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REPRESENTATIONS OF THE BRAID GROUP B_4

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ABSTRACT. In this work, the irreducible complex representations of degree 4 of B_4, the braid group on 4 strings, are classified. There are 4 families of representations: A two-parameter family of representations for which the image of P_4, the pure braid group on 4 strings, is abelian; two families of representations which are the composition of an irreducible representation of B_3, the braid group on 3 strings, with a certain special homomorphism $\pi : B_4 \longrightarrow B_3$; a family of representations which are the tensor product of 2 irreducible two-dimensional representations of B_4.

1. Introduction

The braid groups were first studied systematically by E. Artin in 1925 [1], and he continued his work in 1947 [2]. Among other results, he gave generators $\sigma_1, \ldots, \sigma_{n-1}$ and defining relations $\sigma_i\sigma_j = \sigma_j\sigma_i$, $|i-j| \geq 2$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $1 \leq i \leq n-2$ for B_n, the braid group on n strings, and showed that it has a faithful representation as a group of automorphisms of a free group of rank n.

As for matrix representations of the braid groups, the first ones were given by W. Burau in 1936 [3]. For each n, the Burau representation is a homomorphism from B_n to $\mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$, the group of invertible $n \times n$ matrices over a Laurent polynomial ring. The Burau representation is not irreducible, but it has a composition factor of degree $n-1$, which is called the reduced Burau representation.

The problem of completely classifying the matrix representations of B_n seems out of reach at the present time. A first step toward classifying the irreducible complex representations of B_n was taken by E. Formanek [7]. Note that if the variable t in the reduced Burau representation is

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specialized to a nonzero complex number, then a representation of B_n in $\mathrm{GL}_{n-1}(\mathbb{C})$ is obtained. Moreover, except for a finite set of roots of unity, the representation so obtained is irreducible. The main result of Formanek [7] is that, with a few exceptions, any irreducible complex representation of B_n of degree $n-1$ or less is equivalent to the tensor product of a one-dimensional representation with a composition factor of a specialization of the reduced Burau representation.

Westbury [11] identified the set of equivalence classes of irreducible representations of $\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}$, which is the quotient of B_3 by the center of B_3, as a variety and classified its components by a one-to-one correspondence between them and ordered lists of 5 non-negative integers which satisfy certain conditions. Thus B_4 is the first of the braid groups whose representation theory is an open problem. It is not known if the Burau representation of B_4 is faithful, or even if B_4 has a faithful linear representation of any degree. It is known that the Burau representation of B_n is faithful for $n \leq 3$ [9], and is not faithful for $n \geq 6$ [8], [10]. The group B_4 has particular interest because, letting Z_4 be its center, B_4/Z_4 is isomorphic to a subgroup of index 2 in $\mathrm{Aut}(F_2)$, the automorphism group of a free group of rank 2. It was shown in [6] that $\mathrm{Aut}(F_2)$ has a faithful linear representation if and only if B_4 has one.

The main result of this work is a classification, up to equivalence, of the irreducible complex representations of B_4 of degree 4. One consequence is that no irreducible four-dimensional complex representation of B_4 is faithful. Throughout this article, let η denote a representation of B_4 of degree 4, and let μ denote a root of $t^2 - t + 1 = 0$ and $\mathcal{V} = (\eta(\sigma_1)\eta(\sigma_2))^3$.

2. Preminaries

In this section, the fundamental results are listed.

THEOREM 2.1. [1, p. 51]. *The braid group B_n is generated by σ_1, ..., σ_{n-1} with the relations:*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{for } |i-j| \geq 2, \quad 1 \leq i, j \leq n-1,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \text{for } 1 \leq i \leq n-2.$$

EXAMPLE 2.2. (1) $B_1 = \{1\}$ is the trivial group.
(2) $B_2 = <\sigma_1>$ is infinite cyclic.

(3) $B_3 = < \sigma_1, \ \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 >$
$= < \alpha, \ \beta \mid \alpha^3 = \beta^2 >$ where $\alpha = \sigma_1\sigma_2$, $\beta = \sigma_1\sigma_2\sigma_1$.

(4) $B_4 = < \sigma_1, \ \sigma_2, \ \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1, \ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 >$.

Among the many important results, the following are essential to our analysis.

LEMMA 2.3. *Let B_n be the braid group on n strings.*

(1) [4, p. 655]. *Let $\theta_n = \sigma_1 \cdots \sigma_{n-1}$. Then $\theta_n\sigma_i\theta_n^{-1} = \sigma_{i+1}$ for $i = 1, \ldots, n-2$. Hence θ_n and any σ_i generate B_n.*

(2) [4, p. 656]. *For $n \geq 3$, the center of B_n is infinite cyclic, with generator θ_n^n.*

There is an exceptional homomorphism of B_4 onto B_3 denoted by π that sends both σ_1 and σ_3 to σ_1. The next lemma collects the facts about this homomorphism.

LEMMA 2.4. [6, p. 406]. *Let $\pi : B_4 \to B_3$ be the homomorphism defined by $\pi(\sigma_1) = \sigma_1$, $\pi(\sigma_2) = \sigma_2$, $\pi(\sigma_3) = \sigma_1$.*

(1) *The kernel of π is $F_2 = < \sigma_1\sigma_3^{-1}, \ \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1} >$, a free group of rank 2.*

(2) *Let $p = \sigma_1\sigma_3^{-1}$, $q = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$. Then the action of B_4 on $< p, \ q >$ by conjugation is given by*

$$\sigma_1 p \sigma_1^{-1} = p, \qquad \sigma_2 p \sigma_2^{-1} = q, \qquad \sigma_3 p \sigma_3^{-1} = p,$$
$$\sigma_1 q \sigma_1^{-1} = qp^{-1}, \qquad \sigma_2 q \sigma_2^{-1} = qp^{-1}q, \qquad \sigma_3 q \sigma_3^{-1} = p^{-1}q.$$

All the irreducible representations of B_3 and B_4 of degree 1,2,3 are listed now (cf. [7]). For $y \in \mathbb{C}^*$, we define a one-dimensional representation,

$$\chi(y) : B_n \to \mathbb{C}^*,$$

where $\chi(y)(\sigma_i) = y$ for $1 \leq i \leq n-1$.

THEOREM 2.5. [7, Theorem 3]. *For $n \geq 2$, the representations $\chi(y) : B_n \to \mathbb{C}^*$ ($y \in \mathbb{C}^*$) are a complete set of one-dimensional representations of B_n.*

For the irreducible representations of B_3 of degree 2, the following theorem is in order.

