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EXTENSIONS OF THE BORSUK-ULAM THEOREM

IN-SOOK KIM

ABSTRACT. In this paper we give a generalization of the well-known Borsuk-Ulam theorem and its extensions to countably many products of spheres.

1. Introduction

Borsuk's antipodal theorem which was conjectured by Ulam has equivalent formulations in great variety. In the present paper it is natural to investigate the Borsuk-Ulam theorem on more general conditions, by replacing odd map by equivariant map with respect to group actions on the spheres.

This observation is motivated by the almost periodicity of orbits as a generalization of the periodicity. In contrast to actions of compact Lie group, a new result on $\mathbb{Z}$-actions is of considerable practical importance. Furthermore, our result is extended to finitely as well as countably many products of spheres, where the idea lies in the approximation theorem for almost periodic functions, see [4]. In particular, we show that this extension holds for the case of rationally independent numbers. We know that there is a close relationship between Borsuk-Ulam theorem and index theory for group actions, see [7], [10]. Therefore, we here provide an appropriate framework for an index theory. This kind of application is one of the main reasons to study a $\mathbb{Z}$-version of the Borsuk-Ulam theorem.

2. Preliminaries

From elementary observations we will see that the approach to the

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Borsuk-Ulam theorem can be extended to other group actions, for instance $\mathbb{Z}_q$- or $\mathbb{Z}$-actions.

To present these facts, we introduce the concept of equivariance.

DEFINITION 1. Let α_j be a real number and let k_j, l_j be natural numbers for $j = 1, \cdots, n$ and $\tilde{\alpha} := (e^{2\pi i\alpha_1}, \cdots, e^{2\pi i\alpha_n})$. A map $h = (h_1, \cdots, h_n) : S^{2k_1-1} \times \cdots \times S^{2k_n-1} \to S^{2l_1-1} \times \cdots \times S^{2l_n-1}$ will be called *equivariant* with respect to $\tilde{\alpha}$, if

$$h(e^{2\pi i\alpha_1}x_1, \cdots, e^{2\pi i\alpha_n}x_n) = (e^{2\pi i\alpha_1}h_1(x), \cdots, e^{2\pi i\alpha_n}h_n(x))$$

for every $x = (x_1, \cdots, x_n) \in S^{2k_1-1} \times \cdots \times S^{2k_n-1}$. In this case we write $h : (S^{2k_1-1} \times \cdots \times S^{2k_n-1}, \tilde{\alpha}) \to (S^{2l_1-1} \times \cdots \times S^{2l_n-1}, \tilde{\alpha})$.

The classical Borsuk-Ulam theorem says that given a continuous map $h : S^n \to \mathbb{R}^n$, there exists an $x \in S^n$ such that $h(x) = h(-x)$. It was well-known that this result is equivalent to the following theorem.

THEOREM 2. *For every $n, m \in \mathbb{N}$ with $n > m$ there is no continuous odd map $h : S^n \to S^m$.*

For the proof we refer to Bredon [3].

The following result which we will need later can be deduced from the so-called Borsuk-Ulam theorem.

THEOREM 3. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Z}$. For every $k, l \in \mathbb{N}$ with $k > l$ there is no continuous equivariant map $(S^{2k-1}, e^{2\pi i\alpha}) \longrightarrow (S^{2l-1}, e^{2\pi i\alpha})$.*

Proof. For $\alpha \in \mathbb{Q}\backslash\mathbb{Z}$ the theorem is well-known, see [5]. Now let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Suppose that there is a continuous equivariant map $h : (S^{2k-1}, e^{2\pi i\alpha}) \longrightarrow (S^{2l-1}, e^{2\pi i\alpha})$ for $k, l \in \mathbb{N}$. Since $\overline{\{e^{2\pi in\alpha} : n \in \mathbb{Z}\}} = S^1$, there exists a sequence (n_j) of integers such that $\lim_{j\to\infty} e^{2\pi in_j\alpha} = -1$. As h is continuous and equivariant, we have that for every $z \in S^{2k-1}$

$$h(-z) = h(\lim_{j\to\infty} e^{2\pi in_j\alpha}z) = \lim_{j\to\infty} e^{2\pi in_j\alpha}h(z) = -h(z).$$

Therefore, it follows from Theorem 2 that $k \leq l$. □

Thus Theorem 3 is a generalization of the Borsuk-Ulam theorem. The basic idea is to use the fact that $\overline{\{e^{2\pi in\alpha} : n \in \mathbb{Z}\}} = S^1$ if α is an irrational number. It will be shown in Theorem 10 more generally.

