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A CLASS OF THE OPERATOR-VALUED FEYNMAN INTEGRAL

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ABSTRACT. We investigate the existence of the operator-valued Feynman integral when a Wiener functional is given by a Fourier transform of complex Borel measure.

1. Introduction

Fix $t > 0$. Denote by C^t the space of $\mathbb{R}^N$-valued continuous functions x on $[0, t]$. Let C_0^t be the Wiener space - $C_0^t = \{x \in C^t : x(0) = 0\}$ - equipped with Wiener measure m . Let F be a function from C^t to $\mathbb{C}$. Given $\lambda > 0, \psi \in L^2(\mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$, let

$$(K_\lambda(F)\psi)(\xi) = \int_{C_0^t} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)\, dm(x). \tag{1.1}$$

DEFINITION. The operator- valued function space integral $K_\lambda(F)$ exists for $\lambda > 0$ if (1.1) defines $K_\lambda(F)$ as a bounded linear operator on $L^2(\mathbb{R}^N)$. If, in addition, the operator-valued function $K_\lambda(F)$, as a function of λ, has an extension to an analytic function in $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re\lambda > 0\}$ and a strongly continuous function in $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : Re\lambda \geq 0, \lambda \neq 0\}$, we say that $K_\lambda(F)$ exists for $\lambda \in \tilde{\mathbb{C}}_+$. When λ is purely imaginary, $K_\lambda(F)$ is called the operator-valued Feynman integral of F.

For $s > 0, \lambda \in \tilde{\mathbb{C}}_+$ and $\psi \in L^2(\mathbb{R}^N)$, let

$$(\exp[-s(H_0/\lambda)]\psi)(\xi) = (\frac{\lambda}{2\pi s})^{\frac{N}{2}} \int_{\mathbb{R}^N} \psi(u) \exp(-\frac{\lambda||u - \xi||^2}{2s})\, du. \tag{1.2}$$

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The integral in (1.2) exists as an ordinary Lebesgue integral for $\lambda \in \mathbb{C}_+$, but, when λ is purely imaginary and ψ is not integrable, the integral should be interpreted in the mean as in the theory of the Fourier-Plancherel transform.

$M(0,t)$ will denote the space of complex Borel measures η on $(0,t)$. Then $\eta \in M(0,t)$ has a unique decomposition $\eta = \mu + \eta_d$ into a continuous part μ and a discrete part η_d [6].

Let $\eta \in M(0,t)$. A $\mathbb{C}$ -valued, Borel measurable function θ on $(0,t) \times \mathbb{R}^N$ is said to belong to $L_{\infty 1;\eta}$ if

$$(1.3) \qquad ||\theta||_{\infty 1;\eta} := \int_{(0,t)} ||\theta(s,\cdot)||_\infty \, d|\eta|(s) < \infty.$$

Let $M(\mathbb{C})$ be the space of complex Borel measures on $\mathbb{C}$. The Fourier transform of $\nu \in M(\mathbb{C})$ is the function $\hat{\nu}$ defined by

$$(1.4) \qquad \hat{\nu}(u) = \int_{\mathbb{C}} e^{-iuv} \, d\nu(v), \qquad u \in \mathbb{C}$$

Consider the functional

$$(1.5) \qquad F(x) = \hat{\nu}\Big(\int_{(0,t)} \theta(s, x(s)) \, d\eta(s)\Big), \qquad x \in C^t.$$

2. A existence theorem

A first result is easily proved and is essentially contained in [1].

LEMMA. *For every $\lambda > 0$, $K_\lambda(F)$ is a bounded linear operator on $L^2(\mathbb{R}^N)$. In fact,*

$$(2.1) \qquad ||K_\lambda(F)|| \leq ||\nu||.$$

Let $\eta \in M(0,t)$. As usual, $\eta = \mu + \eta_d$ will be the decomposition of η into its continuous and discrete parts. The case where η_d has finite support is most likely to be of interest. Hence we restrict ourselves to that setting.

