J. Korean Math. Soc. **34** (1997), No. 3, pp. 633–651

ON *UDL* DECOMPOSITIONS IN SEMIGROUPS

YONGDO LIM

ABSTRACT. For a non-degenerate symmetric bilinear form σ on a finite dimensional vector space E, the Jordan algebra of σ-symmetric operators has a symmetric cone Ω_σ of positive definite operators with respect to σ. The cone C_σ of elements $(x, y) \in E \times E$ with $\sigma(x, y) \geq 0$ gives the compression semigroup. In this work, we show that in the automorphism group of the tube domain over Ω_σ, this semigroup has a *UDL* and Ol'shanskii decompositions and is exactly the compression semigroup of Ω_σ.

1. Introduction

Let σ be a non-degenerate symmetric bilinear form on a finite dimensional vector space E. Let $\mathfrak{g} := \mathcal{L}(E)$ be the Banach algebra of the linear maps on E. Then

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-,$$

where $\mathfrak{g}^-$ is the Lie subalgebra of all self-adjoint operators with respect to σ and $\mathfrak{g}^+$ is the space of all skew-symmetric operators on E. The space $\mathfrak{g}$ is also a Jordan algebra with the anti-commutator product $x \circ y = \frac{1}{2}(xy + yx)$. Then $\mathfrak{g}^-$ is a Jordan subalgebra of $\mathfrak{g}$. If σ is positive definite, then $\mathfrak{g}^-$ is a simple Euclidean Jordan algebra which is isomorphic to $Sym(n, \mathbb{R})$, the symmetric $n \times n$ matrices with the corresponding symmetric cone Ω of positive definite symmetric operators. However, if σ is not positive definite, then the Jordan algebra $\mathfrak{g}^-$ is non-Euclidean but still simple. In this case we let Ω_σ be the set of all positive definite operators with respect to the bilinear form σ. Then there is an isomorphism (not Jordan algebra isomorphism) from $\mathfrak{g}^-$ to $Sym(n, \mathbb{R})$ which sends Ω_σ to Ω. Therefore Ω_σ is a symmetric cone.

Received March 18, 1997.
1991 Mathematics Subject Classification: 22A15.
Key words and phrases: semigroup, Lie group, Jordan algebra.

For a closed cone C in a Euclidean vector space E, the compression semigroup

$$\mathrm{Compr}(C) = \{g \in GL(E) \mid gC \subset C\}$$

is a closed subsemigroup in $GL(E)$. A non-degenerate symmetric bilinear form σ induces the cone $C_\sigma = \{x \in E \mid \sigma(x,x) \geq 0\}$ and there are two different semigroups which are cannonically related to the bilinear form σ : the expansion and contraction semigroups

$$\begin{aligned} S^{\geq} &= \{g \in GL(E) \mid \sigma(gx, gx) \geq \sigma(x,x)\ \forall x \in E\}, \\ S^{\leq} &= \{g \in GL(E) \mid \sigma(gx, gx) \leq \sigma(x,x)\ \forall x \in E\}. \end{aligned}$$

These semigroups are known as Ol'shanskii semigroups [3],[6]. However, the bilinear form σ gives a skew-symmetric bilinear form $(\cdot|\cdot)$ on $E \times E$:

$$(u_1|u_2) := \sigma(x_1, y_2) - \sigma(x_2, y_1),$$

for $u_i = (x_i, y_i)$ which gives a cone $C_\sigma = \{(x,y) \in E \times E \mid (x|y) \geq 0\}$. The compression semigroup S_σ of C_σ in the symplectic group $\mathbf{Sp}^\sigma(E)$ with respect to σ is completely characterized by using Wojtkowski's method and has UDL (upper triangular, diagonal, lower triangular) decomposition which plays a role to estimate Lyapunov exponents[10],[11].

From the one-to-one correspondence between symmetric cones and Siegel domains of tube type, we consider the tube domain $T_{\Omega_\sigma} = V_\sigma + \Omega_\sigma$, where V_σ is the Jordan algebra of self-adjoint operators with respect to σ. The compression semigroup Γ_{Ω_σ} in the automorphism group of the tube domain which can be extended to Ω_σ and carries Ω_σ into itself is a closed semigroup and is exactly equal to the compression semigroup S_σ. Furthermore, it is an Ol'shanskii semigroup with the decomposition $S_\sigma = H_\sigma \cdot \exp W_\sigma$, where H_σ is the group of units in S_σ and W_σ is a closed convex cone in the Lie algebra of the symplectic group $\mathbf{Sp}^\sigma(E)$ which is invariant under adjoint action of H_σ.

For positive definite symmetric bilinear form σ, Koufany [5] has proved the same results in his dissertation. But we give more direct proofs and consider any non-degenerate symmetric bilinear form.

2. Ol'shanskii decompositions

Let G be a Lie group with Lie algebra $\mathcal{L}(G)$ and S be a closed subsemigroup of G with identity. The *tangent wedge* of S is defined by

$$\mathcal{L}(S) = \{X \in \mathcal{L}(G) \mid \exp \mathbb{R}^{+} X \subseteq S\}.$$

Then it is a closed convex cone containing zero and is a Lie wedge, i.e.,

$$e^{adX}\mathcal{L}(S) = \mathcal{L}(S) \quad \forall X \in \mathcal{L}(S) \cap -\mathcal{L}(S).$$

The largest group $H(S) := S \cap S^{-1}$ contained in S is called the *group of units* of S. The systematic groundwork for a Lie theory of semigroups was worked out by K. H. Hofmann, J. Hilgert and J. D. Lawson [2] (cf.[3]). An important class of semigroups is Ol'shanskii semigroups that play the role of noncommutative analogue of tube domains in the harmonic analysis of hermitian semisimple Lie groups.