THEOREM 2.6. [7, Theorem 11]. *Let $\rho : B_3 \to \mathrm{GL}_2(\mathbb{C})$ be an irreducible representation. Then ρ is equivalent to $\chi(y) \otimes \beta_3(z)$ for some y, $z \in \mathbb{C}^*$ where z is not a root of $f_3(t) = t^2 + t + 1$,*

$$\beta_3(z)(\sigma_1) = \begin{pmatrix} -z & 0 \\ -1 & 1 \end{pmatrix}, \qquad \beta_3(z)(\sigma_2) = \begin{pmatrix} 1 & -z \\ 0 & -z \end{pmatrix}.$$

Moreover, $\chi(y) \otimes \beta_3(z)$ is equivalent to the following representation $\rho' = \rho'(a, b)$,

$$\rho'(\sigma_1) = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}, \qquad \rho'(\sigma_2) = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix}$$

for $b \neq a\mu$, where μ is a root of $t^2 - t + 1 = 0$, if we conjugate with $\begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix}$, where $-yg^2 = 1$ and $a = y$, $b = -yz$.

The irreducible representations of B_4 of degree 2 have been classified in the following theorem.

THEOREM 2.7. [7, Theorem 12]. *Let $\rho : B_4 \to \mathrm{GL}_2(\mathbb{C})$ be an irreducible representation. Then ρ is equivalent to one of the following, for some y, $z \in \mathbb{C}^*$.*

(1) *$\chi(y) \otimes \widehat{\beta}_4(z)$ where z is either $\pm i$ or -1,*

$$\widehat{\beta}_4(z)(\sigma_1) = \begin{pmatrix} -z & 0 \\ -1 & 1 \end{pmatrix}, \quad \widehat{\beta}_4(z)(\sigma_2) = \begin{pmatrix} 1 & -z \\ 0 & -z \end{pmatrix},$$

$$\widehat{\beta}_4(z)(\sigma_3) = \begin{pmatrix} 1 & 0 \\ -z^2 & -z \end{pmatrix}.$$

(2) *$(\chi(y) \otimes \beta_3(z))\pi$ where z is not a root of $f_3(t) = t^2 + t + 1$ and $\pi : B_4 \to B_3$ is the special homomorphism, i.e.*

$$\beta_3(z)\pi(\sigma_1) = \begin{pmatrix} -z & 0 \\ -1 & 1 \end{pmatrix} = \beta_3(z)\pi(\sigma_3), \qquad \beta_3(z)(\sigma_2) = \begin{pmatrix} 1 & -z \\ 0 & -z \end{pmatrix}.$$

Representations in (1) are not equivalent to representations in (2) except $\chi(y) \otimes \widehat{\beta}_4(-1) = (\chi(y) \otimes \beta_3(-1))\pi$.

For the representations of B_3, B_4 of degree 3, we have the following theorems. Let $\mathbb{C}^3$, $\bar{\mathbb{C}}^3$ denote the sets of three-dimensional column and row vectors respectively.

THEOREM 2.8. [7, Theorems 24, 25]. *Let $B_3 = < \alpha, \beta >$ where $\alpha = \sigma_1\sigma_2$, $\beta = \sigma_1\sigma_2\sigma_1$. Let $y \in \mathbb{C}^*$, let $A = (a_1, a_2, a_3)^t \in \mathbb{C}^3$, and let $B = (1, 1, 1) \in \bar{\mathbb{C}}^3$ where $a_1 + a_2 + a_3 = 2$.*

(1) *$\tau(\alpha) = y^2\mathrm{diag}(1, \omega, \omega^2)$, $\tau(\beta) = y^3(I - AB)$ defines a representation $\tau(y,\ a_1,\ a_2,\ a_3)$ of B_3.*
(2) *The representation $\tau(y, a_1, a_2, a_3)$ is irreducible if and only if $a_1a_2a_3 \neq 0$.*
(3) *Every irreducible representation $\tau : B_3 \longrightarrow \mathrm{GL}_3(\mathbb{C})$ is equivalent to some $\tau(y, a_1, a_2, a_3)$.*

Then $\tau(y, a_1, a_2, a_3)$, $\tau(y\omega, a_2, a_3, a_1)$ and $\tau(y\omega^2, a_3, a_1, a_2)$ are equivalent and these are the only equivalences among $\tau(y, a_1, a_2, a_3)$.

THEOREM 2.9. [7, Theorem 13]. *Let $\rho : B_4 \longrightarrow \mathrm{GL}_3(\mathbb{C})$ be an irreducible representation. Then ρ is equivalent to one of the following, for some $y,\ z \in \mathbb{C}^*$.*

(1) *$\chi(y) \otimes \beta_4(z)$ where z is not a root of $f_4(t) := t^3 + t^2 + t + 1$,*

$$\beta_4(z)(\sigma_1) = \begin{pmatrix} -z & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta_4(z)(\sigma_2) = \begin{pmatrix} 1 & -z & 0 \\ 0 & -z & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\beta_4(z)(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -z \\ 0 & 0 & -z \end{pmatrix}.$$

(2) *$\tau\pi$ where $\tau : B_3 \longrightarrow \mathrm{GL}_3(\mathbb{C})$ is an irreducible representation and $\pi : B_4 \longrightarrow B_3$ is the special homomorphism.*
(3) *$\chi(y) \otimes \mathcal{E}(z)$ where $\mathcal{E}(z) : B_4 \longrightarrow \mathrm{GL}_3(\mathbb{C})$ is defined by*

$$\mathcal{E}(z)(\sigma_1) = \begin{pmatrix} 0 & 1 & 0 \\ z & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathcal{E}(z)(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & z & 0 \end{pmatrix},$$

$$\mathcal{E}(z)(\sigma_3) = \begin{pmatrix} 0 & -1 & 0 \\ -z & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $\chi(y) \otimes \mathcal{E}(z)$ are y, $\pm y\sqrt{z}$, so distinct parameters y, z give inequivalent representations. Representations in (1), (2) and

(3) *are mutually inequivalent, except that* $\chi(y) \otimes \beta_1(1)$ *is equivalent to* $\chi(y) \otimes \mathcal{E}(1)$.

3. Reduction of the classification of representations.

To classify all the irreducible representation η of B_4 in $\mathrm{GL}_4(\mathbb{C})$, we consider the restriction of η to B_3 since it must be a representation of B_3 also. Since $(\sigma_1\sigma_2)^3$ is in the center of B_3, the image $\mathcal{V}$ of $(\sigma_1\sigma_2)^3$ must centralize the image of B_3. Hence we consider the various possible Jordan canonical forms for $\mathcal{V}$.