THEOREM 1. *Let μ be a continuous measure in $M(0,t)$ and let*

$$\eta_d = \sum_{n=1}^{N} \omega_j \delta_{\tau_j}$$

where δ_{τ_j} is as usual the Dirac measure at $\tau_j \in (0,t)$, $0 < \tau_1 < \cdots < \tau_N < t$ and $\omega_j \in \mathbb{C}$ for $j = 1, 2, \ldots, N$.

Then for every $\lambda > 0$ the operator $K_\lambda(F)$ on $L^2(\mathbb{R}^N)$ whose existence was established in Lemma is given by the formula

(2.2)
$$K_\lambda(F) = \int_{\mathbb{C}} \sum_{n=0}^{\infty} (-i)^n u^n \Bigg[\sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}} L_0 L_1 \cdots L_N \, d\mu(s_1)\cdots d\mu(s_{q_0}) \Bigg] d\nu(u)$$

where $q_0, \ldots, q_N, k_1, \ldots, k_{N+1}$ are nonnegative integers,

(2.3)
$$\begin{aligned}\Delta_{q_0;k_1,\ldots,k_{N+1}} = \{(s_1,\ldots,s_{q_0}) \in (0,t)^{q_0} : 0 < s_1 < \cdots < s_{k_1} \\ < \tau_1 < s_{k_1+1} < \cdots < s_{k_1+k_2} < \tau_2 < s_{k_1+k_2+1} < \cdots \\ < s_{k_1+\cdots+k_N} < \tau_N < s_{k_1+\cdots+k_N-1} < \cdots < s_{q_0} < t\}\end{aligned}$$

and, for $(s_1,\ldots,s_{q_0}) \in \Delta_{q_0;k_1,\ldots,k_{N+1}}$ and $r \in \{0,1,\ldots,N\}$

(2.4)
$$\begin{aligned}L_r = [\theta(\tau_r)]^{q_r} e^{-(s_{k_1+\cdots+k_r+1}-\tau_r)(H_0/\lambda)} \theta(s_{k_1+\cdots+k_r+1}) \\ e^{-(s_{k_1+\cdots+k_r+2}-s_{k_1+\cdots+k_r+1})(H_0/\lambda)} \theta(s_{k_1+\cdots+k_r+2}) \cdots \\ \theta(s_{k_1+\cdots+k_{r+1}}) e^{-(\tau_{r+1}-s_{k_1+\cdots+k_{r+1}})(H_0/\lambda)}\end{aligned}$$

We use the conventions $\tau_0 = 0, \tau_{N+1} = t$ and $[\theta(\tau_0)]^{q_0} = 1$.

Proof. Let $\lambda > 0$ and $\psi \in L^2(\mathbb{R}^N)$ be given. Set

$$\Delta_{q_0} = \{(s_1,\ldots,s_{q_0}) \in (0,t)^{q_0} : 0 < s_1 < \cdots < s_{q_0} < t\}. \tag{2.5}$$

We will go through the formal argument which establishes (2.6), numbering steps along the way. Finally, we explain the numbered steps. For Leb.-a.e. $\xi \in \mathbb{R}^N$,

(2.6)