Let (G, τ) be an involutive Lie group, and let its derivative $\tau : \mathcal{L}(G) \to \mathcal{L}(G)$ have $+1$-eigenspace $\mathfrak{h}$ and -1-eigenspace $\mathfrak{q}$. If H is an open subgroup of $G_\tau := \{g \in G \mid \tau(g) = g\}$, if W is an $Ad(H)$-invariant cone in $\mathfrak{q}$, and if $S := H(\exp W)$ is a subsemigroup of G for which the mapping $(h, X) \to h(\exp X) : H \times W \to S$ is a homeomorphism, then S is called an *Ol'shanskii semigroup*, and the factorization $s = h(\exp X)$ for $s \in S$ is called the *Ol'shanskii polar factorization*.

The following theorm, which can be applied to the polar decomposition of matrices, will be a useful tool for this work.

THEOREM 2.1. *Let (G, τ) be an involutive Lie group, and let $H \subset G_\tau$ be a closed subgroup containing the identity component of G_τ. Let W be a wedge in $\mathfrak{q}$ which is invariant under the adjoint action of H and for which adX has real spectrum for each $X \in W$. Then the following conditions are equivalent.*

(1) *$(h, X) \to h(\exp X) : H \times W \to H(\exp W)$ is a diffeomorphism onto a closed subset of G.*

(2) *The mapping* Exp $: \mathfrak{q} \to G/H$ *defined by* $\mathrm{Exp}(X) = H(\exp X)$ *restricted to W is a diffeomorphism onto a closed subset of G/H.*

(3) *The mapping* exp *restricted to W is a diffeomorphism onto a closed subset of G.*

(4) *If $Z \in \mathfrak{z} \cap (W - W)$ satisfies $\exp Z = e$, then $Z = 0$. For each non-zero $X \in W \cap \mathfrak{z}$, the closure of* $\exp(\mathbb{R}X)$ *is not compact.*

If these conditions hold, then $S := H(\exp W)$ *is a closed semigroup with the tangent wedge* $\mathcal{L}(S) = \mathfrak{h} \oplus W$.

Proof. ([6], Theorem 3.1). □

In a real or complex vector space V with a cone C, we are very interested in a semigroup associated to the cone C, namely the *compression semigroup*

$$\mathrm{Compr}(C) = \{T \in GL(V) \mid T(C) \subset C\}.$$

This semigroup is always closed in $GL(V)$. Let V be a finite dimensional real (complex) vector space endowed with a non-degenerate symmetric or skew- symmetric bilinear form $\sigma(u, v)$. Then one of cones which is canonically related to the form is

$$C_\sigma = \{u \in V \mid \sigma(u, u) \geq 0\}.$$

There are two different semigroups which are canonically related to the form, the *contraction semigroup*

$$S^{\leq} = \{T \in GL(V) \mid \sigma(Tu, Tu) \leq \sigma(u, u), \forall u \in V\}$$

and the *expansion semigroup*

$$S^{\geq} = \{T \in GL(V) \mid \sigma(Tu, Tv) \geq \sigma(u, u), \forall u \in V\}.$$

For $T \in \mathfrak{g} = gl(V)$, let T^* be the adjoint operator of T associated to the bilinear form $\sigma(u, v)$. Then $\sigma(Tu, v) = \sigma(u, T^*v)$ for all $u, v \in V$. We may assume that the bilinear form $\sigma(u, v)$ is

$$j_{p,q}(u, v) = \sum_{i=1}^{p} r_i s_i - \sum_{i=p+1}^{n} r_i s_i, \quad j^{\mathbb{C}}_{p,q}(x, y) := \sum_{i=1}^{p} r_i \bar{s}_i - \sum_{i=p+1}^{n} r_i \bar{s}_i$$

by Sylvester's law of inertia. Let $\Omega_{p,q}$ be the open cone of self adjoint positive definite operators with respect to $\sigma(u, v)$ and $W_{p,q}$ be the closure of $\Omega_{p,q}$. Then

THEOREM 2.2. (Ol'shanskii Decomposition)
(1) *Real case:*

$$\begin{aligned} S^{\leq} &= O(p, q) \exp(W_{p,q}), \\ S^{\geq} &= O(p, q) \exp(-W_{p,q}) \end{aligned}$$

where $O(p, q)$ *is the pseudo-orthogonal group of the bilinear form* $j_{p,q}$.

(2) *Complex case:*

$$S^{\leq} = U(p,q)\exp(W_{p,q}),$$
$$S^{\geq} = U(p,q)\exp(-W_{p,q})$$

where $U(p,q)$ is the unitary group of the bilinear form $j^{\mathbb{C}}_{p,q}$.

Proof. (cf. [6], [7]). □

3. Ol'shanskii semigroups in symplectic groups

Let E be a real Hilbert space with inner product $\langle x|y\rangle$. Let $\mathcal{L}(E)$ be the Banach algebra of bounded operators on E. For $M \in \mathcal{L}(E)$, we denote M^t be the adjoint operator of M. If $M = M^t$, we say M is symmetric. A symmetric matrix M is *positive definite (positive semidefinite)*, written $M > 0 (M \geq 0)$, if $\langle Mx|x\rangle > 0$ ($\langle Mx|x\rangle \geq 0$) whenever $x \neq 0$.

