $(\overline{x}, \overline{y}) \in T$ such that

$$0 \leq \langle \overline{x}, \overline{y}\rangle \leq \langle x, \overline{y}\rangle \quad \text{for all} \quad x \in X,$$

and hence, for all $x \in P$. Therefore, we have $\overline{y} \in P^+$. This completes our proof. □

EXAMPLES. 1. Gwinner [9, Corollary to Theorem 8]: E and F are locally convex and $T \in \mathbb{K}(X, F)$.

2. Gale, Nikaido, and Debreu: $P = \{x \in \mathbf{R}^n : x_i \geq 0,\ 1 \leq i \leq n\}$, $X = \{x \in P : x_1 + \cdots + x_n = 1\}$, the standard $(n-1)$-simplex, and $T \in \mathbb{K}(X, \mathbf{R}^n)$. For the references, see [9].

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