$$
\begin{aligned}
&(K_\lambda(F)\psi)(\xi) \\
&= \int_{C_0} \hat{\nu}\Big(\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi) d\eta(s)\Big) \psi(\lambda^{-\frac{1}{2}} x(t) + \xi)\, dm(x) \\
&= \int_{C_0} \Big\{ \int_{\mathbb{C}} e^{-iu \int_{(0,t)} \theta(s,\lambda^{-\frac{1}{2}} x(s)+\xi) d\eta(s)} d\nu(u) \Big\} \psi(\lambda^{-\frac{1}{2}} x(t) + \xi)\, dm(x) \\
&\overset{(I)}{=} \int_{\mathbb{C}} \Big[\int_{C_0} e^{-iu \int_{(0,t)} \theta(s,\lambda^{-\frac{1}{2}} x(s)+\xi) d\eta(s)} \psi(\lambda^{-\frac{1}{2}} x(t) + \xi)\, dm(x) \Big] d\nu(u) \\
&= \int_{\mathbb{C}} \Big[\int_{C_0} \sum_{n=0}^{\infty} \frac{(-iu)^n}{n!} \Big\{ \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi) d\eta(s) \Big\}^n \\
&\quad \psi(\lambda^{-\frac{1}{2}} x(t) + \xi) dm(x) \Big] d\nu(u) \\
&\overset{(II)}{=} \int_{\mathbb{C}} \sum_{n=0}^{\infty} \frac{(-i)^n u^n}{n!} \Big[\int_{C_0} \Big\{ \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi) d\eta(s) \Big\}^n \\
&\quad \psi(\lambda^{-\frac{1}{2}} x(t) + \xi) dm(x) \Big] d\nu(u) \\
&\overset{(III)}{=} \int_{\mathbb{C}} \sum_{n=0}^{\infty} \frac{(-i)^n u^n}{n!} \Big[\int_{C_0} \Big\{ \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi) d\mu(s) + \\
&\quad \sum_{j=1}^{N} \omega_j \theta(\tau_j, \lambda^{-\frac{1}{2}} x(\tau_j) + \xi) \Big\}^n \psi(\lambda^{-\frac{1}{2}} x(t) + \xi) dm(x) \Big] d\nu(u) \\
&= \int_{\mathbb{C}} \sum_{n=0}^{\infty} \frac{(-i)^n u^n}{n!} \Big[\int_{C_0} \sum_{q_0+q_1+\cdots+q_N=n} \frac{n!}{q_0! q_1! \cdots q_N!} \\
&\quad \Big\{ \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi) d\mu(s) \Big\}^{q_0} \{\omega_1 \theta(\tau_1, \lambda^{-\frac{1}{2}} x(\tau_1) + \xi)\}^{q_1} \cdots \\
&\quad \{\omega_N \theta(\tau_N, \lambda^{-\frac{1}{2}} x(\tau_N) + \xi)\}^{q_N} \psi(\lambda^{-\frac{1}{2}} x(t) + \xi) dm(x) \Big] d\nu(u) \\
&= \int_{\mathbb{C}} \sum_{n=0}^{\infty} \frac{(-i)^n u^n}{n!} \Big[\int_{C_0} \sum_{q_0+q_1+\cdots+q_N=n} \frac{n! \omega_1^{q_1} \cdots \omega_N^{q_N}}{q_0! q_1! \cdots q_N!} \\
&\quad \int_{(0,t)^{q_0}} \theta(s_1, \lambda^{-\frac{1}{2}} x(s_1) + \xi) \theta(s_2, \lambda^{-\frac{1}{2}} x(s_2) + \xi) \ldots \theta(s_{q_0}, \lambda^{-\frac{1}{2}} x(s_{q_0}) + \xi) \\
&\quad d\mu(s_1) \cdots d\mu(s_{q_0}) \prod_{j=1}^{N} \{\theta(\tau_j, \lambda^{-\frac{1}{2}} x(\tau_j) + \xi)\}^{q_j} \psi(\lambda^{-\frac{1}{2}} x(t) + \xi) dm(x) \Big] d\nu(u)
\end{aligned}
$$

$$\overset{(IV)}{=} \int_{\mathbb{C}} \sum_{n=0}^{\infty} \frac{(-i)^n u^n}{n!} \Big[\int_{C_0} \sum_{q_0+\cdots+q_N=n} \frac{n!\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_0!q_1!\cdots q_N!}$$