Members $M \in \mathcal{L}(E \times E)$ have a block decomposition

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in \mathcal{L}(E).$$

Let $J \in \mathcal{L}(E)$ be defined in block form by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Note that $J^2 = -I$ and hence $J^{-1} = -J = J^t$. We define the skew-symmetric form on $E \times E$ by

$$(x|y) = \langle Jx|y\rangle, x, y \in E \times E.$$

We denote by M^* for $M \in \mathcal{L}(E \oplus E)$ the unique linear operator such that

$$(Mx|y) = (x|M^*y)$$

for all $x, y \in E \times E$. Then $M^* = -JM^tJ$.

Let $G = \{M \in GL(E \oplus E) \mid (Mu|Mv) = (u|v)\}$.

Proposition 3.1. *Let E be a real Hilbert space and let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(E \times E)$. Then the following are equivalent:*

(1) *$M \in G$, i.e., M preserves $(\cdot|\cdot)$.*

(2) $M^tJM = J$.
(3) A^tC, B^tD *are symmetric and* $A^tD - C^tB = I$.

If $E = \mathbb{R}^n$, then the group G is called the *symplectic group* which is denoted by $\mathbf{Sp}(2n, \mathbb{R})$. Let τ be the involution on $G = Sp(2n, \mathbb{R})$ defined by

$$\tau(\begin{pmatrix} A & B \\ C & D \end{pmatrix}) = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

Then (G, τ) is an involutive Lie group with

$$H := G_\tau = \{\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \mid A \in GL(n, \mathbb{R})\}.$$

Let $W = \{\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mid A, B \geq 0\}$. Then W is a closed convex cone in the Lie algebra of G. The Lie algebra $\mathfrak{h}$ of H is

$$\mathfrak{h} = \{\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \mid A \in M_n(\mathbb{R})\}.$$

It is easy to show that W is invariant under the adjoint action of H. By lemma 4.1 [6], if X has real spectrum, then adX has real spectrum for a matrix Lie algebra. Let $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in W$. To show adX has real spectrum, it is enough to consider when A and B are positive definite by continuity. Write $A = CC^t$, for some $C \in GL(n, \mathbb{R})$. Then $C^tBC > 0$. In this case, we may write $C^tBC = \exp Z$, for some symmetric matrix Z. Then X is similar to $\begin{pmatrix} 0 & I \\ C^tBC & 0 \end{pmatrix}$ by the matrix $\begin{pmatrix} C & 0 \\ 0 & (C^t)^{-1} \end{pmatrix}$. And $\begin{pmatrix} 0 & I \\ C^tBC & 0 \end{pmatrix}$ is similar to $\begin{pmatrix} 0 & \exp \frac{1}{2}Z \\ \exp \frac{1}{2}Z & 0 \end{pmatrix}$ by the matrix $\begin{pmatrix} \exp \frac{1}{4}Z & 0 \\ 0 & \exp \frac{-1}{4}Z \end{pmatrix}$. Therefore adX has real spectrum. Note that the symplectic group is simple and hence all the conditions of theorem 2.1 hold.

THEOREM 3.1. *We have* $S := H(\exp W)$ *is an Ol'shanskii semigroup in* $Sp(2n, \mathbb{R})$ *with* $\mathcal{L}(S) = \mathfrak{h} \oplus W$.

4. The compression semigroup of C_j

Note that for any non-degenerate symmetric bilinear form σ on $\mathbb{R}^n$, it is isometric to the following form of signature (p, q):

$$\langle x | j_{p,q} y \rangle,$$

where $j_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. From now on, we assume that $\sigma(x, y) = \langle x | j_{p,q} y \rangle$. Now let

$$J_{p,q} = \begin{pmatrix} 0 & -j_{p,q} \\ j_{p,q} & 0 \end{pmatrix}.$$

From now on, we fix p, q and let $j = j_{p,q}$, $J = J_{p,q}$ for convenience. Define a non-degenerate skew-symmetric bilinear form on $E \times E$ by

$$(u_1 | u_2) = \sigma(x_1, y_2) - \sigma(x_2, y_1) = \langle J u_1 | u_2 \rangle.$$

We define the symplectic group with respect to the bilinear form σ. Let $\mathbf{Sp}^j(E) = \{g \in GL(E \times E) \mid (gu|gv) = (u|v)\}$. Then we have

$$\begin{aligned} \mathbf{Sp}^j(E) &= \{g \in GL(E \times E) \mid (gu|gv) = (u|v)\} \\ &= \{g \in GL(E \times E) \mid g^t J g = J\} \\ &= \{g \in GL(E \times E) \mid g^* = g^{-1}\}. \end{aligned}$$

Here g^* is the adjoint operator of g with respect to the symplectic form $(u|v)$. Furthermore, note that every element g in $GL(E \times E)$ can be written as a block matrix:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A, B, C, D \in \mathcal{L}(E)$. So by solving the equation $g^t J g = J$, $\mathbf{Sp}^j(2n, \mathbb{R})$ consists of all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(E \times E)$ satisfying

$$A^t j C,\ D^t j B \quad \text{are} \quad \text{symmetric}, \quad D^t j A - B^t j C = j.$$

Let Q_j be the quadratic form on $E \times E$ associated to σ, $Q_j(u) = \sigma(x, y)$ for $u = (x, y) \in E \times E$. Then the cone C_j corresponding to the bilinear form σ is

$$C_j = \{u \in V \mid Q_j(u) \geq 0\} = \{u = (x, y) \in E \times E \mid \sigma(x, y) \geq 0\}.$$

The compression semigroup of the cone C_j on the group $\mathbf{Sp}^j(2n,\mathbb{R})$ is defined by

$$S_j := \mathrm{Compr}(C_j) \cap \mathbf{Sp}^j(2n,\mathbb{R}).$$

When the bilinear form σ is positive definite or equivalently j is the identity matrix, the structure of an element in S_I is completely characterized by Wojtkowski [10]. We follow his method for a generalization. Note that S_j is a closed subsemigroup of $\mathbf{Sp}^j(2n,\mathbb{R})$.