Let λ be a partition of a natural number, denoted $\{1^{\lambda_1}2^{\lambda_2}\cdots s^{\lambda_s}\}$. Corresponding to λ is the nilpotent Jordan matrix $J(\lambda)$ which has λ_i elementary $i \times i$ nilpotent Jordan blocks for $i = 1, \ldots, s$. For $a \in \mathbb{C}$, set $J(a, \lambda) = aI + J(\lambda)$. Now assume the Jordan canonical form of $\mathcal{V}$ is the direct sum of Jordan blocks $J(a_1, \lambda(a_1)), \ldots, J(a_k, \lambda(a_k))$, where $a_1, \ldots, a_k$ are the distinct eigenvalues. The centralizer of $\mathcal{V}$ is the direct sum of the centralizers of the distinct $J(a_i, \lambda(a_i))$. Thus one needs to find the centralizer of a block $J(a, \lambda)$, and we notice that this is the same as the centralizer of the nilpotent Jordan block $J(0, \lambda)$. In other words, the structure of the centralizer does not depend on a.

Lemma 3.1. *Let* $\lambda = \{1^{\lambda_1}2^{\lambda_2}\cdots s^{\lambda_s}\}$. *The centralizer of* $J(0, \lambda)$ *has an invariant subspace of dimension* λ_s *where* s *is the largest part of* λ. *In particular, if* $\lambda_s = 1$, *then the centralizer of* $J(0, \lambda)$ *has a one-dimensional invariant subspace. Thus the centralizer of a matrix* $\mathcal{V}$ *has a one-dimensional invariant subspace unless, for every Jordan block* $J(a, \lambda)$ *occurring in the Jordan canonical form of* $\mathcal{V}$, *the largest part of the partition* λ *has multiplicity two or more.*

We have 14 Jordan canonical forms of the 4×4 matrix $\mathcal{V}$ according to its partition type. Among these, there are only three for which the centralizer of $\mathcal{V}$ does not have a one-dimensional invariant subspace, namely (A) $\mathcal{V}$ has two distinct eigenvalues u, w, each of multiplicity two with corresponding partitions (1^2), (1^2). The Jordan canonical form of $\mathcal{V}$ is the diagonal matrix $\mathrm{diag}(u, u, w, w)$, $u \neq w$. (B) $\mathcal{V}$ has one eigenvalue w with corresponding partition (2^2). The Jordan canonical form of $\mathcal{V}$ has two identical 2×2 blocks $(w, 1, 0, w)$. (C) $\mathcal{V}$ has one eigenvalue w with corresponding partition (1^4), i.e. $\mathcal{V} = wI$.

THEOREM 3.2. *Let $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$ be a representation, not necessarily irreducible. Then either $\eta|_{B_3}$, the restriction of η to B_3, has a one-dimensional invariant subspace or $\mathcal{V}$ has one of the following Jordan canonical forms*

$$\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix}, \quad \begin{pmatrix} w & 1 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 1 \\ 0 & 0 & 0 & w \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix},$$

where u, w are distinct.

4. The restriction of η to B_3 has an invariant one-dimensional subspace.

Let $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation and assume $\eta|_{B_3}$ has an invariant one-dimensional subspace, span $\{v\}$, i.e., $\eta(\sigma_1)v = xv = \eta(\sigma_2)v$ for some $x \in \mathbb{C}^*$. Define $\theta = \sigma_1\sigma_2\sigma_3 \quad \sigma_0 = \theta\sigma_3\theta^{-1}$, $V_i =$ Ker $(\eta(\sigma_i) - x \cdot I)$ for $0 \leq i \leq 3$.

Conjugation by θ permutes σ_0, σ_1, σ_2, σ_3 cyclically. Thus left multiplication by $\eta(\theta)$ permutes V_0, V_1, V_2, V_3 cyclically. In particular, all the V_i, $0 \leq i \leq 3$, have the same dimension. Moreover, if v is an x-eigenvector for $\eta(\sigma_1)$ and $\eta(\sigma_2)$, then $\eta(\theta)v$ is an x-eigenvector for $\eta(\sigma_2)$ and $\eta(\sigma_3)$.

PROPOSITION 4.1. dim $V_1 = 2$.

Proof. By one of the hypotheses, dim $V_1 \geq 1$, since $v \in V_1$.

If dim $V_1 = 1$, then dim $V_2 = 1$ since as noted above, $V_2 = \eta(\theta)(V_1)$. Since both v and $\eta(\theta)v$ are in V_2, and dim $V_2 = 1$, one must be a nonzero scalar multiple of the other, i.e. $\eta(\theta)v = yv$ for some $y \in \mathbb{C}^*$. But then span $\{v\}$ is invariant under $\eta(\sigma_1)$ and $\eta(\theta)$, and hence invariant under B_4, since σ_1 and θ generate B_4. This contradicts the irreducibility of $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$.

If dim $V_1 \geq 3$, then dim $(V_1 \cap V_2 \cap V_3) \geq 1$. This follows from two applications of the formula, dim P + dim Q = dim $(P \cap Q)$ + dim $(P + Q)$ for subspaces of a vector space, since V_1, V_2, V_3 are subspaces of $\mathbb{C}^4$. But any element of $V_1 \cap V_2 \cap V_3$ is a common x-eigenvector for σ_1, σ_2, σ_3 and again the irreducibility of $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$ is contradicted.

Finally, dim $V_1 = 2$ since this is the only remaining possibility. □

Now let $W = \text{span}\,\{\eta(\theta^3)v,\ v,\ \eta(\theta)v,\ \eta(\theta^2)v\}$. If dim W were less than or equal to 3, then W would be a proper invariant subspace since W is invariant under $\eta(\sigma_1)$ and $\eta(\theta)$. Hence dim $W = 4$ since η is irreducible, in other words, $\{\eta(\theta^3)v,\ v,\ \eta(\theta)v,\ \eta(\theta^2)v\}$ forms a basis for W. Since θ^4 is in the center of B_4 and η is irreducible, $\eta(\theta^4) = z \cdot I$ for some $z \in \mathbb{C}^*$. Since the action of $\eta(\theta)$ on W sends v to $\eta(\theta)v$, $\eta(\theta)v$ to $\eta(\theta^2)v$, $\eta(\theta^2)v$ to $\eta(\theta^3)v$, and $\eta(\theta^3)v$ to $z \cdot v$, we have

$$\eta(\theta) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ z & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with respect to the ordered basis $\{\eta(\theta^3)v,\ v,\ \eta(\theta)v,\ \eta(\theta^2)v\}$. The action of $\eta(\sigma_1)$ on W sends $\eta(\theta^3)v$ to $x \cdot \eta(\theta^3)v$, v to $x \cdot v$ and $\eta(\sigma_1)(V_3) \subseteq V_3$ since σ_1, σ_3 commute. Thus there are a_1, a_2, a_3, $a_4 \in \mathbb{C}$ such that the matrices of $\eta(\sigma_1)$, $\eta(\sigma_2)$, $\eta(\sigma_3)$ with respect to the ordered basis $\{\eta(\theta^3)v,$ $v,\ \eta(\theta)v,\ \eta(\theta^2)v\}$, are

$$\eta(\sigma_1) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_3 & a_4 \end{pmatrix}, \quad \eta(\sigma_2) = \eta(\theta\sigma_1\theta^{-1}) = \begin{pmatrix} a_4 & 0 & 0 & a_3 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ a_2 & 0 & 0 & a_1 \end{pmatrix},$$

and

$$\eta(\sigma_3) = \eta(\theta\sigma_2\theta^{-1}) = \begin{pmatrix} a_1 & a_2 z^{-1} & 0 & 0 \\ a_3 z & a_4 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}.$$

Now comparing $\eta(\theta)$ and $\eta(\sigma_1\sigma_2\sigma_3)$, we have $a_1 = 0$, $a_4 = 0$, and $a_3 = x^{-2}$, $z = a_2^3$. By conjugating with the diagonal matrix diag $(1,\ a_2^{-2},\ a_2^{-2},$ $a_2^{-1})$, and setting $u = a_2 x^{-2}$, we get the following theorem.