$$q_0! \int_{\Delta_{q_0}} \theta(s_1, \lambda^{-\frac{1}{2}}x(s_1)+\xi)\theta(s_2, \lambda^{-\frac{1}{2}}x(s_2)+\xi)\cdots\theta(s_{q_0}, \lambda^{-\frac{1}{2}}x(s_{q_0})+\xi)$$

$$d\mu(s_1)\cdots d\mu(s_{q_0}) \prod_{j=1}^{N} \{\theta(\tau_j, \lambda^{-\frac{1}{2}}x(\tau_j)+\xi)\}^{q_j}\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)dm(x)\Big] d\nu(u)$$

$$\overset{(V)}{=} \int_{\mathbb{C}} \sum_{n=0}^{\infty} (-i)^n u^n \Big[\int_{C_0} \sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1}\omega_2^{q_2}\cdots\omega_N^{q_N}}{q_1!q_2!\cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0}$$

$$\int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} \theta(s_1, \lambda^{-\frac{1}{2}}x(s_1)+\xi)\cdots\theta(s_{q_0}, \lambda^{-\frac{1}{2}}x(s_{q_0})+\xi)\, d\mu(s_1)\cdots d\mu(s_{q_0})$$

$$\prod_{j=1}^{N} \{\theta(\tau_j, \lambda^{-\frac{1}{2}}x(\tau_j)+\xi)\}^{q_j}\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)dm(x)\Big] d\nu(u)$$

$$\overset{(VI)}{=} \int_{\mathbb{C}} \sum_{n=0}^{\infty} (-i)^n u^n \Big[\sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}}$$

$$\Big\{ \int_{C_0} \theta(s_1, \lambda^{-\frac{1}{2}}x(s_1)+\xi)\cdots\theta(s_{k_1}, \lambda^{-\frac{1}{2}}x(s_{k_1})+\xi)[\theta(\tau_1, \lambda^{-\frac{1}{2}}x(\tau_1)+\xi)]^{q_1}$$

$$\theta(s_{k_1+1}, \lambda^{-\frac{1}{2}}x(s_{k_1+1})+\xi)\cdots\theta(s_{k_1+k_2}, \lambda^{-\frac{1}{2}}x(s_{k_1+k_2})+\xi)[\theta(\tau_2, \lambda^{-\frac{1}{2}}x(\tau_2)+\xi)]^{q_2}$$

$$\cdots\theta(s_{k_1+\cdots+k_r}, \lambda^{-\frac{1}{2}}x(s_{k_1+\cdots+k_r})+\xi)[\theta(\tau_r, \lambda^{-\frac{1}{2}}x(\tau_r)+\xi)]^{q_r}$$

$$\theta(s_{k_1+\cdots+k_r+1}, \lambda^{-\frac{1}{2}}x(s_{k_1+\cdots+k_r+1})+\xi)\cdots\theta(s_{k_1+\cdots+k_N}, \lambda^{-\frac{1}{2}}x(s_{k_1+\cdots+k_N})+\xi)$$

$$[\theta(\tau_N, \lambda^{-\frac{1}{2}}x(\tau_N)+\xi)]^{q_N}\theta(s_{k_1+\cdots+k_N+1}, \lambda^{-\frac{1}{2}}x(s_{k_1+\cdots+k_N+1})+\xi)\cdots$$

$$\theta(s_{q_0}, \lambda^{-\frac{1}{2}}x(s_{q_0})+\xi)\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)dm(x)\Big\}\, d\mu(s_1)\cdots d\mu(s_{q_0})\Big] d\nu(u)$$

$$\overset{(VII)}{=} \int_{\mathbb{C}} \sum_{n=0}^{\infty} (-i)^n u^n \Big[\sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}}$$

$$(L_0L_1\cdots L_N\psi)(\xi)d\mu(s_1)\cdots d\mu(s_{q_0})\Big] d\nu(u).$$

Since

$$\int_{C_0}\Big\{\int_{\mathbb{C}} |e^{-iu\int_{(0,t)}\theta(s,\lambda^{-\frac{1}{2}}x(s)+\xi)\,d\eta(s)}|\, d|\nu|(u)\Big\}|\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)|\, dm(x)$$
$$\le ||\nu|| \int_{C_0} |\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)|\, dm(x)$$
$$= ||\nu||(e^{-t(H_0/\lambda)}|\psi|)(\xi)$$

step(I) follows from Fubini's Theorem.