THEOREM 4.1. *Let* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}^j(2n,\mathbb{R})$. *Then the following are equivalent:*

(a) $Q_j(gu) \geq Q_j(u)$, *for all* $u \in E \times E$.
(b) $g \in S_j$.
(c) A *is invertible and* $A^t jC \geq 0$ *and* $BjA^t \geq 0$.
(d) D *is invertible and* $CjD^t \geq 0$ *and* $D^t jB \geq 0$.

By definition, (a) implies (b). The proof of the theorem is from the following lemmas.

LEMMA 4.1. *Let* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}^j(2n,\mathbb{R})$. *If* $g \in S_j$, *then* A *and* D *are invertible.*

Proof. Suppose $g \in S_j$ and $Ax_0 = 0$. Since $D^t jA - B^t jC = j$, $B^t jC(x_0) = -j(x_0)$. Let $y = sj(x_0)$. Then $(x_0|y) = s\langle x_0|x_0\rangle \geq 0$, for all $s \geq 0$. Hence $v = (x_0, y) \in C_j$ and $gv \in C_j$. But $gv = (By, Cx_0 + Dy) \in C_j$ implies that

$$(By|Cx_0 + Dy) = -\langle y|jx_0\rangle + \langle y|B^t jDy\rangle \geq 0.$$

Hence $\langle jx_0|jx_0\rangle \leq s\langle jx_0|BjDjx_0\rangle \to 0$, as $s \to 0$. Hence $jx_0 = 0$. Therefore A is invertible. Using the same method of the case A, we show that D is invertible. □

REMARK In the proof of lemma 4.1, one may want to generalize this result to any (infinite-dimensional) real or complex Hilbert spaces. It looks quite non-trival to prove that A and D are surjective. We leave this as an open problem.

LEMMA 4.2. *If* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_j$, *then*

(1) $A^t jC \geq 0$, $jA^{-1}B \geq 0$ *and hence* $BjA^t \geq 0$.
(2) $D^t jB \geq 0$, $jD^{-1}C \geq 0$ *and hence* $CjD^t \geq 0$.
(3) $BD^{-1}j \geq 0$.

Proof. (1) $A^t jC \geq 0$. By definition, $A^t jC$ is symmetric. Set

$$g_0 = \begin{pmatrix} jA^{-1} & 0 \\ 0 & A^t j \end{pmatrix}.$$

Then

$$g_1 = g_0 g = \begin{pmatrix} j & jR \\ P & j + PR \end{pmatrix} \in S_j,$$

where

$$R = A^{-1}B, P = A^t jC.$$

For $u = (x, 0)$,

$$Q_j(g_1 u) = (jx | A^t jCx) = \langle x | A^t jCx \rangle \geq 0.$$

Hence $P = A^t jC \geq 0$.

(2) $R = A^{-1}B \geq 0$. Since $g_1 = g_0 g = \begin{pmatrix} j & jR \\ P & j + PR \end{pmatrix} \in S_j \subset \mathbf{Sp}^j(2n, \mathbb{R})$, from the definition of $\mathbf{Sp}^j(2n, \mathbb{R})$, we have that jR is symmetric. To show that $jR \geq 0$, suppose that $\langle jA^{-1}By_0 | y_0 \rangle < 0$. Choose $x_0 \in E$ such that $\langle x_0 | y_0 \rangle < 0$. Let $v = (sx_0 - Ry_0, y_0)$. Then

$$(sx_0 - Ry_0 | y_0) = s(x_0 | y_0) - (Ry_0 | y_0) > 0$$

for sufficiently small $s > 0$. So $v \in C_j$ for sufficiently small $s > 0$. Since $g_1 \in S_j$, $g_1 v \in C_j$. But $g_1 v = (sx_0, sA^t jCx_0 + y_0)$ and

$$\begin{aligned} Q_j(g_1 v) &= (sjx_0 | sPx_0 + jy_0) \\ &= s^2 (jx_0 | Px_0) + s(jx_0 | jy_0) \\ &= s^2 \langle x_0 | Px_0 \rangle + s(x_0 | y_0) < 0 \end{aligned}$$

for sufficiently small $s > 0$ which leads a contradiction. Therefore $jA^{-1}B \geq 0$.

(3) $BjA^t \geq 0$. $BjA^t = (Aj)(jR)(Aj)^t \geq 0$. The proof of the remaining part is similar to that of the previous one. □

LEMMA 4.3. *If* A *is invertible and* $A^t jC \geq 0, BjA^t \geq 0$ *then* $Q_j(gu) \geq Q_j(u), \forall u \in E \oplus E$.