3. Extensions to finitely many products

By modifying Theorem 3, we can now obtain the following results on finitely many products of spheres. For $j = 1, \cdots, n$, let k_j, l_j be natural numbers, let $\mathcal{J} := \{j \in \{1, \cdots, n\} : k_j > l_j\} \neq \emptyset$, and let p_j, q_j be nonzero integers such that $\frac{p_j}{q_j}$ is in reduced form and $q_j \neq 1$.

THEOREM 4. *Let $\alpha_j = \frac{p_j}{q_j}$ be in reduced form for $j = 1, \cdots, n$. Suppose that there is a $j_0 \in \mathcal{J}$ such that the number q_{j_0} does not divide the least common multiple of $\{q_1, \cdots, q_{j_0-1}, q_{j_0+1}, \cdots, q_n\}$. Then there is no continuous equivariant map*

$$h : (S^{2k_1-1} \times \cdots \times S^{2k_n-1}, \tilde{\alpha}) \longrightarrow (S^{2l_1-1} \times \cdots \times S^{2l_n-1}, \tilde{\alpha}),$$

where $\tilde{\alpha} = (e^{2\pi i\alpha_1}, \cdots, e^{2\pi i\alpha_n})$.

Proof. Suppose that there is a continuous equivariant map h. Let $c := \mathrm{lcm}\{q_1, \cdots, q_{j_0-1}, q_{j_0+1}, \cdots, q_n\}$. Hence, for every $z = (z_1, \cdots, z_n) \in S^{2k_1-1} \times \cdots \times S^{2k_n-1}$, we obtain

$$\begin{aligned} & h(z_1, \cdots, z_{j_0-1}, e^{2\pi ic\alpha_{j_0}} z_{j_0}, z_{j_0+1}, \cdots, z_n) \\ &= h(e^{2\pi ic\alpha_1} z_1, \cdots, e^{2\pi ic\alpha_{j_0}} z_{j_0}, \cdots, e^{2\pi ic\alpha_n} z_n) \\ &= (e^{2\pi ic\alpha_1} h_1(z), \cdots, e^{2\pi ic\alpha_{j_0}} h_{j_0}(z), \cdots, e^{2\pi ic\alpha_n} h_n(z)) \\ &= (h_1(z), \cdots, h_{j_0-1}(z), e^{2\pi ic\alpha_{j_0}} h_{j_0}(z), h_{j_0+1}(z), \cdots, h_n(z)). \end{aligned}$$

Let $a_j \in S^{2k_j-1}$ be fixed for each $j \in \{1, \cdots, n\} \setminus \{j_0\}$. Define $\tilde{h} : S^{2k_{j_0}-1} \to S^{2l_{j_0}-1}$ by

$$\tilde{h}(x) := h_{j_0}(a_1, \cdots, a_{j_0-1}, x, a_{j_0+1} \cdots, a_n)$$

for $x \in S^{2k_{j_0}-1}$. Then $\tilde{h}$ is continuous and equivariant, because

$$\begin{aligned} \tilde{h}(e^{2\pi ic\alpha_{j_0}} x) &= e^{2\pi ic\alpha_{j_0}} h_{j_0}(a_1, \cdots, a_{j_0-1}, x, a_{j_0+1}, \cdots, a_n) \\ &= e^{2\pi ic\alpha_{j_0}} \tilde{h}(x). \end{aligned}$$

for every $x \in S^{2k_{j_0}-1}$. Since q_{j_0} does not divide c, by Theorem 3, this contradicts the fact that $j_0 \in \mathcal{J}$. □

We now turn our attention to the case when $\alpha_1, \cdots, \alpha_n$ are taken as rational and irrational numbers together.

THEOREM 5. *Suppose α_{j_1}, $j_1 \in \{1, \cdots, n\}$, is the only one irrational number and $\alpha_j = \frac{p_j}{q_j}$, $j \in \{1, \cdots, n\}\backslash\{j_1\}$, are rational numbers in reduced form.*

(1) *If $j_1 \in \mathcal{J}$, then there is no continuous equivariant map*

$$h : (S^{2k_1-1} \times \cdots \times S^{2k_n-1}, \beta) \to (S^{2l_1-1} \times \cdots \times S^{2l_n-1}, \tilde{\alpha}).$$

(2) *If $j_1 \notin \mathcal{J}$ and if the number q_{j_0} does not divide the least common multiple of $\{q_1, \cdots, q_n\} \setminus \{q_{j_0}, q_{j_1}\}$ with $j_0 \in \mathcal{J}$, then the same conclusion holds.*

Proof. For simplicity, we may suppose that α_1 is an irrational number and $\alpha_2, \cdots, \alpha_n$ are rational numbers. There are two cases to consider.