$$\int_{C_0}\sum_{n=0}^{\infty}\left|\frac{(-iu)^n}{n!}\right|\left|\left\{\int_{(0,t)}\theta(s,\lambda^{-\frac{1}{2}}x(s)+\xi)d\eta(s)\right\}^n\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)\right|dm(x)$$

$$(2.7)\qquad \le\int_{C_0}\sum_{n=0}^{\infty}\frac{\|\theta\|_{\infty 1;\eta}^n|u|^n}{n!}|\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)|dm(x)$$

$$=e^{\|\theta\|_{\infty 1;\eta}|u|}\int_{C_0}|\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)|dm(x)<\infty.$$

Since $|\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)|$ is Wiener integrable, (II) follows. Step (III) results from writing η as $\mu+\sum_{j=1}^{N}\omega_j\delta_{\tau_j}$ and carrying out the integral with respect to $\omega_j\delta_{\tau_j}$ for $j=1,2,\ldots,N$. Step (IV) follows from the simplex trick. Since μ is a continuous measure, one can see by a sectioning argument that

$$\Delta_{q_0}=\cup_{k_1+\cdots+k_{N+1}=q_0}\Delta_{q_0;k_1,\ldots,k_{N+1}}$$

except for a set of $\mu\times\cdots\times\mu$ measure zero. So, step (V) follows. Step (VI) follows from Fubini's theorem. Using Wiener's integration formula, a simple change of variables and (1.2), a calculation yields (2.2). □

It is not possible in general in Theorem 1 to take the sum over n outside the integral with respect to ν. However, with the assumption the measure $|\nu|$ dies off rapidly at ∞, this can be done.

THEOREM 2. *Let $\eta=\mu+\sum_{j=1}^{N}\omega_j\delta_{\tau_j}$ where $\mu\in M(0,t)$ is continuous, $0<\tau_1<\cdots<\tau_N<t$ and $\omega_j\in\mathbb{C}$ for $j=1,2,\ldots,N$. Also let $\nu\in M(\mathbb{C})$ be such that*

$$(2.8)\qquad \int_{\mathbb{C}}e^{\|\theta\|_{\infty 1;\eta}|u|}d|\nu|(u)<\infty.$$

Then for every $\lambda>0$, the operator $K_\lambda(F)$ on $L^2(\mathbb{R}^N)$ which is given by the series (2.2) from Theorem 1, can also be expressed as the time-ordered operator norm convergent series

(2.9)

$$K_\lambda(F)=\sum_{n=0}^{\infty}n!a_n\sum_{q_0+\cdots+q_N=n}\frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N!}\sum_{k_1+\cdots+k_{N+1}=q_0}$$

$$\int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}}L_0L_1\cdots L_N\,d\mu(s_1)\cdots d\mu(s_{q_0})$$

where $q_0, \ldots, q_N, k_1, \ldots, k_{N+1}$ *are nonnegative integers, and* $\Delta_{q_0:k_1,\cdots,k_{N+1}}$ *is given by (2.3); for* $(s_1, \cdots, s_{q_0}) \in \Delta_{q_0:k_1,\cdots,k_{N+1}}$ *and* $r \in \{0, \cdots, N\}$ L_r *is given by (2.4) and*

$$a_n = \frac{1}{n!}\int_{\mathbb{C}} (-i)^n u^n \, d\nu(u). \tag{2.10}$$

The series (2.9) also converges Leb-a.e. on $\mathbb{R}$, *that is, for every* $\psi \in L^2(\mathbb{R}^N)$ *and Leb-a.e.* ξ,