Proof.

$$\begin{aligned} Q_j(gu) &= Q_j(g_1u) \\ &= (jx + jRy|Px + jy + PRy) \\ &= \langle x + Ry|jy\rangle + \langle x + Ry|P(x + Ry)\rangle \\ &= \langle x|jy\rangle + \langle Ry|jy\rangle + \langle P(x + Ry)|x + Ry\rangle \\ &\geq (x|y) = Q_j(u). \end{aligned}$$

□

5. The *UDL* decomposition of S_j

Let $Sym(n, \mathbb{R})$ be the space of symmetric $n \times n$-matrices and let

$$\begin{aligned} \Gamma_j^- &= \{\begin{pmatrix} j & A \\ 0 & j \end{pmatrix} \mid A \in Sym(n, \mathbb{R}),\ A \geq 0\}, \\ \Gamma_j^- &= \{\begin{pmatrix} j & 0 \\ A & j \end{pmatrix} \mid A \in Sym(n, \mathbb{R}), A \geq 0\}, \\ H_j &= \{\begin{pmatrix} A^{t-1} & 0 \\ 0 & jAj \end{pmatrix} \mid A \in GL(n, \mathbb{R})\}. \end{aligned}$$

Then $\Gamma_j^{\pm}$ are closed subsemigroups $\mathbf{Sp}^j(2n, \mathbb{R})$ and H_j is the group units of S_j. Therefore, $\Gamma_j^+ \cdot H_j \cdot \Gamma_j^- \subset S_j$.

THEOREM 5.1 (*UDL* decomposition of S_j). *We have*

$$S_j = \Gamma_j^+ \quad H_j \cdot \Gamma_j^-.$$

Proof. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_j$. Then by theorem 4.1 and lemma 4.2,

$$\begin{pmatrix} j & BD^{-1}j \\ 0 & j \end{pmatrix}, \begin{pmatrix} (D^{-1})^t & 0 \\ 0 & jDj \end{pmatrix}, \begin{pmatrix} j & 0 \\ jD^{-1}C & j \end{pmatrix} \in S_j.$$

Hence

$$g = \begin{pmatrix} j & BD^{-1}j \\ 0 & j \end{pmatrix} \begin{pmatrix} (D^{-1})^t & 0 \\ 0 & jDj \end{pmatrix} \begin{pmatrix} j & 0 \\ jD^{-1}C & j \end{pmatrix} \in \Gamma_j^+ \cdot H_j \cdot \Gamma_j^-.$$

□

Let

$$\alpha = \begin{pmatrix} j & 0 \\ 0 & I \end{pmatrix}, \ \beta = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}.$$

Then $\alpha, \beta \in GL(2n, \mathbb{R})$ are involutions and $\gamma := \alpha\beta = \begin{pmatrix} I & 0 \\ 0 & j \end{pmatrix}$. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}^j(2n, \mathbb{R})$,

$$\alpha g \alpha = \begin{pmatrix} jAj & jB \\ Cj & D \end{pmatrix}.$$

THEOREM 5.2. *The mapping*

$$g \in \mathbf{Sp}^j(2n, \mathbb{R}) \to \alpha g \alpha \in \mathbf{Sp}(2n, \mathbb{R})$$

gives an isomorphism between $\mathbf{Sp}^j(2n, \mathbb{R})$ *and* $\mathbf{Sp}(2n, \mathbb{R})$.

Proof. Since $A^t jC = C^t jA$,

$$(jAj)^t Cj = jA^t jCj = jC^t jAj = (Cj)^t (jAj).$$

Note that $D^t jB = B^t jD = (jB)^t D$ and

$$D^t (jAj) - (jB)^t Cj = (D^t jA - B^t jC)j = j^2 = I.$$

Hence the mapping is well-defined. Because α is an involution, it is not hard to see that α is an isomorphism. □

For $j = I$, we let $\Gamma^{\pm} = \Gamma_I^{\pm}$ and $H = H_I$. Then $S_I = \Gamma^+ \cdot H \cdot \Gamma^-$.

COROLLARY 5.1. *We have* $\gamma S_j \gamma = S_I$.

Proof. Note that

$$\begin{aligned} \alpha \Gamma_j^+ \alpha &= \beta \Gamma^+, \\ \alpha \Gamma_j^- \alpha &= \Gamma^- \beta, \\ \alpha H_j \alpha &= \beta H \beta = H. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha S_j \alpha &= \alpha \Gamma_j^+ \alpha \cdot \alpha H \alpha \cdot \alpha \Gamma_j^- \alpha \\ &= \beta \Gamma^+ \cdot H \cdot \Gamma^- \beta. \end{aligned}$$

Hence $\beta \alpha S_j \alpha \beta = S$. □

Set

$$\begin{aligned} S_j^+ &= \{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \mid Aj \geq 0 \}, \\ S_j^- &= \{ \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \mid jA \geq 0 \}. \end{aligned}$$

Then by theorem 4.1, $S_j^{\pm} \subset S_j$.

THEOREM 5.3. *The semigroup S_j can be decomposed as*

$$S_j = S_j^+ H_j S_j^-.$$

Proof. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_j$. Then by lemma 4.2, $BD^{-1}j, jD^{-1}C$ are symmetric. Hence $\begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \in S_j^+$ and $\begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \in S_j^-$. Using $D^t jA - B^t jC = j$,

$$g = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} j(D^{-1})^t j & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \in S_j^+ \cdot H_j \cdot S_j^-.$$

□

We recall the Ol'shanskii semigroup $S = H(\exp W)$ which appears in section 3.

