J. Korean Math. Soc. **34** (1997), No. 3, pp. 653–659

AN EXTENSION OF GOTTLIEB GROUPS

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ABSTRACT. In this paper, we extend the Gottlieb groups of a space to the Gottlieb groups of a map and show some properties of those groups. Especially, We show the 2nd Gottlieb group of a map is contained in the center of the homotopy group of the map and show $G_n(f) = \pi_n(f)$ for an H-map f between H-spaces. We also show the Gottlieb subgroups $G_n(A)$, $G_n(X)$ and $G_n(f)$ make a sequence if the map $f : A \to X$ has a right homotopy inverse.

1. Introduction

Gottlieb [1,2] defined and studied the Gottlieb groups $G_n(X)$ of homotopy groups $\pi_n(X)$. The importance of Gottlieb groups is that these subgroups have many applications on topology, especially, on the fibration theory, on the fixed point theory, and on the theory of identification of spaces.

The homotopy sequence of a topological pair plays an important role in computing homotopy groups. In [3,4], Lee, Kim and Woo introduced the notion of generalized evaluation subgroups, relative evaluation subgroups and G-sequences of a CW-pair consisting of Gottlieb groups, generalized evaluation subgroups and relative evaluation subgroups. They made some improvements in computing evaluation subgroups.

In this paper, we introduce Gottlieb groups of a map by the same method as Gottlieb groups of a space(or evaluation subgroups of homotopy groups) given by Gottlieb and try to constract a sequence consisting of Gottlieb groups.

After the brief introduction of Gottlieb groups, we give the definition of Gottlieb groups of a map and discuss the main results in section 3.

Received April 12, 1997.
1991 Mathematics Subject Classification: 55P45.
Key words and phrases: evaluation groups, evaluation group of a map.
*Supported by TGRC96.
**Supported by KOSEF and BSRI 96-1409.

2. Preliminaries

Let X be a CW-complex and $\alpha : (S^n, s_0) \to (X, x_0)$ be a map. If there is a map $A : X \times S^n \to X$ with a homotopy commutative diagram

$$\begin{array}{ccc} X \times S^n & \xrightarrow{A} & X \\ \uparrow & \nearrow 1_X \vee \alpha & \\ X \vee S^n & & \end{array}$$

then A is called an *affiliated* map of $[\alpha]$. The Gottlieb group, $G_n(X)$, of a space X consists of all $[\alpha] \in \pi_n(X, x_0)$ such that there exists an affiliated map $[\alpha]$.

This group, $G_n(X)$, is also characterized by $G_n(X) = \omega_\sharp(\pi_n(X^X, 1_X)) \subset \pi_n(X, x_0)$ where $\omega : X^X \to X$ is an evaluation map at $x_0 \in X$. Thus $G_n(X)$ is also called an *evaluation subgroup* of $\pi_n(X, x_0)$. Gottlieb extensively studied $G_1(X)$ in [1] and $G_n(X)$ for $n \geq 2$ in [2]. Among other things he has shown that if X is an H-space then $G_n(X) = \pi_n(X)$ for all n. He also had computed

$$G_n(S^n) = \begin{cases} 0 & \text{for } n \text{ even} \\ Z & \text{for } n = 1, 3, 7 \\ 2Z & \text{for other odd } n\text{'s} \end{cases}$$

To define the Gottlieb groups of a map, we introduce *the category of pairs* whose objects are maps from a pointed space to a pointed space and whose morphism from a map f to g is a pair of maps (α_1, α_2) such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{\alpha_2} & Y \end{array}$$

commutes. A homotopy of (α_1, α_2) is just *a pair of homotopies* $(\alpha_{1t}, \alpha_{2t})$ such that $g\alpha_{1t} = \alpha_{2t} f$. This category reduces to the category of ordinary pairs of spaces if we restrict ourselves to maps which are inclusions.