(2.11)
$$(K_\lambda(F)\psi)(\xi) = \sum_{n=0}^{\infty} n! a_n \sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N!}\int_{\Delta_{q_0:k_1,\cdots,k_{N+1}}} (L_0\cdots L_N\psi)(\xi)\, d\mu(s_1)\cdots d\mu(s_{q_0})$$

Proof. Let $\lambda > 0$ and $\psi \in L^2(\mathbb{R}^N)$ be given. By the previous theorem and (II) of (2.6) we know that for Leb-a.e. $\xi \in \mathbb{R}^N$,

(2.12)
$$(K_\lambda(F)\psi)(\xi) = \int_{\mathbb{C}} \sum_{n=0}^{\infty} \frac{(-i)^n u^n}{n!}\left[\int_{C_0}\left\{\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}}x(s)+\xi)\, d\eta(s)\right\}^n \psi(\lambda^{-\frac{1}{2}}x(t)+\xi)\, dm(x)\right] d\nu(u)$$

However,
(2.13)
$$\begin{aligned}
&\int_{\mathbb{C}} \sum_{n=0}^{\infty}\left|\frac{(-i)^n u^n}{n!}\right|\left|\int_{C_0}\left\{\int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}}x(s)+\xi)\, d\eta(s)\right\}^n \psi(\lambda^{-\frac{1}{2}}x(t)+\xi)\, dm(x)\right| d|\nu|(u) \\
&\leq \int_{\mathbb{C}} \sum_{n=0}^{\infty} \frac{|u|^n}{n!}||\theta||_{\infty 1;\eta}^n \int_{C_0} |\psi(\lambda^{-\frac{1}{2}}x(t)-\xi)|\, dm(x)\, d|\nu|(u) \\
&= \int_{C_0} |\psi(\lambda^{-\frac{1}{2}}x(t)+\xi)|\, dm(x) \int_{\mathbb{C}} e^{||\theta||_{\infty 1;\eta}|u|}\, d|\nu|(u) < \infty
\end{aligned}$$

where the last inequality comes from (2.8.)

Now (2.13) permits us to use Fubini's theorem and we obtain (2.11) as follow;

(2.14)

$$(K_\lambda(F)\psi)(\xi) = \int_{\mathbb{C}} \sum_{n=0}^{\infty} (-i)^n u^n \Big[\sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1} \cdots \omega_N^{q_N}}{q_1! \cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0}$$

$$\int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}} (L_0 \cdots L_N\psi)(\xi)\, d\mu(s_1) \cdots d\mu(s_{q_0}) \Big] \, d\nu(u)$$

$$= \sum_{n=0}^{\infty} \int_{\mathbb{C}} (-i)^n u^n \Big[\sum_{q_0+\cdots q_N=n} \frac{\omega_1^{q_1} \cdots \omega_N^{q_N}}{q_1! \cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0}$$

$$\int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}} (L_0 \cdots L_N\psi)(\xi)\, d\mu(s_1) \cdots d\mu(s_{q_0}) \Big] \, d\nu(u)$$

$$= \sum_{n=0}^{\infty} n! a_n \sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1} \cdots \omega_N^{q_N}}{q_1! \cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0}$$

$$\int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}} (L_0 \cdots L_N\psi)(\xi)\, d\mu(s_1) \cdots d\mu(s_{q_0}).$$

