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AN INEQUALITY OF SUBHARMONIC FUNCTIONS

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Abstract. We prove a norm inequality of the form $\|v\|_p \le (r-1)\|u\|_p$, $1 < p < \infty$, between a non-negative subharmonic function u and a smooth function v satisfying $|v(0)| \le u(0)$, $|\nabla v| \le |\nabla u|$ and $|\Delta v| \le \alpha \Delta u$, where α is a constant with $0 \le \alpha \le 1$. This inequality extends Burkholder's inequality where $\alpha = 1$.

1. Introduction

Let $T = \{z \in \mathbb{C} : |z| = 1\}$ and μ be the normalized Lebesgue measure on T, that is, $\mu(T) = 1$. Let $1 < p < \infty$. For each $f \in L^p(\mu)$ one can consider its conjugate function g defined by the following steps:

(a) First solve the Dirichlet problem to get a harmonic function u on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ such that

$$\lim_{r \uparrow 1} u(rt) = f(t)$$

for almost all $t \in T$ and

$$\lim_{r \uparrow 1} \int_T |u(rt) - f(t)|^p \, d\mu(t) = 0.$$

(b) Find the conjugate harmonic function v of u on D; that is, $u + iv$ is analytic on D and $v(0) = 0$.

(c) It is well known that $v(rt)$ has radial limit for almost all $t \in T$. We write

$$\lim_{r \uparrow 1} v(rt) = g(t).$$

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For the basic facts of harmonic analysis used in the above one may refer to [7].

In the beginning of the 20th century it was a hot issue whether or not $g \in L^p(\mu)$. The question was answered by M. Riesz:

THEOREM 1.1. (Riesz, [5]) *For $1 < p < \infty$ there is a constant c_p such that*

$$\int_T |g(t)|^p \, d\mu(t) \le c_p \int_T |f(t)|^p \, d\mu(t)$$

whenever $f \in L^p(\mu)$ and g is the conjugate function of f.

Later Burkholder studied the conjugate functions in terms of harmonic functions. The following is a special case of Burkholder's inequality. For $1 < p < \infty$ we set $p^* = \max\{p, p/(p-1)\}$.

THEOREM 1.2. *Let $1 < p < \infty$ and $\rho > 1$. If u and v are harmonic functions on the disk $D_\rho = \{z \in \mathbb{C} : |z| < \rho\}$ such that*

(i) $$|v(0)| \le |u(0)|,$$
(ii) $$|\nabla v| \le |\nabla u| \quad on \quad D_\rho,$$

then

$$\int_T |v(t)|^p \, d\mu(t) \le (p^* - 1)^p \int_T |u(t)|^p \, d\mu(t).$$

It is simple to check that Theorem 1.2 implies Theorem 1.1. Indeed, let $1 < p < \infty$, $f \in L^p(\mu)$ and u, v and g be as in (a), (b) and (c). For each $0 < r < 1$ we consider harmonic functions u_r and v_r on the disk $\{z \in \mathbb{C} : |z| < 1/r\}$ given by $u_r = u(rz)$ and $v_r(z) = v(rz)$. Clearly u_r and v_r satisfy (i) and (ii); in fact, the Cauchy-Riemann equations give $|\nabla v_r| = |\nabla u_r|$. Thus from Theorem 1.2 we get

$$\int_T |v(rt)|^p \, d\mu(t) \le (p^* - 1)^p \int_T |u(rt)|^p \, d\mu(t).$$

Now let $r \uparrow 1$ and use the facts from (a) and (c), Fatou's lemma to get the Riesz inequality in Theorem 1.1 with $c_p = (p^* - 1)^p$.

In order to consider Burkholder's inequality in its full generality we consider an open set Ω in $\mathbb{R}^n$, where n is a positive integer, and a bounded domain D such that $0 \in D$ and $\overline{D} \subset \Omega$. Assume that ∂D admits the harmonic measure μ with respect to 0.

THEOREM 1.3. (Burkholder, [1]) *Let $1 < p < \infty$. If u and v are harmonic functions on Ω with values in a Hilbert space and*

(i) $$|v(0)| \le |u(0)|,$$

(ii) $$|\nabla v| \le |\nabla u| \quad on \quad \Omega,$$

then

$$\int_{\partial D} |v|^p \, d\mu \le (p^* - 1)^p \int_{\partial D} |u|^p \, d\mu.$$

Burkholder also considered the case that u is a non-negative smooth subharmonic function and v is simply smooth. For $1 < p < \infty$ we set $p^{**} = \max\{2p, p/(p-1)\}$.

