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ON THE SQUARE OF BROWNIAN DENSITY PROCESS

NHANSOOK CHO

ABSTRACT. The square of Brownian density process, Q^λ is defined where λ is a parameter. Applying limit theorems of stochastic integrals w.r.t. martingale measure, we prove a weak limit theorem for Q^λ in $D_{\mathcal{S}'(R^d)}[0,1]$.

1. Introduction

Let $\{X^\alpha, \alpha \in N\}$ be a family of i.i.d. standard Brownian motions in R^d with initial distribution given by a Poisson point process Π^λ of parameter λ. Set for any test function ϕ,

$$\eta_t(\phi) = \sum_\alpha \phi(X_t^\alpha), \qquad {\eta_t}^2(\phi) = \sum_{\alpha,\beta} \phi(X_t^\alpha)\phi(X_t^\beta) \tag{1.1}$$

We will first symmetrize this, then throw away the terms with $\alpha = \beta$ to get a new process, Q_t^λ of which limit we want to consider.

Let $\{\xi^\alpha, \alpha \in A\}$ be i.i.d. random variables independent of the X^α such that $\mathbf{P}\{\xi^\alpha = 1\} = \mathbf{P}\{\xi^\alpha = -1\} = \frac{1}{2}$.

Define for any test function ψ on $R^d \times R^d$

$$Q_t^\lambda(\psi) = \frac{1}{\lambda} \sum_{\alpha \neq \beta} \xi^\alpha \xi^\beta \psi(X^\alpha, X^\beta). \tag{1.2}$$

The study of this process is related to the intersection local times of super processes and is said (by Dynkin and Mandelbaum[2]), and Walsh[5]) to be connected with U-statistics.

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Define

$$
\begin{aligned}
(1.3) \qquad \tilde{\eta}_t(\phi) &= \sum_{\alpha \in A} \xi^\alpha \phi(X_t^\alpha) \\
W_t(A) &= \frac{1}{\lambda} \sum_\alpha \int_0^t I_A(X_s^\alpha) dX_s^\alpha \\
\tilde{W}_t(A) &= \frac{1}{\lambda} \sum_\alpha \xi^\alpha \int_0^t I_A(X_s^\alpha) dX_s^\alpha
\end{aligned}
$$

Obviously, $E[\tilde{\eta}(\phi)] = 0$. It is known (see[G],[5]) that if $\phi \in L^1(R^d)$, then

$$
(1.4) \qquad E[\eta_t(\phi)] = \lambda \int \phi(x)dx, \quad \text{and Var } \eta_t(\phi) \sim O(\lambda)
$$

Now, let λ_n be the sequence of parameter values and we consider the corresponding processes, η^n, $\tilde{\eta}^n$, W_n and $\tilde{W}^n$.

Let $\tilde{\Pi}^n(dx) = \frac{1}{\sqrt{\lambda_n}}(\Pi^{\lambda_n}(dx) - \lambda_n\, dx)$ be the normalized initial measure. It is known that $\tilde{\Pi}^n \Rightarrow V^0$, where V^0 is a white noise based on R^d.

Walsh[5] studied the limiting behavior of Q^λ and proved the following theorem in his famous note.

THEOREM 1.1. [5] *The process Q^λ converges weakly in $D_{S'(R^{2d})}[0,1]$ to a solution of the SPDE*

$$
\begin{aligned}
\frac{\partial Q}{\partial t}(x,y) &= \frac{1}{2}\Delta Q(x,y) + \eta(x)\nabla_2 \cdot W_{ys}^0 + \eta(y)\nabla_1 \cdot W_{xs}^0 \\
Q_0 &= V^0 \times V^0
\end{aligned}
$$

where V^0 and W^0 are independent white noises on R^d and $R^d \times R_+$ respectively.

We dare to say that the proof in [5] is somehow wrong and try to give an alternative proof using our previous theorem w.r.t. martingale measure.