THEOREM 4.2. *Let $\xi : B_4 \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation. If the restriction of ξ to B_3 has an invariant one-dimensional subspace, then ξ is equivalent to the following representation $\eta = \eta(x, u)$, for some x, $u \in \mathbb{C}^*$, $u \neq x^2$, defined by*

$$\eta(\sigma_1) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & u & 0 \end{pmatrix}, \quad \eta(\sigma_2) = \begin{pmatrix} 0 & 0 & 0 & u \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \eta(\sigma_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ u & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}.$$

Distinct choices of x, u give inequivalent representations.

All nonzero x, u define representations, and η is irreducible unless $u = x^2$. If $u = x^2$, then $(x^2, x^3, 1, x)^t$ is a common eigenvector for η.

THEOREM 4.3. *The image of the pure braid group P_4 of the representation η in* Theorem 4.2 *consists of the diagonal matrices, and is abelian.*

Proof. The pure braid group P_4 is the normal subgroup of B_4 generated by σ_i^2 for $1 \leq i \leq 3$ [2, Theorem 17]. Since $\eta(\sigma_i^2)$ and $\eta(\sigma_j D \sigma_j^{-1})$ are diagonal matrices for any diagonal matrix D, $1 \leq i,\ j \leq 3$, the image of the pure braid group P_4 under the representation η consists of diagonal matrices, and hence is abelian. □

REMARK 4.1. For the representation η in Theorem 4.2, we have $\mathcal{V} = \text{diag}\ (x^2u^2,\ x^6,\ x^2u^2,\ x^2u^2)$.

(1) If $u \neq \pm x^2$, then $\mathcal{V}$ has eigenvalues, x^2u^2 of multiplicity 3, and x^6 of multiplicity 1.
(2) If $u = x^2$, then the representation is reducible.
(3) If $u = -x^2$, then $\mathcal{V}$ is a scalar matrix.

Thus any irreducible representation of B_4 of degree 4 whose restriction to B_3 has an invariant one-dimensional subspace, has $\mathcal{V}$ either a scalar matrix or a diagonal matrix with one eigenvalue of multiplicity 3 and another eigenvalue of multiplicity 1.

5. The Jordan canonical form of $\mathcal{V}$ is the diagonal matrix $\text{diag}(u, u, w, w)$, $u \neq w$.

Let's assume $\mathcal{V}$ is the diagonal matrix diag $(u,\ u,\ w,\ w)$, $u \neq w$. Since $\eta(\sigma_1)$ and $\eta(\sigma_2)$ commute with $\mathcal{V}$, they have the form, $\eta(\sigma_i) = \begin{pmatrix} \rho(\sigma_i) & \mathbf{0} \\ \mathbf{0} & \tau(\sigma_i) \end{pmatrix}$, $i = 1,\ 2$, where $\rho,\ \tau : B_3 \to \text{GL}_2(\mathbb{C})$ are representations. If either ρ or τ is reducible, then $\eta|_{B_3}$, the restriction of η to B_3, has an invariant one-dimensional subspace, but then $\mathcal{V}$ can not have Jordan canonical form $\text{diag}(u, u, w, w)$, $u \neq w$, by Remark 4.1. Therefore the representations ρ, τ are irreducible. Then, by Theorem 2.6, there exist a, b, c, d such that $b \neq a\mu$, $d \neq c\mu$ (where μ is a primitive cube root of

-1), and

$$\eta(\sigma_1) = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & d \end{pmatrix}, \qquad \eta(\sigma_2) = \begin{pmatrix} b & 0 & 0 & 0 \\ -ab & a & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & -cd & c \end{pmatrix}.$$

DEFINITION 5.1. A matrix is called *non-derogatory* if it generates its own centralizer over $\mathbb{C}$.

If X is non-derogatory, then any matrix which centralizes X is a polynomial in X over $\mathbb{C}$. In particular, if $\eta(\sigma_1)$ is non-derogatory, then $\eta(\sigma_3)$ has the form,

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix},$$

and thus η is reducible.

REMARK 5.1. An $n \times n$ matrix is non-derogatory if it has only 1 Jordan block for each eigenvalue.

For $n = 4$, the following are the "types" of non-derogatory matrices,

$$\begin{pmatrix} s & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & w \end{pmatrix}, \qquad \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & w & 1 \\ 0 & 0 & 0 & w \end{pmatrix}, \qquad \begin{pmatrix} u & 1 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & w & 1 \\ 0 & 0 & 0 & w \end{pmatrix},$$

$$\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & w & 1 & 0 \\ 0 & 0 & w & 1 \\ 0 & 0 & 0 & w \end{pmatrix}, \qquad \begin{pmatrix} w & 1 & 0 & 0 \\ 0 & w & 1 & 0 \\ 0 & 0 & w & 1 \\ 0 & 0 & 0 & w \end{pmatrix}.$$

where s, t, u, w are distinct complex numbers. Let

$$\eta(\sigma_3) = \begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ n_{41} & n_{42} & n_{43} & n_{44} \end{pmatrix}.$$

The commutativity of σ_1 and σ_3 implies

$$\eta(\sigma_1)\eta(\sigma_3) = \begin{pmatrix} an_{11}+n_{21} & an_{12}+n_{22} & an_{13}+n_{23} & an_{14}+n_{24} \\ bn_{21} & bn_{22} & bn_{23} & bn_{24} \\ cn_{31}+n_{41} & cn_{32}+n_{42} & cn_{33}+n_{43} & cn_{34}+n_{44} \\ dn_{41} & dn_{42} & dn_{43} & dn_{44} \end{pmatrix}$$

$$= \begin{pmatrix} an_{11} & n_{11}+bn_{12} & cn_{13} & n_{13}+dn_{14} \\ an_{21} & n_{21}+bn_{22} & cn_{23} & n_{23}+dn_{24} \\ an_{31} & n_{31}+bn_{32} & cn_{33} & n_{33}+dn_{34} \\ an_{41} & n_{41}+bn_{42} & cn_{43} & n_{43}+dn_{44} \end{pmatrix} = \eta(\sigma_3)\eta(\sigma_1).$$

Let's consider the following cases according to the number of distinct eigenvalues of $\eta(\sigma_1)$.