Let $f : (A, a_0) \to (X, x_0)$ be a map. In [3], the *homotopy group of the map* f is defined by $\pi_n(f) = \{[(\alpha_1, \alpha_2)] | (\alpha_1, \alpha_2) : i_n \to f\}$, where $i_n : S^{n-1} \to CS^{n-1}$ is the canonical inclusion, CS^{n-1} is the cone over S^{n-1} and $[\;]$ denotes the homotopy class. Especially, if $A \subset X$ and f is the inclusion map from A to X, then $\pi_n(f)$ is $\pi_n(X, A)$. For a map $f : (A, a_0) \to (X, x_0)$, there is the homotopy sequence of f

$$\xrightarrow{J} \pi_{n+1}(f) \xrightarrow{\partial} \pi_n(A) \xrightarrow{f_\sharp} \pi_n(X) \to \cdots \xrightarrow{J} \pi_1(f) \xrightarrow{\partial} \pi_0(A) \xrightarrow{f_\sharp} \pi_0(X)$$

3. The Gottlieb groups of a map

Let $[(\alpha_1, \alpha_2)]$ be an element of $\pi_n(f)$ and $i_n : S^{n-1} \to CS^{n-1}$. If there exists a pair (F_1, F_2) of affiliated maps F_i of α_i with the commutative diagram

$$\begin{array}{ccc} A \times S^{n-1} & \xrightarrow{F_1} & A \\ \downarrow f \times i_n & & \downarrow f \\ X \times CS^{n-1} & \xrightarrow{F_2} & X \end{array}$$

then (F_1, F_2) is called an *affiliated pair* of $[(\alpha_1, \alpha_2)]$. The Gottlieb group, $G_n(f)$, of a map f consists of all $[(\alpha_1, \alpha_2)] \in \pi_n(f)$ such that there exists an affiliated map of $[(\alpha_1, \alpha_2)]$.

THEOREM 3.1. *$G_n(f)$ is a subgroup of the homotopy group $\pi_n(f)$ for $n > 1$.*

Proof. Let $[(\alpha_1, \alpha_2)] \in G_n(f)$ and $[(\beta_1, \beta_2)] \in G_n(f)$. Then there are affiliated pairs (F_1, F_2) of (α_1, α_2) and (G_1, G_2) of (β_1, β_2). Define an affiliated pair (H_1, H_2) by

$$H_i(x, h(t_1, ..., t_n)) = \begin{cases} F_i(x, h(2t_1, ..., t_n)), & \text{if } 0 \le t_1 \le \frac{1}{2} \\ G_i(x, (h(2t_1 - 1, .., t_n)), & \text{if } \frac{1}{2} \le t_1 \le 1. \end{cases}$$

where $h : (I^n, \partial I^n, J^{n-1}) \to (CS^{n-1}, S^{n-1}, *)$ is a map such that the restriction of h on $I^n - J^{n-1}$ is a homeomorphism. Then (H_1, H_2) is an affiliated pair of $[(\alpha_1\beta_1, \alpha_2\beta_2)] = [(\alpha_1, \alpha_2)][(\beta_1, \beta_2)]$.
By the similar way, we can show the inverse of any element of $G_n(f)$ belong to $G_n(f)$. □

REMARK. $G_n(f)$ is the Gottlieb group $G_n(X)$ in [2] if A is the trivial space and $G_n(f)$ is the relative evaluation subgroup $G_n^{Rel}(X, A)$ in [3] if $f : A \to X$ is the inclusion map.

In [1], Gottlieb showed that $G_1(X)$ is contained in the center of $\pi_1(X)$.

THEOREM 3.2. *Let $f : (A, a_0) \to (X, x_0)$ be a map. Then $G_2(f)$ is contained in the center of $\pi_2(f)$.*

Proof. Let $[(\alpha_1, \alpha_2)] \in G_2(f)$ and $[(\beta_1, \beta_2)] \in \pi_2(f)$. It is sufficient to show that there exists a pair (H_1, H_2) of homotopies such that $fH_1(,t) = H_2(,t)i_1$, where $H_1(,0) = \beta_1\alpha_1$ and $H_1(,1) = \alpha_1\beta_1$ and $H_2(,0) = \beta_2\alpha_2$ and $H_2(,1) = \alpha_2\beta_2$. Since $[(\alpha_1, \alpha_2)] \in G_2(f)$, there exists an affiliated pair (F_1, F_2) of $[(\alpha_1, \alpha_2)]$.