THEOREM 1.4. (Burkholder, [2]) *Let $1 < p < \infty$. If u is a non-negative smooth subharmonic function on Ω, v is a smooth function on Ω with values in $\mathbb{R}^\nu$ where ν is a positive integer and*

(i) $$|v(0)| \le u(0),$$

(ii) $$|\nabla v| \le |\nabla u| \quad on \quad \Omega,$$

(iii) $$|\Delta v| \le \Delta u \quad on \quad \Omega,$$

then

$$\int_{\partial D} |v|^p \, d\mu \le (p^{**} - 1)^p \int_{\partial D} |u|^p \, d\mu.$$

In this paper we want to generalize Theorem 1.4. We replace the assumption (iii) by $|\Delta v| \le \alpha \, \Delta u$, where $0 \le \alpha \le 1$ is a constant. Also, we assume that v has value in a Hilbert space.

2. A norm inequality

Let Ω be an open subset of $\mathbb{R}^n$ where n is a positive integer. Let D be a bounded subdomain of Ω with $0 \in D$ and $\partial D \subset \Omega$. Let μ be the harmonic measure on ∂D with respect to 0. Let $\mathbb{H}$ be a Hilbert space over $\mathbb{R}$. For $x, y \in \mathbb{H}$ we denote by $x \cdot y$ the inner product of x and y and put $|x|^2 = x \cdot x$. We consider two smooth functions u and v on Ω; that is, u and v have continuous partial derivatives up to the second order. Here, u is real-valued and v is $\mathbb{H}$-valued. By ∇u we denote the gradient of u and by Δu, the Laplacian of u. Write u_i for the partial derivative of u with respect to the i-th variable. Thus, $\nabla v = (v_1, \cdots, v_n) \in \mathbb{H}^n$, the standard product Hilbert space. Let α and p be constants with $0 \le \alpha \le 1$ and $1 < p < \infty$. Set $r = r(\alpha, p) = \max\{(\alpha + 1)p, p/(p-1)\}$.

THEOREM 2.1. *If u is a non-negative subharmonic function on Ω and*

$$\begin{aligned} &\text{(i)} && |v(0)| \le u(0), \\ &\text{(ii)} && |\nabla v| \le |\nabla u| \quad on \quad \Omega, \\ &\text{(iii)} && |\Delta v| \le \alpha\,\Delta u \quad on \quad \Omega, \end{aligned}$$

then

$$\int_{\partial D} |v|^p\, d\mu \le (r-1)^p \int_{\partial D} |u|^p\, d\mu.$$

3. Technical lemmas

Put $S = \{(x, y) : x > 0 \text{ and } y \in \mathbb{H} \text{ with } |y| > 0\}$. Define two functions U and V on S by

$$\begin{cases} U(x,y) &= \big(|y| - (r-1)x\big)\big(x + |y|\big)^{p-1}, \\ V(x,y) &= |y|^p - (r-1)^p x^p. \end{cases}$$

LEMMA 3.1. *There is a constant $c > 0$ such that $V \le cU$ on S.*

Proof. Put $c = p(1 - 1/r)^{p-1}$. We want to show that $V - cU \le 0$ on S. By the homogeneity we may consider only those $(x, y) \in S$ with $x + |y| = 1$. Thus, with

$$F(x) = (1-x)^p - (r-1)^p x^p - c(1 - rx),$$

we need to show that $F(x) \le 0$ if $0 < x < 1$.

Observe that F is continuous on $[0, 1]$ and smooth on the open interval $(0, 1)$. Thus, for $0 < x < 1$, we have

$$F'(x) = -p\Big((1-x)^{p-1} + (r-1)^p x^{p-1}\Big) + rc,$$

$$F''(x) = p(p-1)\Big((1-x)^{p-2} - (r-1)^p x^{p-2}\Big).$$

Notice that $0 < 1/r < 1$. One can check that $F(1/r) = F'(1/r) = 0$.

We divide the rest of the proof into three cases.

In case $p = 2$ we have $F'' = 2\big(1 - (r-1)^2\big) \le 0$ on $(0, 1)$ because $r \ge 2$. Hence F has the maximum over $[0, 1]$ at $t = 1/r$, which implies that $F \le 0$ on $[0, 1]$.

Now let $1 < p < 2$. From the formula of F'' we see that $F''(x) < 0$ if and only if $x < x^*$ where $1/x^* = 1 + (r-1)^{p/(p-2)}$. Here $0 < 1/r < x^*$. Thus, $F \le 0$ on $[0, x^*]$ for the same reason as in the previous case. On the interval $[x^*, 1]$ the function F is convex. Hence it suffices to check $F(1) \le 0$. For this we use the concavity of $\log x$ to get

$$(p-1)\log(p-1) + (2-p)\log p \le \log 1, \quad \text{or} \quad (p-1)^{p-1} \le p^{p-2}.$$

Hence we have $r^{p-1} \ge \big(p/(p-1)\big)^{p-1} = pp^{p-2}/(p-1)^{p-1} \ge p$ and

$$F(1) = -(r-1)^p + p(r-1)(1-1/r)^{p-1} = (r-1)^p r^{1-p}(p - r^{p-1}) \le 0.$$

The case $p > 2$ is proved similarly. This time one needs to check $F(0) \le 0$ for which the inequality $(p-1)^{p-1} \ge p^{p-2}$ could be used.