(1) If $\eta(\sigma_1)$ has a single eigenvalue i.e. $a = b = c = d$, then $\mathcal{V}$ is a scalar matrix. Contradiction to the hypotheses.

(2) $\eta(\sigma_1)$ has 2 eigenvalues.

(a) $a = b = c \neq d$.
The commutativity of σ_1 and σ_3 implies $n_{21} = n_{23} = n_{24} = n_{31} = n_{41} = n_{42} = n_{43} = 0$, $n_{11} = n_{22}$, $n_{13} = (a-d)n_{14}$, and $n_{34} = \frac{n_{33}-n_{44}}{a-d}$. The equality of the $(4,1)$-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$ forces $a^3dn_{32} = 0$. But if n_{32} vanishes, then η is reducible.

(b) The same reasoning as in (2a) applies when any three of a, b, c, d are equal, since the role of a, b, c, d are symmetric.

(c) $a = b \neq c = d$.
$\eta(\sigma_1)$ is non-derogatory.

(d) $a = c \neq b = d$.
Again we have a scalar matrix for $\mathcal{V}$. Contradiction to the hypotheses.

(3) $\eta(\sigma_1)$ has 3 eigenvalues.

(a) $a = b$ and b, c, d are distinct.
$\eta(\sigma_1)$ is non-derogatory.

(b) $a = c$ and b, c, d are distinct.
The commutativity of σ_1 and σ_3 implies $n_{21} = n_{23} = n_{24} = n_{41} = n_{42} = n_{43} = 0$, $n_{12} = \frac{n_{11}-n_{22}}{a-b}$, $n_{14} = \frac{n_{13}}{a-d}$, $n_{32} = \frac{n_{31}}{a-b}$, and $n_{34} = \frac{n_{33}-n_{44}}{a-d}$. Comparing $(2,4)$-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$, we have $abn_{13}(n_{22}-a) = 0$. If n_{13} is equal to 0, then the representation is reducible. Thus $n_{22} = a$. From the equality of the $(4,2)$-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$, we have

$abn_{31}(n_{44}-a)=0$. Thus $n_{44}=a$, for if n_{31} vanishes, then the corresponding representation becomes reducible. Then the equality of the $(2,3)$-entries of $\eta(\sigma_2\sigma_3\sigma_2)=\eta(\sigma_3\sigma_2\sigma_3)$ forces $abn_{13}(a^2-ad+d^2)=0$. So $a^2-ad+d^2=0$. Thus $d=a\mu=c\mu$, where μ is a root of $t^2-t+1=0$, and $\tau : B_3 \to \mathrm{GL}_2(\mathbb{C})$ is reducible (cf. Theorem 2.6). Contradiction to the hypotheses.

(4) All eigenvalues of $\eta(\sigma_1)$ are distinct to each other. Then $\eta(\sigma_1)$ is non-derogatory again.

Thus we have shown

THEOREM 5.2. *Let $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation. Then the Jordan canonical form of $\mathcal{V}=(\eta(\sigma_1)\eta(\sigma_2))^3$ is not a diagonal matrix* $\mathrm{diag}(u,u,w,w)$ *where $u \neq w$.*

6. The Jordan canonical form of $\mathcal{V}$ has two identical 2×2 blocks $(w,1,0,w)$.

When $\mathcal{V}$ has two identical 2×2 blocks $(w,1,0,w)$, we first conjugate the representations so that $\mathcal{V}$ is equal to $\begin{pmatrix} w\cdot I & w\cdot I \\ 0 & w\cdot I\end{pmatrix}$ where I is the 2×2 identity matrix. The centralizer C of $\mathcal{V}$ in $\mathrm{M}_4(\mathbb{C})$ is the set of block matrices $\begin{pmatrix} A & B \\ 0 & A\end{pmatrix}$ where $A, B \in \mathrm{M}_2(\mathbb{C})$. Since $\eta(\sigma_1)$ and $\eta(\sigma_2)$ commute with $\mathcal{V}$, the image $\eta(B_3) \subseteq C$. Moreover, the map $\gamma : C \to \mathrm{M}_2(\mathbb{C})$ that sends $\begin{pmatrix} A & B \\ 0 & A\end{pmatrix}$ to A is a $\mathbb{C}$-algebra homomorphism. For the representation $\rho = \eta|_{B_3}$ of B_3 into $C \cap \mathrm{GL}_4(\mathbb{C})$, we consider the composition $\delta = \gamma\rho : B_3 \to \mathrm{GL}_2(\mathbb{C})$. If δ is reducible, $\eta(\sigma_1)$ and $\eta(\sigma_2)$ have a one-dimensional invariant subspace, and then $\mathcal{V}$ can not have Jordan canonical form with two identical 2×2 blocks $(w,1,0,w)$ by the Remark 4.1. Thus δ is irreducible. Then by Theorem 2.6,

$$\eta(\sigma_1)=\begin{pmatrix} a & 1 & * & p \\ 0 & b & * & q \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & b\end{pmatrix}, \qquad \eta(\sigma_2)=\begin{pmatrix} b & 0 & * & * \\ -ab & a & * & * \\ 0 & 0 & b & 0 \\ 0 & 0 & -ab & a\end{pmatrix};$$

where $b \neq a\mu$ (where μ is a primitive cube root of -1), $a, b \in \mathbb{C}^*$, $p, q \in \mathbb{C}$, and each " $*$ " denotes an element of $\mathbb{C}$. By conjugating with

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -q & p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we may assume

$$\eta(\sigma_1) = \begin{pmatrix} a & 1 & c & 0 \\ 0 & b & d & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & b \end{pmatrix}, \qquad \eta(\sigma_2) = \begin{pmatrix} b & 0 & g & h \\ -ab & a & k & l \\ 0 & 0 & b & 0 \\ 0 & 0 & -ab & a \end{pmatrix},$$

where $b \neq a\mu$, $a, b \in \mathbb{C}^*$. The braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and $\mathcal{V}$ as in the hypotheses implies

$$\begin{aligned} d &= b(c - \frac{a}{3}), \\ h &= \frac{(b-a)g - d}{ab}, \\ k &= abh - bc = (b-a)g - d - bc, \\ l &= c - g. \end{aligned}$$

To determine $\eta(\sigma_3)$, let

$$\eta(\sigma_3) = \begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ n_{41} & n_{42} & n_{43} & n_{44} \end{pmatrix}.$$