THEOREM 5.4. *We have*

$$S = H \cdot \exp W = S_I = \Gamma^+ H \Gamma^-.$$

Proof. Note that

$$\Gamma^+ = \exp\{\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \mid A \geq 0\} \subset \exp W,$$

$$\Gamma^- = \exp\{\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \mid A \geq 0\} \subset \exp W.$$

Hence $\Gamma^+ \cdot H \cdot \Gamma^- \subset S^3 = S = H \cdot \exp W$. Conversely, let $Z = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in W$. Then $A, B \geq 0$. If $X = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$, then $Z = X + Y$ and

$$\exp(Z) = \exp(X + Y) = \lim_{n \to \infty} (\exp(\frac{1}{n}X) \exp(\frac{1}{n}Y))^n.$$

Since for each $n \geq 0, \exp(\frac{X}{n}) \in \Gamma^+ \exp(\frac{Y}{n}) \in \Gamma^-$ and $S_I = \Gamma^+ H \Gamma^-$ is closed, $\exp(Z) \in S_I$. Hence $H(\exp W) \subset S_I S_I \subset S_I$. □

6. The Euclidean Jordan algebra $Sym(n, \mathbb{R})$

Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. A commutative algebra V over $\mathbb{F}$ with product xy is said to be a *Jordan algebra* if for all elements x and y in V:

$$x(x^2y) = x^2(xy).$$

This identity is called the *Jordan identity*. For $x \in V$, we denote $L(x)y := xy$, the multiplication operator representation. Then the Jordan identity can be written $[L(x), L(x^2)] = 0$, where the bracket is usual Lie bracket on $\mathcal{L}(V)$, the set of all bounded linear operators on the vector space V. For $x \in V$, we define $P(x) = 2L(x)^2 - L(x^2)$. The map P is called the *quadratic representation* of V. Every associative algebra V with product xy becomes a Jordan algebra with the anti-commutator product:

$$x \circ y = \frac{1}{2}(xy + yx).$$

An element x of a Jordan algebra V with identity e is called *invertible* with inverse y if $xy = e$ and $x^2y = x$. One can see that an element x in a Jordan algebra V is invertible if and only if $P(x)$ is invertible. In this case, $P(x)x^{-1} = x$ and $P(x)^{-1} = P(x^{-1})$.

Let V be a finite dimensional Jordan algebra and let $\tau(x, y) = TrL(xy)$. Then $\tau(x, y)$ is an associative symmetric bilinear form on V, that is,

$$\tau(xy, z) = \tau(y, xz),$$

for all x, y, z in V. A Jordan algebra is said to be *semi-simple* if the bilinear form $\tau(x, y)$ is non-degenerate on it. A semi-simple Jordan algebra is called *simple* if it has no non-trivial ideal. It is well-known that every semisimple Jordan algebra has a unit and every ideal is a semi-simple Jordan algebra [4]. A semi-simple Jordan algebra over $\mathbb{R}$ or $\mathbb{C}$ is, in a unique way, a direct sum of simple ideals [4].

A real Jordan algebra V is called a *Jordan-Hilbert algebra* if V is a real Hilbert space with inner product $\langle x|y\rangle$ such that

$$\langle xy|z\rangle = \langle y|xz\rangle,$$

for all $x, y, z \in V$. In addition, if V is finite dimensional and has an identity, then it is called a *Euclidean Jordan algebra*. In general a Jordan-Hilbert algebra does not contain a unit element [9]. In [8], it was proved that a finite dimensional Jordan-Hilbert Jordan algebra has an identity if and only if $L(x) = 0 \Longrightarrow x = 0$. It is easy to show that a Euclidean

Jordan algebra is formally real in the following sense: $x^2+y^2=0$ implies $x=y=0$. The converse is also true, i.e., a finite dimensional formally real Jordan algebra becomes a Euclidean Jordan algebra [1].

Examples (1) The algebra $Sym(n,\mathbb{R})$ of $n\times n$ real symmetric matrices with the Jordan product

$$x\circ y=\frac{1}{2}(xy+yx)$$

is a Euclidean Jordan algebra since the bilinear form $Tr(xy)$ is positive definite and associative.

(2) (Non-Euclidean Jordan algebra) Let $V_{1,1}$ be the space of 2×2-matrices of the form:

$$V_{1,1}:=\{\begin{pmatrix} x & y \\ -y & z\end{pmatrix} \mid x,y,z\in\mathbb{R}\}.$$

Then $V_{1,1}$ is a 3-dimensional Jordan algebra with the anti-commutative product. Let

$$A=\begin{pmatrix} 1 & 1 \\ -1 & -1\end{pmatrix}.$$

Then $A^2=0$ and hence $V_{1,1}$ is not formally real, hence not a Euclidean Jordan algebra.