Define homotopies

$$H_1 : S^1 \times I \longrightarrow A$$

by

$$H_1(h(t_1, t_2), s) = \begin{cases} F_1(\beta_1(h(2t_1(1-s), t_2)), h(2t_1 s, t_2)), & \text{if } 0 \le t_1 \le \frac{1}{2} \\ F_1(\beta_1(h(1-(2-2t_1)s, t_2), h((2-2t_1)s + 2t_1 - 1, t_2)), \\ \text{if } \frac{1}{2} \le t_1 \le 1, \end{cases}$$

and

$$H_2 : CS^1 \times I \longrightarrow X$$

by

$$H_2(h(t_1, t_2), s) = \begin{cases} F_2(\beta_2(h(2t_1(1-s), t_2)), h(2t_1 s, t_2)), & \text{if } 0 \le t_1 \le \frac{1}{2} \\ F_2(\beta_2(h(1-(2-2t_1)s, t_2), h((2-2t_1)s + 2t_1 - 1, t_2)), \\ \text{if } \frac{1}{2} \le t_1 \le 1, \end{cases}$$

where $h : (I^2, \partial I^2, J^1) \to (CS^1, S^1, (1,0))$ is a map such that $h_{(I^2 - J^1)}$ is a homeomorphism.

Then we have

$$\begin{aligned}
H_1(h(t_1,t_2),0) &= (\beta_1 * \alpha_1)(h(t_1,t_2)),\\
H_1(h(t_1,t_2),1) &= (\alpha_1 * \beta_1)(h(t_1,t_2))\\
H_2(h(t_1,t_2),0) &= (\beta_2 * \alpha_2)(h(t_1,t_2)),\\
H_2(h(t_1,t_2),1) &= (\alpha_2 * \beta_2)(h(t_1,t_2))
\end{aligned}$$

and

$$fH_1(h(t_1,t_2),s) = H_2(i_1 \times id)(h(t_1,t_2),s).$$

This completes the proof. □

In [3], we show the following corollary.

COROLLARY 3.3. *Let (X,A) be a CW-pair. Then $G_2^{Rel}(X,A)$ is contained in the center of $\pi_2(X,A)$.*

THEOREM 3.4. *Let (A,a_0) and (X,x_0) be CW-complexes. If $f: (A,a_0) \to (X,x_0)$ has a right homotopy inverse, then $G_n(A), G_n(X)$ and $G_n(f)$ make a sequence*

$$\cdots \xrightarrow{\partial} G_n(A) \xrightarrow{f_\sharp} G_n(X) \xrightarrow{J} G_n(f) \to \cdots \xrightarrow{\partial} G_1(A) \xrightarrow{f_\sharp} G_1(X).$$

Proof. Consider the homotopy sequence of the map f

$$\cdots \longrightarrow \pi_n(A) \xrightarrow{f_\sharp} \pi_n(X) \xrightarrow{J} \pi_n(f) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots.$$

Since (A,a_0) and (X,x_0) are CW-complexes, we can take a right homotopy inverse $r:(X,x_0)\to(A,a_0)$ of the map f such that $frf=f$. Let $[\alpha]$ be an element of $G_n(A)$. Then there exists an affiliated map $F: A\times S^n \to A$ such that $[F(a_0,)] = [\alpha]$ and $[F(,s_0)] = [id]$. Let $H = fF(r\times 1): X\times S^n \to X$. Then $H(x_0,) = f\alpha$ and $H(,s_0) = fF(r,s_0) = fr \cong id$. Thus $f_\sharp(G_n(A))$ is contained in $G_n(X)$. Let $[\alpha]\in G_n(X)$. Then there exists an affiliated map $F: X\times S^n \to X$ and a homotopy commutative diagram

$$\begin{array}{ccc}
X\times S^n & \xrightarrow{F} & X\\
\uparrow & \nearrow 1_X \vee \alpha & \\
X\vee S^n & &
\end{array}$$