DEFINITION 1.1. Let $(R^d, \mathcal{B}(R^d), \nu)$ be a $\sigma-$finite measure space. A white noise based on ν is a random set function W on the sets $A \in \mathcal{B}(R^d)$ of finite $\nu-$measure such that

(1) $W(A)$ is a $N(0, \nu(A))$ random variable,

(2) if $A \cap B = \emptyset$, then $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$.

Let $\mathcal{S}'(R^d)$ be the dual of Schwartz space, $\mathcal{S}(R^d)$ which is the space of infinitely differentiable functions vanishing at infinity.

The folllowing definition is for the martingale measure established by Walsh[5].

DEFINITION 1.2. Let $(\Omega, \mathcal{F}_t, P)$ be a filtered space, and $\mathcal{B}(R^d)$ be the Borel $\sigma-$field. Let $M(\cdot, \cdot)$ be a random real-valued function on $R^d \times R_+$. M is called an $(\mathcal{F}_t, P)$-martingale measure if it satisfies the following properties.

(1) For each $A \in \mathcal{B}(R^d)$, $M(A, \cdot)$ is a $(\mathcal{F}_t, P)-$ square integrable martingale and $M(A, 0) = 0$.

(2) For any $A, B \in \mathcal{B}(R^d)$ such that $A \cap B = 0$, $M(A \cup B, t) = M(A, t) + M(B, t), P$ a.s. for every $t > 0$.

(3) For every $t > 0$, $M(\cdot, t)$ is a σ-finite L^2-valued measure in a certain sense. (See in detail [5]).

For $A, B \in \mathcal{B}(R^d)$, there exists a unique predictable process, $\langle M(A), M(B)\rangle_t$ such that $M(A, t)M(B, t) - \langle M(A), M(B)\rangle_t$ is a martingale.

DEFINITION 1.3. Let the covariance functional of martingale measure, M be $Cov_t(A, B) = \langle M(A), M(B)\rangle_t$ where $A, B \in \mathcal{B}(R^d)$. Define a set function U by

$$U(A \times B \times (s, t]) = Cov_t(A, B) - Cov_s(A, B)$$

DEFINITION 1.4. A martingale measure is worthy if there is a σ-finite L^2-valued measure $K(\Gamma, \omega), \Gamma \in \mathcal{B}(R^d \times R^d \times R_+), \omega \in \Omega$ such that for fixed A, B $\{K(A \times B \times (0, t], t \geq 0\}$ is predictable, and $|U(\Gamma)| \leq K(\Gamma)$. We call K a dominating measure.

The processes W^n in (1.3) are good examples of martingale measure.

PROPOSITION 1.1. [1] *If $\lambda_n \to \infty$, then for each $\phi(x,y) \in \mathcal{S}(R^{2d})$, along the appropriate subsequence*

$$\frac{1}{\sqrt{\lambda_n}} \int \phi(x,y)\tilde{\eta}_s^n(dx)\tilde{W}^n(ds,dy) \Longrightarrow \int \phi(x,y)\tilde{\eta}_s(dx)\tilde{W}(ds,dy),$$

where $\tilde{W}$ is a white noise based on Lebesgue measure, and $\tilde{\eta}$ is the $\mathcal{S}'(R^d)$-valued Gaussian process.

2. Main theorem

The following is our version of Theorem 1.1.

THEOREM 2.1. *The process $\{Q_t^{\lambda_n}\}$ is relatively compact in $D_{\mathcal{S}'(R^d)}[0,\infty)$, and converges to the solution of the following equation;*
(2.0)

$$\begin{aligned} Q_t(\psi) &= Q_0(\psi) + \frac{1}{2}\int_0^t Q_s(\Delta\psi)ds + \int_{R^d\times[0,t]}\int_{R^d} \psi(x,y)\tilde{\eta}_s(dy)\tilde{W}(ds,dx) \\ &\quad + \int_{R^d\times[0,t]}\int_{R^d} \psi(x,y)\tilde{\eta}_s(dx)\tilde{W}(ds,dy), \end{aligned}$$