It remains only to establish the operator norm convergence of the series (2.9). This comes from the following calculation;
(2.15)

$$\sum_{n=0}^{\infty} n!|a_n| \sum_{q_0+\cdots+q_N=n} \frac{|\omega_1|^{q_1} \cdots |\omega_N|^{q_N}}{q_1! \cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0} \Big\| \int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}}$$

$$L_0 \cdots L_N \, d\mu(s_1) \cdots d\mu(s_{q_0}) \Big\|$$

$$\leq \sum_{n=0}^{\infty} n! \frac{1}{n!} \int_{\mathbb{C}} |u|^n \, d|\nu|(u) \sum_{q_1+\cdots+q_N=n} \frac{|\omega_1|^{q_1} \cdots |\omega_N|^{q_N}}{q_1! \cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0}$$

$$\int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}} \|L_0 \cdots L_N\| \, d|\mu|(s_1) \cdots d|\mu|(s_{q_0})$$

$$= \int_{\mathbb{C}} \sum_{n=0}^{\infty} |u|^n \Big[\sum_{q_0+\cdots+q_N=n} \frac{|\omega_1|^{q_1} \cdots |\omega_N|^{q_N}}{q_1! \cdots q_N!} \sum_{k_1+\cdots-k_{N+1}=q_0}$$

$$\int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}} \|L_0 \cdots L_N\| d|\mu|(s_1) \cdots d|\mu|(s_{q_0}) \Big] d|\nu|(u)$$

$$\leq \int_{\mathbb{C}} \sum_{n=0}^{\infty} |u|^n \Big[\sum_{q_0+\cdots+q_N=n} \frac{|\omega_1|^{q_1} \cdots |\omega_N|^{q_N}}{q_1! \cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0}$$

$$\int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}} ||L_0|| \cdots ||L_N|| d|\mu|(s_1) \cdots d|\mu|(s_{q_0}) \Big] d|\nu|(u)$$

$$= \int_{\mathbb{C}} \sum_{n=0}^{\infty} |u|^n \Big[\sum_{q_0+\cdots+q_N=n} \frac{|\omega_1|^{q_1} \cdots |\omega_N|^{q_N}}{q_1! \cdots q_N!}$$

$$\int_{\Delta_{q_0}} ||L_0|| \cdots ||L_N|| d|\mu|(s_1) \cdots d|\mu|(s_{q_0}) \Big] d|\nu|(u)$$

$$= \int_{\mathbb{C}} \sum_{n=0}^{\infty} |u|^n \Big[\sum_{q_0+\cdots+q_N=n} \frac{|\omega_1|^{q_1} \cdots |\omega_N|^{q_N}}{q_1! \cdots q_N!}$$

$$\{ \int_{\Delta_{q_0}} ||\theta(s_1)|| \cdots ||\theta(s_{q_0})|| \, d|\mu|(s_1) \cdots d|\mu|(s_{q_0}) \} ||\theta(\tau_1)||^{q_1} \cdots ||\theta(\tau_N)||^{q_N} \Big] d|\nu|(u)$$

$$= \int_{\mathbb{C}} \sum_{n=0}^{\infty} |u|^n \Big[\sum_{q_0+\cdots+q_N=n} \frac{1}{q_1! \cdots q_N!} \frac{1}{q_0!}$$

$$\{ \int_{(0,t)} ||\theta(s)|| \, d|\mu|(s) \}^{q_0} (|\omega_1| ||\theta(\tau_1)||)^{q_1} \cdots (|\omega_N| ||\theta(\tau_N)||)^{q_N} \Big] \, d|\nu|(u)$$

$$= \int_{\mathbb{C}} \sum |u|^n \frac{1}{n!} \Big[\sum_{q_0+\cdots+q_N=n} \frac{n! ||\theta||_{\infty 1;\mu}^{q_0} (|\omega_1| ||\theta(\tau_1)||)^{q_1} \cdots (|\omega_{q_N}| ||\theta(\tau_N)||)^{q_0}}{q_0! q_1! \cdots q_N!} \Big] \, d|\nu|(u)$$

$$= \int_{\mathbb{C}} \sum_{n=0}^{\infty} \frac{1}{n!} |u|^n ||\theta||_{\infty 1;\eta}^{n} \, d|\nu|(u)$$

$$= \int_{\mathbb{C}} e^{||\theta||_{\infty 1;\eta} |u|} d|\nu|(u) \; < \; \infty$$

□

Under the assumption (2.8) that ν dies off rapidly at ∞, we show that $K_\lambda(F)$ exists and is given by the series (2.9) for all $\lambda \in \tilde{\mathbb{C}}_+$.