Case 1 : $1 \in \mathcal{J}$; that is $k_1 > l_1$. Assume that there is a continuous equivariant map h. Since $\alpha_1 \in \mathbb{R}\setminus\mathbb{Q}$, there exists a sequence (n_k) of integers such that

$$\lim_{k\to\infty} e^{2\pi i n_k q_2 \cdots q_n \alpha_1} = e^{2\pi i \alpha_1}.$$

Furthermore, $e^{2\pi i n_k q_2 \cdots q_n \alpha_j} = 1$ for $j = 2, \cdots, n$ and for all $k \in \mathbb{N}$. Since h is continuous and equivariant, it follows that

$$h(e^{2\pi i \alpha_1} z_1, z_2, \cdots, z_n) = (e^{2\pi i \alpha_1} h_1(z), h_2(z), \cdots, h_n(z))$$

for all $z = (z_1, \cdots, z_n) \in S^{2k_1-1} \times \cdots \times S^{2k_n-1}$. Hence the function $\tilde{h}_1 : S^{2k_1-1} \to S^{2l_1-1}$, $x \mapsto h_1(x, a_2, \cdots, a_n)$, for fixed $a_j \in S^{2k_j-1}$, $j = 2, \cdots, n$, is continuous and equivariant. By Theorem 3, this contradicts the fact that $k_1 > l_1$. Thus the proof of the statement (1) is complete.

Case 2 : $1 \notin \mathcal{J}$. Without loss of generality we may suppose that $2 \in \mathcal{J}$; that is $k_2 > l_2$. Assume that there is a continuous equivariant map h. Let c be the least common multiple of $\{q_3, \cdots, q_n\}$ such that q_2 does not divide c. Hence there exists a sequence (n_k) of integers such that

$$m_k := c(q_2 n_k + 1) \quad \text{and} \quad \lim_{k\to\infty} e^{2\pi i m_k \alpha_1} = 1.$$

It follows that

$$e^{2\pi i m_k \alpha_2} = e^{2\pi i c \alpha_2} \qquad \text{and} \qquad e^{2\pi i m_k \alpha_j} = 1 \quad \text{for } j = 3, \cdots, n.$$

Therefore, for all $z = (z_1, \cdots, z_n) \in S^{2k_1 - 1} \times \cdots \times S^{2k_n - 1}$, we have

$$\begin{aligned} & h(z_1, e^{2\pi i c \alpha_2} z_2, z_3, \cdots, z_n) \\ & = (h_1(z), e^{2\pi i c \alpha_2} h_2(z), h_3(z), \cdots, h_n(z)) \end{aligned}$$

which, as in the proof of Theorem 4, leads to a contradiction to $k_2 > l_2$. This proves the statement (2). □

Theorem 5 can be formulated in more general situation as follows.

THEOREM 6. *Let α_j be a real number for $j = 1, \cdots, n$. Suppose that for some $j_0 \in \mathcal{J}$, there exists a sequence (m_k) of integers such that*

$$\lim_{k \to \infty} e^{2\pi i m_k \alpha_j} = \begin{cases} e^{2\pi i \alpha_{j_0}} \neq 1 & \text{for } j = j_0 \\ 1 & \text{for } j \in \{1, \cdots, n\} \setminus \{j_0\} \end{cases}.$$

Then there is no continuous map

$$h: S^{2k_1 - 1} \times \cdots \times S^{2k_n - 1} \longrightarrow S^{2l_1 - 1} \times \cdots \times S^{2l_n - 1}$$

such that h is equivariant with respect to $\tilde{\alpha}$.

Proof. Without loss of generality we may suppose that $j_0 = 1$. Assume that there is a continuous equivariant map h. Then, for all $z = (z_1, \cdots, z_n) \in S^{2k_1 - 1} \times \cdots \times S^{2k_n - 1}$, we obtain

$$\begin{aligned} & h(e^{2\pi i \alpha_1} z_1, z_2, \cdots, z_n) \\ & = h(\lim_{k \to \infty} e^{2\pi i m_k \alpha_1} z_1, \cdots, \lim_{k \to \infty} e^{2\pi i m_k \alpha_n} z_n) \\ & = \lim_{k \to \infty} (e^{2\pi i m_k \alpha_1} h_1(z), \cdots, e^{2\pi i m_k \alpha_n} h_n(z)) \\ & = (e^{2\pi i \alpha_1} h_1(z), h_2(z), \cdots, h_n(z)). \end{aligned}$$