for every $\psi \in \mathcal{S}(R^{2d})$

Proof. Using Ito's formula for $Q_t^\lambda(\psi)$ and letting

$$\chi(x) = (\nabla_1\psi)(x,x) + (\nabla_2\psi)(x,x),$$

we rewrite $Q_t^\lambda(\psi)$ as the following:
(2.1)

$$\begin{aligned} Q_t^\lambda(\psi) &= Q_0^\lambda(\psi) + \frac{1}{2}\int_0^t Q_s^\lambda(\Delta\psi)\,ds + \frac{1}{\sqrt{\lambda}}\int_{R^d\times[0,t]} \tilde{\eta}_s[\nabla_1\psi(x,\cdot)]\tilde{W}(dx,ds) \\ &\quad + \frac{1}{\sqrt{\lambda}}\int_{R^d\times[0,t]} \tilde{\eta}_s[\nabla_2\psi(\cdot,y)]\tilde{W}(dy,ds) - \frac{1}{\sqrt{\lambda}}\int_{R^d\times[0,t]} \chi(x)W(dx,ds) \end{aligned}$$

Let

$$p_t(x,x') = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-x'|^2}{2t}}, \quad G_t(x,x';y,y') := p_t(x,x')p_t(y,y')$$

$$G_t(\psi,x,y) = \int_{R^{2d}} p_t(x,x')p_t(y,y')\psi(x',y')dx'dy'$$

Then G is the green function on R^{2d} for this problem. Write $Q^\lambda = \tilde{Q}^\lambda + R^\lambda$, where

(2.2)

$$\tilde{Q}_t^\lambda(\psi) = Q_0^\lambda(\psi) + \frac{1}{2}\int_0^t \tilde{Q}_s^\lambda(\Delta\psi)\,ds$$

$$+ \frac{1}{\sqrt{\lambda}}\int_{R^d\times[0,t]} \tilde{\eta}_s(\nabla_1\psi(x,\cdot))\tilde{W}(ds,dx)$$

$$+ \frac{1}{\sqrt{\lambda}}\int_{R^d\times[0,t]} \tilde{\eta}_s(\nabla_2\psi(\cdot,y))\tilde{W}(ds.dy)$$

$$R_t^\lambda(\psi) = \frac{1}{2}\int_0^t R_s(\Delta\psi)ds - \frac{1}{\sqrt{\lambda}}\int_{R^d\times[0,t]} \chi(x)W(ds,dx).$$

$$= -\frac{1}{\lambda}\int_{R^d\times[0,t]} (\nabla_1 G_{t-s}(\psi)(y,y) + \nabla_2 G_{t-s}(\psi)(y,y))W(ds,dy).$$

Define a pair of martingale measures on R^{2d} by

$$M_{1,t}^n(\psi) = \int_{R^d\times[0,t]} (\int_{R^d} \psi(x,y)\tilde{\eta}_s^n(dx))\tilde{W}^\lambda(ds,dy),$$

$$M_{2,t}^n(\psi) = \int_{R^d\times[0,t]} (\int_{R^d} \psi(x,y)\tilde{\eta}_s^n(dy))\tilde{W}^\lambda(ds,dx).$$

Define

$$M_{2,t}(\psi) \equiv \int_{R^d\times[0,t]}\int_{R^d} \psi(x,y)\tilde{\eta}_s(dy)\tilde{W}^\lambda(ds,dx)$$

We write (2.2);

(2.3)

$$\tilde{Q}_t^\lambda(\psi) = Q_0^\lambda(\psi) + \frac{1}{2}\int_0^t \tilde{Q}_s^\lambda(\Delta\psi)ds + \frac{1}{\sqrt{\lambda}}M_{1,t}(\nabla_2\psi) + \frac{1}{\sqrt{\lambda}}M_{2,t}(\nabla_1\psi).$$

Now we apply Theorem 5.1 of Walsh[5], replacing λ with λ_n, for every $\psi \in \mathcal{S}(R^{2d})$.