THEOREM 3. *Let $\eta = \mu + \sum_{j=1}^{N} \omega_j \delta_{\tau_j}$ where $\mu \in M(0,t)$ is continuous, $0 < \tau_1 < \cdots < \tau_N < t$ and $\omega_j \in \mathbb{C}$ for $j = 1, 2, \ldots, N$. Also let $\nu \in M(\mathbb{C})$ be such that*

$$\int_{\mathbb{C}} e^{||\theta||_{\infty 1;\eta} |u|} \, d|\nu|(u) < \infty. \tag{2.16}$$

Then the operator $K_\lambda(F)$ exists for all $\lambda \in \tilde{\mathbb{C}}_+$ and is given by (2.9).

Proof. From (2.9) for $\lambda > 0$,

(2.17)

$$K_\lambda(F) = \sum_{n=0}^{\infty} n! a_n \sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N} \sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}} L_0 L_1 \cdots L_N \, d\mu(s_1)\cdots d\mu(s_{q_0})$$

where $\Delta_{q_0;k_1,\ldots,k_{N+1}}$ is given by (2.3), L_r is given by (2.4) and a_n is given by (2.10)

$L_0 L_1 \cdots L_N$ depends on λ and we indicate this dependence now by writing $L_0 \cdots L_N = (L_0 \cdots L_N)(\lambda; s_1, \ldots, s_{q_0})$. For $s > 0$, as a function of λ, $e^{-s(H_0/\lambda)}$ is analytic in $\mathbb{C}_+$ and strongly continuous in $\tilde{\mathbb{C}}_+$. Next, noting that the multiplication operators involved in $(L_0 L_1 \cdots L_N)(\lambda; s_1, \ldots, s_{q_N})$ are independent of λ, we see that $\mu \times \cdots \times \mu - a.e.$ $(s_1, \cdots, s_{q_0}) \in \Delta_{q_0;k_1,\ldots,k_{N+1}}$, $L_0 \cdots L_N$ is strongly continuous for $\lambda \in \tilde{\mathbb{C}}_+$ and analytic for $\lambda \in \mathbb{C}_+$.

Since $\|e^{-s(H_0/\lambda)}\| \le 1$ for $\lambda \in \mathbb{C}_+$ and $s > 0$, we see that, for all $\lambda \in \tilde{\mathbb{C}}_+$ and $\mu \times \cdots \times \mu$ -a.e. $(s_1, \ldots s_{q_0})$,

$$\|L_0 L_1 \cdots L_N\| \le \Big(\prod_{k=1}^{q_0} \|\theta(s_k)\|\Big)\Big(\prod_{j=1}^{N} \|\theta(\tau_j)\|^{q_j}\Big). \tag{2.18}$$

The right-hand side of (2.18) is $\mu \times \cdots \times \mu$-integrable since $\theta \in L_{\infty 1;\eta}$. It thus from Lemma 0.2 of [5] which deals with Bochner integrals depending on a parameter that

$$G(\lambda) := \int_{\Delta_{q_0;k_1,\ldots,k_{N+1}}} (L_0 L_1 \cdots L_N)(\lambda; s_1, \ldots, s_{q_0}) d\mu(s_1)\cdots d\mu(s_{q_0})$$

is strongly continuous in $\tilde{\mathbb{C}}_+$ and analytic in $\mathbb{C}_+$.

For $\lambda \in \tilde{\mathbb{C}}_+$, from the norm inequality (2.15) the right-hand side of (2.17) converges uniformly. So we conclude that $K_\lambda(F)$ exists and is given by (2.9). □

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