Then we have

$$\eta(\sigma_1)\eta(\sigma_3) = \begin{pmatrix} D & E \\ F & G \end{pmatrix},$$

where

$$D = \begin{pmatrix} an_{11} + n_{21} + cn_{31} & an_{12} + n_{22} + cn_{32} \\ bn_{21} + bcn_{31} - \frac{ab}{3}n_{31} & bn_{22} + bcn_{32} - \frac{ab}{3}n_{32} \end{pmatrix},$$
$$E = \begin{pmatrix} an_{13} + n_{23} + cn_{33} & an_{14} + n_{24} + cn_{34} \\ bn_{23} + bcn_{33} - \frac{ab}{3}n_{33} & bn_{24} + bcn_{34} - \frac{ab}{3}n_{34} \end{pmatrix},$$
$$F = \begin{pmatrix} an_{31} + n_{41} & an_{32} + n_{42} \\ bn_{41} & bn_{42} \end{pmatrix},$$
$$G = \begin{pmatrix} an_{33} + n_{43} & an_{34} + n_{44} \\ bn_{43} & bn_{44} \end{pmatrix}.$$

And

$$\eta(\sigma_3)\eta(\sigma_1) = \begin{pmatrix} an_{11} & n_{11} + bn_{12} & cn_{11} + bcn_{12} - \frac{ab}{3}n_{12} + an_{13} & n_{13} + bn_{14} \\ an_{21} & n_{21} + bn_{22} & cn_{21} + bcn_{22} - \frac{ab}{3}n_{22} + an_{23} & n_{23} + bn_{24} \\ an_{31} & n_{31} + bn_{32} & cn_{31} + bcn_{32} - \frac{ab}{3}n_{32} + an_{33} & n_{33} + bn_{34} \\ an_{41} & n_{41} + bn_{42} & cn_{41} + bcn_{42} - \frac{ab}{3}n_{42} + an_{43} & n_{43} + bn_{44} \end{pmatrix}.$$

Comparing $(3,1)$- and $(4,4)$-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$, we have

$$(1) \qquad n_{41} = n_{43} = 0.$$

With the equality of $(1,1)$- and $(2,2)$-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$, we have

$$(2) \qquad n_{21} = -cn_{31},$$

and

$$(3) \qquad n_{21} = bn_{32}(c - \frac{a}{3}).$$

Then with the equality of $(2,1)$-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$, we have

$$\begin{aligned} 0 &= (b-a)n_{21} + bn_{31}(c - \frac{a}{3}) \\ &= (b-a)(-cn_{31}) + bn_{31}(c - \frac{a}{3}) \\ &= n_{31}(-bc + ac + bc - \frac{ab}{3}) \\ &= an_{31}(c - \frac{b}{3}). \end{aligned}$$

(1) $c - \frac{b}{3} \neq 0$.

Then $n_{31} = 0$ implies $n_{21} = 0$ and $bn_{32}(c - \frac{a}{3}) = 0$ by Eq (2) and Eq (3).

(a) $n_{32} = 0$.

If $n_{31} = n_{32} = 0$, then $(a-b)n_{32} - n_{31} + n_{42} = 0$ from the equality of $(3,2)$-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$ implies $n_{42} = 0$. Therefore $\eta(\sigma_3)$has the form,

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix},$$

and the corresponding representation is reducible.

(b) $c - \frac{a}{3} = 0$.

In the equality of $(3,2)$-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$, we have $n_{42} = (b-a)n_{32}$. Now the equality of $(4,1)$-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$ implies $abn_{32}(a^2 - (a-b)n_{11}) = 0$. If n_{32} vanishes, then the representation is reducible by the previous case. So assume $n_{32} \neq 0$, then $n_{11} = \frac{a^2}{a-b}$. In the equality of $(3,1)$-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$, we have $abn_{32}(-b+n_{11}) = 0$. Thus $b = n_{11} = \frac{a^2}{a-b}$. Hence $a^2 - ab + b^2 = 0$. Contradiction to our initial assumption $b \neq a\mu$.

(2) $c - \frac{b}{3} = 0$.

The equality of $(4,3)$-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$ forces $b(b-a)n_{42} = 0$.

(a) $n_{42} = 0$.

If we compare the $(3,2)$-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$, we get $n_{31} = (a-b)n_{32}$. In the equality of $(4,1)$-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$, we have $abn_{32}(b^2 - (b-a)n_{44}) = 0$. Since $n_{32} = 0$ implies reducibility, assume $n_{44} = \frac{b^2}{b-a}$. Then in the equality of $(4,2)$-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$, we have $abn_{32}(-a + n_{44}) = 0$. Therefore $a = n_{44} = \frac{b^2}{b-a}$ implies $b^2 - ab + a^2 = 0$. Contradiction to our initial assumption $b \neq a\mu$ again.

(b) $b = a$.

In Eq (3), we have $n_{21} = bn_{32}(c - \frac{a}{3}) = an_{32}(c - \frac{b}{3}) = 0$. From Eq (2), we have $n_{31} = 0$ since $c = \frac{b}{3} \neq 0$. In the equality of $(3,2)$-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$, $n_{32}(a-b) + n_{42} - n_{31} = 0$ forces $n_{42} = 0$ and the vanishing of n_{42} makes this case belong to the previous case.

Hence we have shown

THEOREM 6.1. *Let $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation. Then the Jordan canonical form of $\mathcal{V} = (\eta(\sigma_1)\eta(\sigma_2))^3$ is not equal to*

$$\begin{pmatrix} w & 1 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 1 \\ 0 & 0 & 0 & w \end{pmatrix}.$$

7. The Jordan canonical form of $\mathcal{V}$ is a scalar matrix.

Since scalar matrices are central in $\mathrm{GL}_4(\mathbb{C})$, all commutators $[(\sigma_1\sigma_2)^3, \mathbf{w}] = (\sigma_1\sigma_2)^3\ \mathbf{w}\ (\sigma_1\sigma_2)^{-3}\ \mathbf{w}^{-1}$ will lie in the kernel of the representation for any word $\mathbf{w}$ in B_4, so it will really be a representation η of B_4/N, where N is the normal subgroup of B_4 generated by all such commutators.