(2.4)

$$\tilde{Q}_t^{\lambda_n}(\psi)$$

$$= Q_0^{\lambda_n}(G_t\psi) + \frac{1}{\sqrt{\lambda_n}}\int_{R^{2d}\times[0,t]} \nabla_1 G_{t-s}(\psi,x,y)M_2^n(dx,dy,ds)$$

$$+ \frac{1}{\sqrt{\lambda_n}}\int_{R^{2d}\times[0,t]} \nabla_2 G_{t-s}(\psi,x,y)M_1^n(dx,dy,ds).$$

The following argument shows the relative compactness of $\{Q_t^{\lambda_n}\}$. Define

$$V_{i,t}^n(\phi) \equiv \frac{1}{\sqrt{\lambda_n}} \int_{R^{2d}\times[0,t]} \nabla_j G_{t-s}(\psi, x, y) M_i^n(dx, dy, ds)$$

for $i, j = 1, 2$. Then

(2.5)

$$\begin{aligned}
&V_{2,t}^n(\psi)\\
=&\frac{1}{\sqrt{\lambda_n}} \int_{R^{2d}\times[0,t]} \nabla_1 G_{t-s}(\psi, x, y) M_2^n(dx, dy, ds)\\
=&\frac{1}{\sqrt{\lambda_n}} \int_{R^{2d}\times[0,t]} \nabla_1 (\int_s^t G_{u-s}(\Delta\psi, x, y) + (\psi, x, y) du) M_2^n(dx, dy, ds)\\
=&\frac{1}{\sqrt{\lambda_n}} (M_{2,t}^n(\nabla_1\psi)\\
&+ \int_0^t [\int_0^u \int_{R^{2d}} \nabla_1 G_{u-s}(\Delta\psi, x, y) M_2^n(dx, dy, ds)] du)
\end{aligned}$$

Recalling that $M_{2,u}^n(\psi) = \int_0^u \int_{R^d} \tilde\eta_s^n(\psi(x,\cdot)) \tilde W^\lambda(ds, dx)$, let

$$\begin{aligned}
V_{2,t}^n(\psi) =& \frac{1}{\sqrt{\lambda_n}} M_{2,t}^n(\nabla_1\psi(x,y))\\
&+ \frac{1}{\sqrt{\lambda_n}} \int_0^t [\int_{R^d\times[0,u]} \tilde\eta_s^n(\nabla_1 G_{t-s}(\Delta\psi, x, \cdot)) \tilde W^\lambda(ds, dx)] du.
\end{aligned}$$

□

LEMMA 2.2. *For any $\phi, \phi \in L^2(R^d) \cap C^1(R^d)$, $\frac{\partial\phi}{\partial x_i} \in L^2(R^d)$ for $i = 1, \cdots, d$,*

$$E[\tilde\eta_t^n(\phi)^2] \le \lambda_n \|\phi\|_2^2 + \lambda_n t \|\nabla\phi\|_2^2$$

Proof. By Proposition 8.4[5], we have

(2.6)

$$\tilde\eta_t^n(\phi) = \tilde\eta_0^n(\phi) + \frac{1}{2}\int_0^t \tilde\eta_s^n(\Delta\phi) ds + \sqrt{\lambda_n} \int_0^t \int_{R^d} \nabla\phi(x) \cdot W(dx, ds)$$

Define $G_t(\phi, y) \equiv \int (2\pi t)^{-\frac{d}{2}} e^{-\frac{|y-x|^2}{2t}} \phi(x)dx$. Then by Theorem 5.1[5], the solution of (2.6) is

(2.7.)

$$\tilde{\eta}_t^n(\phi) = \int_{R^d} G_t(\phi, y)\tilde{\Pi}^{\lambda_n}(dy) + \sqrt{\lambda_n} \int_0^t \int_{R^d} G_{t-s}(\nabla\phi, y) W(dy, ds)$$