As in the proof of Theorem 5, a similar argument leads to a contadiction, and the proof is complete. □

REMARK. In fact, there is no sequence (m_k) of integers such that

$$\lim_{k\to\infty} e^{2\pi i m_k\alpha_1} = e^{2\pi i\alpha_1} \quad \text{and} \quad \lim_{k\to\infty} e^{2\pi i m_k\alpha_2} = 1$$

if $\alpha_1, \alpha_2 \in \mathbb{R}\backslash\mathbb{Q}$ have the property $c_1\alpha_1 + c_2\alpha_2 \in \mathbb{Q}$ for some $c_1, c_2 \in \mathbb{Q}\backslash\{0\}$.

We are now in position to study when there exists a sequence to satisfy the condition of Theorem 6. First, Remark leads us to the following definition.

DEFINITION 7. The n real numbers $\alpha_1, \cdots, \alpha_n$ will be called *rationally independent* if any relation of the form $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$ with rational numbers $c_1, \cdots, c_n$ implies that $c_1 = \cdots = c_n = 0$. Countably many real numbers $\alpha_1, \cdots, \alpha_n, \cdots$ will be called *rationally independent* if any finitely many numbers which belong to $\{\alpha_n : n \in \mathbb{N}\}$ are rationally independent.

Next, the following result is based on the main theorem of Kronecker.

LEMMA 8. *Let $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ be real numbers. If the numbers $1, \alpha_1, \cdots, \alpha_n$ are rationally independent, then the system of the n inequalities*

$$|\alpha_j x_{n+1} - x_j - \beta_j| < \epsilon \qquad \text{for } j = 1, \cdots, n$$

has solutions $x_1, \cdots, x_{n+1} \in \mathbb{Z}$ for any $\epsilon > 0$.

A proof of this lemma can be found in Perron [9].

Using Lemma 8, we can prove a result which gives a sufficient condition for the existence of a sequence stated in Theorem 6. It is also useful for a classification of the type of orbits, see [7].

THEOREM 9. *Let $1, \alpha_1, \cdots, \alpha_n$ be rationally independent. Then, for every $(s_1, \cdots, s_n) \in (S^1)^n$, there exists a sequence (m_k) of integers such that*

$$\lim_{k\to\infty} e^{2\pi i m_k\alpha_j} = s_j \qquad \text{for } j = 1, \cdots, n.$$

Moreover, $\overline{\{(e^{2\pi i m\alpha_1}, \cdots, e^{2\pi i m\alpha_n}) : m \in \mathbb{Z}\}} = (S^1)^n$.

Proof. Let $(s_1, \cdots, s_n) \in (S^1)^n$. Then there is a $\beta_j \in [0,1]$ with $s_j = e^{2\pi i \beta_j}$ for $j = 1, \cdots, n$. For every positive integer k there exists an $\epsilon_k > 0$ such that for all $x \in \mathbb{R}$ with $|x - \beta_j| < \epsilon_k \mod 1$ we have

$$|e^{2\pi i x} - e^{2\pi i \beta_j}| < 2^{-k} \qquad \text{for } j = 1, \cdots, n.$$

By Lemma 8, there are integers $m_k, y_{k1}, \cdots, y_{kn}$ such that

$$|m_k \alpha_j - y_{kj} - \beta_j| < \epsilon_k \qquad \text{for } j = 1, \cdots, n.$$

For $j = 1, \cdots, n$, since $|e^{2\pi i m_k \alpha_j} - e^{2\pi i \beta_j}| < 2^{-k}$, we conclude that

$$\lim_{k \to \infty} e^{2\pi i m_k \alpha_j} = e^{2\pi i \beta_j} = s_j.$$

□

4. An extension to countably many products

The following result is an analogue of Theorem 6 for countably many products of spheres.

THEOREM 10. *Let α_j be a real number and k_j, l_j natural numbers for every $j \in \mathbb{N}$. Suppose that for some $j_0 \in \{j \in \mathbb{N} : k_j > l_j\}$, there exists a sequence (n_k) of integers such that*

$$\lim_{k \to \infty} e^{2\pi i n_k \alpha_j} = \begin{cases} e^{2\pi i \alpha_{j_0}} \neq 1 & \text{for } j = j_0 \\ 1 & \text{for } j \in \mathbb{N} \setminus \{j_0\} \end{cases}.$$

Then there is no continuous map

$$h : \prod_{j \in \mathbb{N}} S^{2k_j - 1} \longrightarrow \prod_{j \in \mathbb{N}} S^{2l_j - 1}$$

which is equivariant with respect to $\tilde{\alpha} = (e^{2\pi i \alpha_1}, \cdots, e^{2\pi i \alpha_j}, \cdots)$.