Now assume $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$ is an irreducible representation and $\mathcal{V}$ is a scalar matrix. There is a short exact sequence $1 \longrightarrow F_2 \longrightarrow B_4 \longrightarrow B_3 \longrightarrow 1$, where the map between B_4 and B_3 is the special homomorphism π which sends σ_1, σ_3 to σ_1 and fixes σ_2. and $F_2 = <\sigma_1\sigma_3^{-1}, \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}>$ is a free group of rank 2 [6, p. 406]. For any word $\mathbf{w} \in B_4$, $\pi(\mathbf{w}) \in B_3$ and $(\sigma_1\sigma_2)^3$ is in the center of B_3. Hence $\pi(N) = 1$ implies $N \subseteq F_2$. Factoring out N gives rise to an exact sequence $1 \longrightarrow K = F_2/N \longrightarrow B_4/N \longrightarrow B_3 \longrightarrow 1$. To figure out F_2/N, set $p = \sigma_1\sigma_3^{-1}$, $q = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$. Then $[(\sigma_1\sigma_2)^3, \sigma_3] = qpq^{-1}p \in N$ and $\sigma_2(qpq^{-1}p)\sigma_2^{-1} = qp^{-1}qp \in N$. Hence F_2/N is isomorphic to the quaternion group of order 8. There are 5 irreducible representations of the quaternion group, four of them, ϕ_i, $1 \le i \le 4$, are of degree 1 and one of them, ψ, is of degree 2. They are as follows, $\phi_1(p) = 1$, $\phi_1(q) = 1$, $\phi_2(p) = 1$, $\phi_2(q) = -1$, $\phi_3(p) = -1$, $\phi_3(q) = 1$, $\phi_4(p) = -1$, $\phi_4(q) = -1$, and $\psi(p) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\psi(q) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For any irreducible representation $\eta : B_4/N \to \mathrm{GL}_4(\mathbb{C})$, the restriction of η to K, $\eta|_K$ is the direct sum of irreducible representations of the same degree, which are conjugates, with the same multiplicity, by Clifford's theorem [5, (49.2) Theorem, (49.7) Theorem], since K is a normal subgroup of B_4/N. And the conjugacy classes of irreducible representations of K are $\{\phi_1\}$, $\{\phi_2, \phi_3, \phi_4\}$, $\{\psi\}$.

LEMMA 7.1. *Let $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation and $\eta|_K$ is the direct sum of irreducible representations. Then one of the following occurs.*

(A) $\eta|_K = \phi_1 \oplus \phi_1 \oplus \phi_1 \oplus \phi_1$.

(B) $\eta|_K = \psi \oplus \psi$.

First of all, we consider the case, in which $\eta|_K$ is the direct sum of 4 one-dimensional representations.

PROPOSITION 7.2. *Let $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation and $\eta|_K$ is the direct sum of 4 one-dimensional representations. Then $\eta = \tau\pi$ where $\pi : B_4 \to B_3$ is the special homomorphism and $\tau : B_3 \to \mathrm{GL}_4(\mathbb{C})$ is an irreducible representation of B_3 of degree 4.*

Proof. By Lemma 7.1, $\eta|_K = \phi_1 \oplus \phi_1 \oplus \phi_1 \oplus \phi_1$. Since $\eta(p) = \eta(\sigma_1\sigma_3^{-1}) = I$, we have $\eta(\sigma_1) = \eta(\sigma_3)$. Hence $\eta = \tau\pi$ where $\pi : B_4 \to B_3$ is the special homomorphism and $\tau : B_3 \to \mathrm{GL}_4(\mathbb{C})$ is an irreducible representation. Since B_3 has a presentation $< \alpha,\ \beta \mid \alpha^3 = \beta^2 >$, $\tau(\alpha^3) = \tau(\beta^2) = z \cdot I$ for some $z \in \mathbb{C}^*$. Thus $\tau(\alpha)$ and $\tau(\beta)$ are diagonalizable. For some $a,\ b \in \mathbb{C}^*$, the eigenvalues of $\tau(\alpha)$ belong to the set $\{a,\ a\omega,\ a\omega^2\}$ where $\omega = \frac{-1+\sqrt{-3}}{2}$, a cube root of unity and the eigenvalues of $\tau(\beta)$ belong to the set $\{b,\ -b\}$. But $\alpha^3 = \beta^2$ implies $a^3 = b^2$. Set $y = a^{-1}b$. If $\tau(\beta)$ has only one eigenvalue, then it is a scalar matrix and this would imply that τ is reducible. So we may suppose that $\tau(\beta)$ is conjugate to either diag $(b, b, b, -b)$ or diag $(b, b, -b, -b)$. But if $\tau(\beta)$ is conjugate to diag $(b, b, b, -b)$, then $\tau(\alpha)$ and $\tau(\beta)$ have a common eigenvector since $\tau(\alpha)$ has an eigenvalue of multiplicity 2 by comparing the dimensions of the null spaces of each eigenvalues. If $\tau(\alpha)$ has an eigenvalue of multiplicity 3, then $\tau(\alpha)$ and $\tau(\beta)$ have a common eigenvector again contradicting to the irreducibility of τ. Hence $\tau(\alpha)$ is conjugate to a diagonal matrix with either one eigenvalue of multiplicity 2 and two of multiplicity 1 or two eigenvalues of multiplicity 2, i.e. we may assume $\tau(\beta) = y^3$ diag $(1, 1, -1, -1)$ and $\tau(\alpha)$ is conjugate to either $y^2\omega^i$ diag $(1,\ 1,\ \omega,\ \omega^2)$ or $y^2\omega^i$ diag $(1,\ 1,\ \omega,\ \omega)$ where $1 \le i \le 3$. But since $y^2\omega^i = (y\omega^{2i})^2$, they correspond to a different choice of y.

According to [11], the set of equivalence classes of irreducible representations of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ of dimension n is a nonempty variety and the components of this variety are indexed by ordered lists of non-negative integers, $(n_1, n_2; m_1, m_2, m_3)$, which satisfy the following conditions, $n_1 + n_2 = n$, $m_1 + m_2 + m_3 = n$ and n_1, n_2 are greater than or equal to all of m_1, m_2, m_3. Furthermore, if an irreducible representation of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} = < s > * < t >$ is given by $s \mapsto S$ and $t \mapsto T$, then for each non-zero scalar c, there is an irreducible representation of B_3 given by $s \mapsto c^3 S$ and $t \mapsto c^2 T$. Thus we have two families of representations corresponding to indices $(2, 2; 2, 1, 1)$ and $(2, 2; 2, 2, 0)$, which are the composition of an irreducible representation of B_3 with the special homomorphism $\pi : B_4 \to B_3$. □

For the second case, we need to consider the restriction of η to K is the direct sum of 2 two-dimensional representations, i.e. $\eta|_K = \psi \oplus \psi$.