Since the two terms on the right hand side of (2.7) are orthogonal, we have

$$\begin{aligned} E[\tilde{\eta}_t^n(\phi)^2] &= E[(\int_{R^d} G_t(\phi, y)\tilde{\Pi}^{\lambda_n}(dy))^2] + \lambda_n \int_0^t \int_{R^d} |G_{t-s}(\nabla\phi, y)|^2 dyds \\ &= \lambda_n \int G_t^2(\phi, y)dy + \lambda_n \int_0^t \int_{R^d} |G_{t-s}(\nabla\phi, y)|^2 dyds \end{aligned} \tag{2.6}$$

Note that since $G_t(\phi, y) = f * \phi(y)$ and $\|f_t\|_1 = 1$

$$\begin{aligned} &|G_t(\phi, y)\|_2 \leq \|\phi\|_2 \qquad \text{and} \\ &|\nabla G_{t-s}(\phi, y)\|_2 = \|G_{t-s}(\nabla\phi, y)\|_2 \leq \|\nabla\phi\|_2, \end{aligned}$$

by Schwartz's inequality. Hence

$$E[|\tilde{\eta}_t^n(\phi)|^2] \leq \lambda_n \|\phi\|_2^2 + \lambda_n t \|\nabla\phi\|_2^2.$$

□

Let $h(x) = (1 + x^2)^{-1}, x \in R^d$, and define an increasing process k_n by

$$k_n(t) = \int_0^t \int_{R^d} (\int_{R^d} h(y)\tilde{\eta}_s^n(dy))^2 h^2(x)dxds.$$

Recall that if $\psi \in \mathcal{S}(R^{2d})$ then $G_t(\psi, x, y) \in \mathcal{S}(R^{2d})$.

LEMMA 2.3. *For each $T > 0$, $\psi(x, y) \in \mathcal{S}(R^{2d})$,*

$$\begin{aligned} &E[\sup_{0 \leq t \leq T} (V_t^n(\psi))^2] \\ &\qquad \leq \{8\|\nabla_1\psi(x, y)(h^2(y)h(x))^{-1}\|_\infty \\ &\qquad + 2T^2 \cdot \sup_{0 \leq t \leq T} \|\nabla_1 G_t(\Delta\psi)(h^2(y)h(x))^{-1}\|_\infty\} \cdot \frac{E[k_n(T)]}{\lambda_n}. \end{aligned}$$

Proof. In (2.5)

$$\sup_t V_t^n(\psi)^2 \le \frac{1}{\lambda_n} 2 \sup_t (M_{2,t}^n(\nabla_1 \psi))^2$$
$$+ 2 \sup_t T(\int_0^t (\int_0^u \int_{R^{2d}} \nabla_1 G_{u-s}(\Delta\phi) M_2^n(dx, dy, ds))^2 du)$$

by the Schwartz inequality. By Doob's inequality

$$E[\sup_{t\le T} V_t^n(\phi)^2]$$

(2.8)

$$\le \frac{1}{\lambda_n} 8E[M_{2,T}^n(\nabla_1(\psi))^2]$$

(2.9)

$$+ 2T \int_0^T E[\int_0^u \int_{R^{2d}} \nabla_1 G_{u-s}(\Delta\psi) M^n(dx, dy, ds)^2 du]$$

Walsh (p410[5]) shows that the covariance measure for M_2^n is

$$\tilde{\eta}_s^n(dy)\tilde{\eta}_s^n(dy')\delta_x(x')dxdx'dsI,$$

where I is the identity matrix, hence its dominating measure (defined in Definition1.4) is

$$K(dxdydx'dy'ds) = \tilde{\eta}_s(dy)\tilde{\eta}_s(dy')\delta_x(x')dxdx'ds$$

Then by theorem 2.5[5]
(2.8)

$$\le \frac{8}{\lambda_n} E[\int (\int \nabla_1 \psi(x,y)\tilde{\eta}_s^n(dy))^2 dxds]$$
$$= \frac{8}{\lambda_n} E[\|\nabla_1 \psi(x,y)(h^2(y)h^2(x))^{-1}\|_\infty \int_{R^d\times[0,T]} (\int h(y)\tilde{\eta}_s^n(dy))^2 h^2(x) dxds]$$
$$\le \frac{8}{\lambda_n} \|\nabla_1 \psi(x,y)(h^2(y)h^2(x))^{-1}\|_\infty E[k_n(T)]$$