Proof. As in the proof of Theorem 6, a similar argument establishes the result for countably many products. □

We can make a modification of Theorem 9 with the aid of a diagonal process.

THEOREM 11. *Let α_j, β_j be real numbers for every $j \in \mathbb{N}$. If $1, \alpha_1, \cdots, \alpha_j, \cdots$ are rationally independent, then there is a sequence (n_k) of integers such that*

$$\lim_{k \to \infty} e^{2\pi i n_k \alpha_j} = e^{2\pi i \beta_j} \qquad \text{for all } j \in \mathbb{N}.$$

Proof. Let $1, \alpha_1, \cdots, \alpha_j, \cdots$ be rationally independent. By Lemma 8, for every $n \in \mathbb{N}$, there is a sequence $(m_{n,k})$ of integers such that

$$| e^{2\pi i m_{n,k} \alpha_j} - e^{2\pi i \beta_j} | < 2^{-k} \qquad \text{for } j = 1, \cdots, n.$$

We now consider the sequence $(m_{k,k})$ on the diagonal. Then, for all $k \in \mathbb{N}$, we have

$$| e^{2\pi i m_{k,k} \alpha_j} - e^{2\pi i \beta_j} | < 2^{-k} \qquad \text{for } j = 1, \cdots, k$$

which implies that $\lim_{k \to \infty} e^{2\pi i m_{k,k} \alpha_j} = e^{2\pi i \beta_j}$ for all $j \in \mathbb{N}$. □

Here we have seen that the Borsuk-Ulam theorem is preserved under certain conditions related to the number theory. The generalized Borsuk-Ulam theorem is fundamental in measuring complexity of almost periodic orbits.

5. Application

In this section we give a definition of a $\mathbb{Z}$-index induced by a homeomorphism of a compact space. Then a $\mathbb{Z}$-version of the Borsuk-Ulam theorem plays a fundamental role in an index theory for $\mathbb{Z}$-actions. Moreover, an index theory for group actions is important from the viewpoint of applications to differential equations, see [1], [2], [6] and [8].

Given a continuous action $\pi : G \times X \to X$ of a topological group G on a topological space X, we denote

$$\Sigma(X, G) := \{A \subset X : A \text{ is } G\text{-invariant}\}.$$

A *G-index* is a mapping

$$i : \Sigma(X, G) \to \mathbb{N} \cup \{0, \infty\}$$

which has the following properties

(1) $i(A) = 0$ if and only if $A = \emptyset$.
(2) If $A, B \in \Sigma(X, G)$ and $\Phi : A \to B$ is a continuous equivariant map, then $i(A) \leq i(B)$.
(3) If $A \in \Sigma(X, G)$ is a closed set, then there exists an open neighborhood $U \in \Sigma(X, G)$ of A such that $i(A) = i(U)$.
(4) If $A, B \in \Sigma(X, G)$ are closed sets, then $i(A \cup B) \leq i(A) + i(B)$.

We consider a compact topological space X and a homeomorphism $f : X \to X$ such that

(1) $\{f^n : n \in \mathbb{Z}\}$ is uniformly equicontinuous; and
(2) there is an irrational number $\alpha \in [0, 1]$ such that for every $x \in X$ there exists a homeomorphism $\Phi : \overline{O(x)} \to S^1$ with the property

$$\Phi(f(u)) = e^{2\pi i\alpha}\Phi(u) \quad \text{for all } u \in \overline{O(x)}$$

where $\overline{O(x)}$ denotes the closure of $O(x) = \{f^n(x) : n \in \mathbb{Z}\}$.

In this framework we define the $\mathbb{Z}$-index $i(A)$ of an invariant subset A of X as the smallest integer k such that there exist an $m \in \mathbb{N}$ and a continuous map $\Phi : A \to S^{2k-1}$ satisfying the following equivariance property

$$\Phi(f(u)) = e^{2\pi i m\alpha}\Phi(u) \quad \text{for all } u \in A.$$

In [7] we show that this is an index in the sense of the above definition.

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Department of Mathematics
Sung Kyun Kwan University
Suwon 440-746, Korea