Let V be a Euclidean Jordan algebra with the associated bilinear form $\langle x|y\rangle$. Let $Q=\{x^2\mid x\in V\}$ be the set of squares Then the set Q is a self dual cone and $Q=\{y\in V\mid L(y)\geq 0\}$. Let Ω be the interior of Q. Then it is a symmetric cone. That is, Ω is a self-dual cone and the group

$$G(\Omega):=\{g\in GL(V)\mid g\Omega=\Omega\}$$

acts on it transitively. Furthermore,

THEOREM 6.1. *The symmetric cone Ω has the following characterizations:*

$$\begin{aligned}\Omega &= \exp V,\\ &= \textit{the identity component of } V^{-1},\\ &= \{u^2\mid u\in V^{-1}\},\\ &= \{u\in V\mid L(u) \textit{ is positive definite}\}.\end{aligned}$$

Proof. (cf. [1]). □

Let E be a finite-dimensional vector space over $\mathbb{R}$ and let $\sigma(x,y)$ be a symmetric bilinear form on E. We also assume that σ is non-degenerate. Then the bilinear form σ is represented as follows by a symmetric matrix $S=(a_{ij})$ for a basis on E

$$\sigma(x,y)=\sum_{i,j}a_{ij}x_iy_j.$$

For $T\in gl(E)$, the σ-adjoint operator T^* is given by $T^*=S^{-1}T^tS$. Let V_σ be the set of all self-adjoint operators with respect to the fixed non-degenerate, symmetric bilinear form σ on V Then V_σ is a Jordan algebra with the product

$$A\circ B=\frac{1}{2}(AB+BA).$$

From now on, we let $V_{p,q}$ denote the Jordan algebra of all self-adjoint matrices on $\mathbb{R}^n$ of dimension $n=p+q$ with respect to the bilinear form

$$j_{p,q}(x,y)=\sum_{i=1}^{p}x_iy_i-\sum_{i=p+1}^{n}x_iy_i.$$

The following result is well-known and easy to prove [cf. 8].

THEOREM 6.2. *Let σ be a non-degenerate symmetric bilinear form on a finite dimensional vector space E. Then the Jordan algebra V_σ is simple and is isomorphic to $V_{p,q}$ for some integers p,q with $p+q=dimV$. Furthermore, the Jordan algebra V_σ is Euclidean if and only if σ is positive or negative definite.*

There is a one-to-one correspondence between Euclidean Jordan algebras and symmetric cones which are the same categories of Siegel domains of tube type [1], [4]. The simple Euclidean Jordan algebra $Sym(n,\mathbb{R})$ of symmetric $n\times n$-matrices has the corresponding symmetric cone Ω_n of all symmetric positive definite matrices. In our notation, $Sym(n,\mathbb{R})=V_n$ for some positive definite symmetric bilinear form on E with $dim(E)=n$. But the non-Euclidean Jordan algebra $V_{p,q}$ for $p\neq 0$ and $q\neq 0$ has a nice cone $\Omega_{p,q}$ which is not appeared in the characterization of symmetric cone in theorem 6.1. Now we remind the notation of $\Omega_{p,q}$ as an open convex cone of the positive definite matrices with respect to $j_{p,q}(x,y)$. We also fix p,q and let $j=j_{p,q}=\begin{pmatrix}I_p & 0\\ 0 & -I_q\end{pmatrix}$.

PROPOSITION 6.1. *We have $jV_{p,q} = Sym(n,\mathbb{R})$. In particular, $j\Omega_{p,q} = \Omega_n$, where Ω_n is the symmetric cone of $Sym(n,\mathbb{R})$.*

Proof. First, note that if $A \in V_{p,q}$, then $jA \in Sym(n,\mathbb{R}) = V_{n,0}$. For

$$jA = jA^* = jjA^tj = A^tj$$

which implies that $(jA)^t = jA$. Hence jA is a symmetric operator. Conversely, if $A \in Sym(n,\mathbb{R})$, then $(jA)^* = jA^t = jA$. Hence $jA \in V_{p,q}$. Therefore $jV_{p,q} = Sym(n,\mathbb{R})$. Now suppose that $A \in \Omega_{p,q}$. Then $(Ax|x) > 0$, for all non-zero element x in V. Since $(Ax|x) = \langle jAx|x\rangle$, jA is a symmetric positive definite operator. So $j\Omega_{p,q} \subset \Omega_n$. Similiary, one can show the converse argument. □

7. The semigroup Γ_Ω

Let $V = Sym(n,\mathbb{R})$ and $\Omega := \Omega_n$ be the open convex cone of positive definite $n \times n$ symmetric matrices. Then V is a simple Euclidean Jordan algebra with the symmetric cone Ω. It is well-known that any biholomorphic automorphisms on the tube domain $T_\Omega = V + i\Omega$ is the following form

$$Z \in T_\Omega \longrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}$$

for some $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}(2n,\mathbb{R})$. Hence the symplectic group

$$\mathbf{Sp}(2n,\mathbb{R}) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n,\mathbb{R}) \mid A^tC, D^tB \in V, D^tA - B^tC = I\}$$

acts on the tube domain $T_{\Omega_n} = V + i\Omega_n$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}.$$

Define a subsemigroup Γ_Ω by the elements of $\mathbf{Sp}(2n,\mathbb{R})$ which can be extended to $\Omega \subset V^{\mathbb{C}}$ and $g \cdot \Omega \subset \Omega$. Since every element in $\mathbf{Sp}(2n,\mathbb{R})$ can be extended to the conformal compactification of $V^{\mathbb{C}}$, we can write

$$\Gamma_\Omega = \{g \in \mathbf{Sp}(2n,\mathbb{R}) \mid g \cdot \Omega \subset \Omega\}.$$

Note that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}(2n, \mathbb{R})$ and $D \in GL(n, \mathbb{R})$ implies that

$$A = (D^t)^{-1} + BD^{-1}C. (*)$$

In this case, g can be decomposed as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} (D^t)^{-1} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}.$$

Let

$$N^+ = \{\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \mid A \in V\},$$
$$N^- = \{\begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \mid A \in V\}.$$

Then N^+ is the abelian subgroup of $\mathbf{Sp}(2n, \mathbb{R})$ of all translations and $\tau \circ N^+ \circ \tau = N^-$, where $\tau = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Lemma 7.1. *For* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}(2n, \mathbb{R})$, *the following properties are equivalent:*

(1) $g \in N^+HN^-$,
(2) $g \cdot 0 \in V$,
(3) $D \in GL(n, \mathbb{R})$.