$$
\begin{aligned}
&= \frac{2T}{\lambda_n}\int_0^T E[\int_0^u (\int_{R^{2d}} \nabla_1 G_{u-s}(\Delta\psi) M^n(dx,dy,ds))^2 du \\
&= \frac{2T}{\lambda_n}\int_0^T E[(\int_0^u \int_{R^{2d}} (\tilde{\eta}^n(\nabla_1 G_{u-s}(\Delta\psi))\tilde{W}^\lambda(dx,ds))^2]du \\
(2.9) \quad &= \frac{2T}{\lambda_n}\int_0^T E[\int_0^u \int_{R^{2d}} (\int \nabla_1 G_{u-s}(\Delta\psi,x,y)\frac{h(y)h(x)}{h(y)h(x)}\tilde{\eta}_s^n(dy))^2 dxds]du \\
&= \frac{2T}{\lambda_n}\sup_{0\le t\le T}\|\nabla_1 G_t(\Delta\psi)(h^2(y)h^2(x))^{-1}\|_\infty \cdot \int_0^T E[k_n(u)]du \\
&\le 2T^2 \cdot \sup_{0\le t\le T}\|\nabla_1 G_t(\Delta\psi)(h^2(y)h^2(x))^{-1}\|_\infty \cdot \frac{E[k_n(T)]}{\lambda_n}
\end{aligned}
$$

□

LEMMA 2.4. *For each $T>0$, $\sup_n \frac{1}{\lambda_n}E[k_n(T)] \le (\|h\|^2 T + \frac{1}{2}T^2 \|\nabla h\|_2^2)\cdot\|h\|_2^2$*

Proof.

$$E[k_n(T)] = \int_0^T E[(\int h(y)\tilde{\eta}_s^n(dy))^2]\int_{R^d} h(x)dxds$$

$$
\begin{aligned}
\int_0^T E[(\int h(y)\tilde{\eta}_s^n(dy))^2]ds &\le \int_0^T (\lambda_n\|h\|_2^2 + \lambda_n s\|\nabla h\|_2^2)ds \\
&= \lambda_n(\|h\|_2^2\cdot T + \frac{1}{2}T^2\|\nabla h\|_2^2),
\end{aligned}
$$

by Lemma 2.2. Therefore,

$$\sup_n \frac{1}{\lambda_n}E[k_n(T)] \le (\|h\|^2 T + \frac{1}{2}T^2\|\nabla h\|_2^2)\cdot\|h\|_1.$$

□

LEMMA 2.5. *For $t\le T$, and $\psi\in\mathcal{S}(R^{2d})$, $\{\tilde{Q}_t^{\lambda_n}(\psi)\}$ is relatively compact.*

Proof. In (2.5), let

$$U_t^n \equiv \frac{1}{\sqrt{\lambda_n}}\int_0^t [\int_0^v \int_{R^{2d}} \nabla_1 G_{u-s}(\Delta\psi)(x,y,s)M_2^n(dx,dy,ds)]dv$$

and

$$S_n \equiv \frac{1}{\sqrt{\lambda_n}}\sup_{v\le T}|\int_0^v \int_{R^{2d}} \nabla_1 G_{u-s}(\Delta\psi)(x,y,s)M_2^n(dx,dy,ds)|$$

Then for $t \le T$, for any $\delta > 0$, and $0 \le u \le \delta$,

$$|U^n_{t+u} - U^n_t| \le \delta \cdot S_n$$

By an argument similar to the proof of Lemma 2.3

$$E[S_n^2] \le C \cdot \frac{1}{\lambda_n} E[k_n(T)]. \qquad \text{for some constant C}$$