PROPOSITION 7.3. *Let $\eta : B_4 \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation and $\eta|_K$ is the direct sum of 2 two-dimensional representations. Then η is equivalent to a representation $\rho\pi \otimes \widehat{\beta}_4(i)$, where π is the special homomorphism, $\rho : B_3 \to \mathrm{GL}_2(\mathbb{C})$ is the irreducible representation defined by $\rho(\sigma_1) = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$, $\rho(\sigma_2) = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix}$ for $a, b \in \mathbb{C}^*$, $b \neq a\mu$, and $\widehat{\beta}_4(i) : B_4 \to \mathrm{GL}_2(\mathbb{C})$ is the irreducible representation defined by*

$$\widehat{\beta}_4(i)(\sigma_1) = \begin{pmatrix} -i & 0 \\ -1 & 1 \end{pmatrix}, \ \widehat{\beta}_4(i)(\sigma_2) = \begin{pmatrix} 1 & -i \\ 0 & -i \end{pmatrix}, \ \widehat{\beta}_4(i)(\sigma_3) = \begin{pmatrix} 1 & 0 \\ 1 & -i \end{pmatrix}.$$

Proof. By conjugation, we may assume

$$\eta(p) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \qquad \eta(q) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Since $\eta(\sigma_1)$ and $\eta(\sigma_3)$ commute with $\eta(p)$, they have the form

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

Let $\eta(\sigma_1) = \begin{pmatrix} L & 0 \\ 0 & L_1 \end{pmatrix}$ where L, L_1 are the upper left and the lower right 2×2 matrices respectively. Since $\sigma_3 = p^{-1}\sigma_1$,

$$\eta(\sigma_3) = \eta(p^{-1})\eta(\sigma_1) = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L_1 \end{pmatrix} = \begin{pmatrix} -iL & 0 \\ 0 & iL_1 \end{pmatrix}.$$

In [6, p. 406], $\sigma_1 q \sigma_1^{-1} = qp^{-1} = q\sigma_3\sigma_1^{-1}$. Then $\sigma_1 q = q\sigma_3$ implies

$$\begin{pmatrix} L & 0 \\ 0 & L_1 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} -iL & 0 \\ 0 & iL_1 \end{pmatrix}.$$

Thus $L_1 = -iL$. Hence we have $\eta(\sigma_1) = \begin{pmatrix} L & 0 \\ 0 & -iL \end{pmatrix}$, $\eta(\sigma_3) = \begin{pmatrix} -iL & 0 \\ 0 & L \end{pmatrix}$.

Let $\eta(\sigma_2) = \begin{pmatrix} M & M_1 \\ M_2 & M_3 \end{pmatrix}$ where M and M_i's are 2×2 matrices. Since $q\sigma_2 = \sigma_2 p$, we have $\eta(\sigma_2) = \begin{pmatrix} M & M_1 \\ iM & -iM_1 \end{pmatrix}$. Furthermore $\sigma_2 q \sigma_2^{-1} = qp^{-1}q = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}$ implies

$$\begin{pmatrix} M & M_1 \\ iM & -iM_1 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} \begin{pmatrix} M & M_1 \\ iM & -iM_1 \end{pmatrix}.$$

Therefore $M_1 = iM$ and consequently $\eta(\sigma_2) = \begin{pmatrix} M & iM \\ iM & M \end{pmatrix}$. The braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ implies $LML = (1+i)MLM$. If we let $M' = (1+i)M$, then $LM'L = M'LM'$. The solutions of the last equation (up to equivalence) are $L = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$, $M' = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix}$ for a, $b \in \mathbb{C}^*$. Then $M = \frac{1}{1+i}M' = \frac{1}{1+i}\begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix}$. We now have

$$\eta(\sigma_1) = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -ia & -i \\ 0 & 0 & 0 & -ib \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix},$$

$$\eta(\sigma_2) = \begin{pmatrix} \frac{b}{1+i} & 0 & \frac{ib}{1+i} & 0 \\ \frac{-ab}{1+i} & \frac{a}{1+i} & \frac{-iab}{1+i} & \frac{ia}{1+i} \\ \frac{ib}{1+i} & 0 & \frac{b}{1-i} & 0 \\ \frac{-iab}{1-i} & \frac{ia}{1+i} & \frac{-ab}{1-i} & \frac{a}{1+i} \end{pmatrix} = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{1+i} & \frac{i}{1+i} \\ \frac{i}{1+i} & \frac{1}{1+i} \end{pmatrix},$$

$$\eta(\sigma_3) = \begin{pmatrix} -ia & -i & 0 & 0 \\ 0 & -ib & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & b \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \otimes \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}.$$

These representations are irreducible for nonzero a and b, $b \neq a\mu$. This family of representations is the tensor product of 2 two-dimensional representations of B_4 and is equivalent to $\rho\pi \otimes \widehat{\beta}_4(i)$. □

8. Conclusion.

Combining Theorem 4.2, Proposition 7.2, and Proposition 7.3, we have the main theorem.

THEOREM 8.1. *Let $\xi : B_4 \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation. Then ξ is equivalent to one of the following irreducible representations.*

(1) *A representation $\eta = \eta(x, u)$, where $x, u \in \mathbb{C}^*$ $u \neq x^2$, defined by*

$$\eta(\sigma_1) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & u & 0 \end{pmatrix}, \ \eta(\sigma_2) = \begin{pmatrix} 0 & 0 & 0 & u \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \eta(\sigma_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ u & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}.$$

(2) *A representation $\tau\pi$ where $\pi : B_4 \to B_3$ is the special homomorphism and $\tau : B_3 \to \mathrm{GL}_4(\mathbb{C})$ is an irreducible representation of B_3 of degree 4.*

(3) *A representation $\rho(a, b)\pi \otimes \widehat{\beta}_4(i)$, where π is the special homomorphism, $\rho(a, b)$ is the irreducible representation of B_3 into $\mathrm{GL}_2(\mathbb{C})$ defined by $\rho(a, b)\ (\sigma_1) = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$, $\rho(a, b)(\sigma_2) = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix}$ for $a, b \in \mathbb{C}^*$, $b \neq a\mu$, and $\widehat{\beta}_4(i) : B_4 \to \mathrm{GL}_2(\mathbb{C})$ is the irreducible representation defined by*

$$\widehat{\beta}_4(i)(\sigma_1) = \begin{pmatrix} -i & 0 \\ -1 & 1 \end{pmatrix}, \ \widehat{\beta}_4(i)(\sigma_2) = \begin{pmatrix} 1 & -i \\ 0 & -i \end{pmatrix}, \ \widehat{\beta}_4(i)(\sigma_3) = \begin{pmatrix} 1 & 0 \\ 1 & -i \end{pmatrix}.$$

There are no equivalences between representations under distinct headings (1), (2) and (3) except $\eta(x, -x^2)$ in (1) is equivalent to $\rho(x, xi)\pi \otimes \widehat{\beta}_4(i)$ in (3)

For representations in (1), the image of P_4, the pure braid group on 4 strings, consists of diagonal matrices, and hence is abelian. In fact, if $x, u \in \mathbb{C}^*$ generate a free abelian group of rank 2 under multiplication, then the kernel of $\eta(x, u)$ is exactly $[P_4, P_4]$, the commutator subgroup of P_4. For representations in (2), the kernel contains F_2, the kernel of the special homomorphism $\pi : B_4 \to B_3$. For representations in (3), the image of F_2 is the quaternion group Q_8, and the kernel of the representation $\rho(a,b)\pi \otimes \hat{\beta}_4(i)$ is the kernel of a specific homomorphism $F_2 \to Q_8$.

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