If we take an affiliated map $F_1 : A \times S^{n-1} \to A$ by $F_1(a, s) = a$ and $F_2 : X \times CS^{n-1} \to X$ by $F_2 = F(fr \times \pi)$, where $\pi : CS^{n-1} \to CS^{n-1}/S^{n-1} \equiv S^n$. Then F_1 and F_2 are affiliated maps of the constant map c_{a_0} and α respectively with the commutative diagram

$$\begin{array}{ccc} A \times S^{n-1} & \xrightarrow{F_1} & A \\ \downarrow f \times i_n & & \downarrow f \\ X \times CS^{n-1} & \xrightarrow{F_2} & X \end{array}$$

This proves $J([\alpha]) = [(*, \alpha)] \in G_n(f)$. Finally if we take $[\alpha_1, \alpha_2] \in G_n(f)$, then we can easily show that $\partial([(\alpha_1, \alpha_2)]) = [\alpha_1] \in G_{n-1}(A)$. □

In [2], Gottlieb has shown that if X is an H-space then $G_n(X) = \pi_n(X)$ for all n. We can expand this result as follows.

THEOREM 3.5. *Let (A, a_0) and (X, x_0) be H-spaces. If $f : (A, a_0) \to (X, x_0)$ is a H-map, then we have $G_n(f) = \pi_n(f)$.*

Proof. Let $[(\alpha_1, \alpha_2)] \in \pi_n(f)$ and let μ_1 and μ_2 be the H-structure of A and X respectively. If we take affiliated maps $F_1 = \mu_1(id_A \times \alpha_1)$ and $F_2 = \mu_2(id_X \times \alpha_2)$, then it is easy to show $G_n(f) = \pi_n(f)$. □

Since any map from the trivial space to an H-space is an H-map, we have the following corollary.

COROLLARY 3.6. *If X is an H-space, then $G_n(X) = \pi_n(X)$ for all n.*

THEOREM 3.7. *Let $f_1, f_2 : (A, a_0) \to (X, x_0)$ be homotopic. Then $G_n(f_1)$ is isomorphic to $G_n(f_2)$.*

Proof. Let $F : (A, a_0) \times I \to (X, x_0)$ be a homotopy between f_1 and f_2 and $[(\alpha_1, \alpha_2)] \in \pi_n(f_1)$. Then the map $K' : CS^{n-1} \times 0 \bigcup S^{n-1} \times I \to X$ given by $K'|S^{n-1} \times I = F(\alpha_1 \times id)$ and $K'|CS^{n-1} \times 0 = \alpha_2$ can be extended to a map $K : CS^{n-1} \times I \to X$. Let $\phi : \pi_n(f_1) \to \pi_n(f_2)$ be a map given by $\phi([(\alpha_1, \alpha_2)]) = [(\beta_1, \beta_2)]$ where $\beta_1 = \alpha_1$ and $\beta_2 = K(\,, 1)$. It is easy to show ϕ is an isomorphism.

Let $[(\alpha_1, \alpha_2)] \in G_n(f_1)$. It is sufficient to show that $\phi([(\alpha_1, \alpha_2)]) \in G_n(f_2)$. If we let (H_1, H_2) be an affiliated pair of $[(\alpha_1, \alpha_2)]$ and take a map

$$\bar{K}' : X \times CS^{n-1} \times 0 \bigcup A \times S^{n-1} \times I \to X$$

given by

$$\bar{K}'|A \times S^{n-1} \times I = F(H_1 \times id), \bar{K}'|X \times CS^{n-1} \times 0 = H_2,$$

then there is an extension $\bar{K} : X \times CS^{n-1} \times I \to X$ by the homotopy extension property. Let $H_1' = H_1$ and $H_2' = \bar{K}(, 1)$. Then (H_1', H_2') is an affiliated pair of $\phi([(\alpha_1, \alpha_2)])$. □

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