Hence if we let $\gamma_n(\delta) = (\delta \cdot S_n)^2$

$$E[|U^n_{t+u} - U^n_t|^2 \,|\mathcal{F}^n_t] \le E[\gamma_n(\delta)\,|\mathcal{F}^n_t]$$

and by Lemma 2.4

$$\lim_{\delta \to 0} \sup_n E[\gamma_n(\delta)] = \lim_{\delta \to 0} \sup_n \delta \cdot C(\frac{1}{\lambda_n} E[k_n(T)]) = 0$$

It is obvious that U^n_t satisfies the condition (a) of Th.3.7.2 in [3], so by Th.3.8.6.in the book U^n_t, the last row in (2.5), is relatively compact. By the same way, we can show the relative compactness of the third term, $V^n_{1,t}(\psi)$ in (2.4). Since $V^n_{1,t}(\psi)$, $V^n_{2,t}(\psi)$ are continuous and $\{\tilde{Q}^{\lambda_n}_0(\psi)\}$ is relative compact, $\{\tilde{Q}^{\lambda_n}_t(\psi)\}$ is relative compact. □

Proof of Theorem 2.1, continued. It is known that(Prop. 8.16[5])

$$(\tilde{\Pi}^n, \tilde{\Pi}^n \times \tilde{\Pi}^n, \tilde{W}^n, \frac{1}{\sqrt{\lambda_n}}\tilde{\eta}^n, R^{\lambda_n}) \Rightarrow (V^0, V^0 \times V^0, \tilde{W}, \tilde{\eta}, 0)$$

Since

$$\frac{1}{\sqrt{\lambda_n}} \int_{R^d} \psi(x,y)\tilde{\eta}^n_s(dy) \Longrightarrow \int_{R^d} \psi(x,y)\tilde{\eta}_s(dy) \qquad \text{on } D_{C_0(R^d)}[0,T]$$

for any test function $\psi(x,y) \in \mathcal{S}(R^{2d})$, by Proposition 1.1,

$$\begin{aligned} \frac{1}{\sqrt{\lambda_n}} M^n_{2,t}(\psi) &= \frac{1}{\sqrt{\lambda_n}} \int_{R^d \times [0,t]} (\int_{R^d} \psi(x,y)\tilde{\eta}^n_s(dy))\tilde{W}^n(ds,dx) \\ &\Longrightarrow \int_{R^d \times [0,t]} \int_{R^d} \psi(x,y)\tilde{\eta}_s(dy)\tilde{W}(ds,dx) \\ &= M_{2,t}(\psi) \end{aligned}$$

Thus the two terms of $V^n_{2,t}(\psi)$ in (2.5) converge.

Furthermore, $Q^{\lambda_n}_0$ is known to be $\tilde{\Pi}^n \times \tilde{\Pi}^n$, and hence $Q^{\lambda_n}_0(\psi) \Rightarrow Q_0(\psi)$, where $Q_0 = V^0 \times V^0$. Since $R^{\lambda_n} \Rightarrow 0$, $\tilde{Q}^{\lambda_n}_t(\psi)$ is relatively compact, and

$$\frac{1}{\sqrt{\lambda_n}} M^n_{1,t}(\nabla_2\psi) \Longrightarrow M_{1,t}(\nabla_2\psi) \quad \frac{1}{\sqrt{\lambda_n}} M^n_{2,t}(\nabla_1\psi) \Longrightarrow M_{2,t}(\nabla_1\psi),$$

in (2.3), $\tilde{Q}^{\lambda_n}_t(\psi)$ converges to $Q_t(\psi)$ satisfying

$$Q_t(\psi) = Q_0(\psi) + \frac{1}{2}\int_0^t Q_s(\Delta\psi)ds + M_1(\nabla_2(\psi)) + M_2(\nabla_1(\psi)).$$

Since $R^{\lambda_n} \Longrightarrow 0$, $Q^{\lambda_n}_t(\psi) \Longrightarrow Q_t(\psi)$, where Q_t is a possible limit of $\tilde{Q}^{\lambda_n}_t$ and in fact, the unique solution of (2.0). □

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Department of general science
Hansung University
Seoul 136-792, Korea