Proof. Obviousely, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Suppose that D is invertible. Then $D^tB = B^tD$ implies that $BD^{-1} = (D^{-1})^tB^t = (BD^{-1})^t$. Therefore BD^{-1} is symmetric. By (*), $A^t = D^{-1} + C^tBD^{-1}$. Since $D^tB = B^tD$, $C^tBD^{-1}C = C^t(D^{-1})^tB^tC$ is symmetric and hence $C^tBD^{-1}C = C^t(D^{-1})^tB^tC$ is symmetic. Therefore $D^{-1}C = A^tC - C^tBD^{-1}C$ is symmetric. So $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} (D^t)^{-1} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \in N^+HN^-$. □

It is easy to see that $\Gamma^+G(\Omega)\Gamma^- \subset \Gamma_\Omega$. Hence $S = \Gamma^+H\Gamma^- \subset \Gamma_\Omega$.

Lemma 7.2. *We have* $N^+HN^- \cap \Gamma_\Omega = S = \Gamma^+H\Gamma^-$.

Proof. Suppose that $g = n^+hn^- = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} (D^t)^{-1} & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} \in \Gamma_\Omega$. Then for $X \in \Omega$, $X(BX+I)^{-1} = (B + X^{-1})$ is an invertible element in $V = Sym(n, \mathbb{R})$. This implies that

$$B + \Omega \subseteq V^{-1}.$$

By the argument in linear algebra or by theorem 6.1, $B + \Omega \subset \Omega$. In particular, $B \in \overline{\Omega}$. To show $A \in \overline{\Omega}$, choose $Z \in B + \Omega \subset \Omega$. Then $nZ \in B + \Omega$, for all natural numbers n. This is from the induction argument. Let $Z = B + X \in B + \Omega$ Then

$$nZ = nB + nX = B + (n-1)B + nX \in B + \Omega + \Omega \subset B + \Omega.$$

But

$$A + \frac{1}{n}h(Z^{-1}) \in n^+h((B+\Omega)^{-1}) = n^+hn^-(\Omega) \subset \Omega.$$

Thus $A \in \overline{\Omega}$. □

THEOREM 7.1. *The two Lie semigroups S and Γ_Ω are the same.*

Proof. Suppose that $g \in \Gamma_\Omega$. Let $t_n = \begin{pmatrix} I & \frac{1}{n}I \\ 0 & I \end{pmatrix}$. Then $gt_n \in \Gamma_\Omega$ and $gt_n(0) = g(\frac{1}{n}I) \in \Omega$. Therefore, by lemma 7.2, $gt_n \in \Gamma^+ H \Gamma^- = S$. Since S is closed, $g \in S$. □

By the isomorphism in theorem 5.2 and theorem 7.1,

COROLLARY 7.1. *We have* $\Gamma_{\Omega_{p,q}} = \Gamma_j^+ H_j \Gamma_j^- = H_j \exp W_j = S_j$, *where*

$$W_j = \{ \begin{pmatrix} 0 & jA \\ Bj & 0 \end{pmatrix} \mid A, B \geq 0 \}.$$

ACKNOWLEDGEMENT This paper is a part of my doctoral dissertation at Louisiana State University. I would like to thank my advisor Jimmie D. Lawson for his mathematical advice and support.

References

[1] J. Faraut and A. Koranyi, *Analysis on Symmetric Cones*, Oxford Press, Oxford, 1994.

[2] J. Hilgert, K. H. Hofmann and J. D. Lawson, *Lie groups, Convex Cones, and Semigroups*, Oxford Press, Oxford, 1989.

[3] J. Hilgert and K. H. Neeb, *Basic theory of Lie semigroups and applications*, Lect. Notes Math. **1552**, Heildelberg-Berlin-New York, 1993.

[4] M. Koecher, *Jordan algebras and their applications*, Lectures notes, Univ. of Minnesota, Minneapolis, 1962.

[5] K. Koufany, *Semigroupe de Lie associe a une algebre de Jordan euclidienne*, Ph.D theses, 1994.

[6] J. D. Lawson, *Polar and Ol'shanskii decompositions*, J. reine angew. Math., **448** (1994), 191-219.

[7] ———, *Semigroups of Ol'shanskii type*, Semigroups in Algebra, Geometry and Analysis, Water de Gruyter, Berlin. York, 1995.

[8] Y. Lim, *Jordan alegbras and Lie semigroups*, Ph.D theses, Louisiana State University, 1996.

[9] T. Nomura, *Manifold of primitive idempotents in a Jordan Hilbert algebra*, J. Math. Soc. Japan, **45** (1993), 37-58.

[10] M. Wojtkowski, *Measure theoretic entropy of the system of hard spheres*, Ergod. Theo. and Dyn. Sys., **8** (1988) 133-153.

[11] ———, *Invariant families of cones and Lyapunov exponents*, Ergod. Theo. and Dyn. Sys., **5** (1985), 145-161.

Department of Mathematics
Kyungpook National University
Taegu, Koera 702-701
E-mail: ylim@math.kyungpook.ac.kr