Basic facts about convex functions can be found in [6]. □

LEMMA 3.2. *$U(x, y) \le 0$ if $(x, y) \in S$ and $x \ge |y|$.*

Proof. Since $r \ge 2$, we have $x - (r-1)|y| \le x - |y|$. Hence Lemma 3.2 follows. □

LEMMA 3.3. *$U_x + \alpha|U_y| \le 0$ on S.*

Proof. Using the chain rule, we get

$$\begin{cases} U_x(x,y) = \Big((p-r)(x+|y|) - r(p-1)x\Big)(x+|y|)^{p-2}, \\ \\ U_y(x,y) = \Big(p(x+|y|) - r(p-1)x\Big)(x+|y|)^{p-2}\dfrac{y}{|y|}. \end{cases}$$

By the homogeneity of U_x and U_y the inequality in Lemma 3.3 is reduced to the inequality that $L \le 0$ on $(0, 1)$, where

$$L(x) = (p-r) - r(p-1)x + \alpha\big|p - r(p-1)x\big|.$$

For this recall that $(\alpha+1)p \le r$, $0 \le \alpha \le 1$ and $1 < p < \infty$. Hence, if $0 < x < 1$ then

$$L(x) \le (\alpha+1)p - r + (\alpha-1)r(p-1)x \le 0.$$

This proves Lemma 3.3. □

Differentiation of vector functions can be found, for example, in [4]. In the following we view $U_{xy}(x, y)$ as a vector in $\mathbb{H}$ and $U_{yy}(x, y)$ as a linear operator on $\mathbb{H}$.

LEMMA 3.4. *If $h \in \mathbb{R}$, $k \in \mathbb{H}$ and $(x, y) \in S$, then*

$$U_{xx}(x,y)h^2 + 2U_{xy}(x,y)\cdot hk + U_{yy}(x,y)k\cdot k \\ \leq \big(|k|^2 - h^2\big) r(p-1)\big(x+|y|\big)^{p-2}.$$

Proof. Put $I = \{t \in \mathbb{R} : x + th > 0 \text{ and } |y + tk| > 0\}$. Observe that $0 \in I$ and I is an open set. Define a function G on I by

$$G(t) = U(x + th, y + tk).$$

Observe that $0 \in I$ and I is an open set. From the chain rule we have

$$G''(0) = U_{xx}(x,y)h^2 + 2U_{xy}(x,y)\cdot hk + U_{yy}(x,y)k\cdot k\,.$$

Thus it suffices to show

$$G''(0) \leq \big(|k|^2 - h^2\big) r(p-1)\big(x+|y|\big)^{p-2}.$$

For this we define more functions K, Q and R on I by $K = K(t) = x + th$, $Q = |y + tk|$ and $R = K + Q$. We omit the argument $t \in I$ in the following computations. Differentiation gives $QQ' = k\cdot(y + tk)$ and $QQ'' = |k|^2 - (Q')^2$, hence, by the Cauchy-Schwarz inequality, we have $Q|Q'| = |QQ'| \leq |k|\ |y + tk| = |k|Q$. Thus, $|Q'| \leq |k|$ and $R'' = Q'' \geq 0$.

Writing $G = R^p - rKR^{p-1}$, we compute

$$\begin{aligned} G' &= pR'R^{p-1} - rhR^{p-1} - r(p-1)KR'R^{p-2}, \\ G'' &= pR''R^{p-1} + p(p-1)(R')^2R^{p-2} - 2r(p-1)hR'R^{p-2} \\ &\quad - r(p-1)KR''R^{p-2} - r(p-1)(p-2)K(R')^2R^{p-3}. \end{aligned}$$

Thus, putting $1/H = (p-1)R^{p-3}$, noting $-rKR''R = -rR''R^2 + rRQR''$, and inserting terms $rR(R')^2 - rR(R')^2$, we have

$$\begin{aligned} HG'' =& \left(\frac{p}{p-1} - r\right) R''R^2 + rR\Big(QR'' - 2hR' + (R')^2\Big) \\ &+ \Big(pR - rR - r(p-2)K\Big)(R')^2 \\ \leq& rR\big(|k|^2 - h^2\big) + \Big((p-r)Q + \big(p - r(p-1)\big)K\Big)(R')^2 \\ \leq& rR\big(|k|^2 - h^2\big) \end{aligned}$$

because $R'' \geq 0$, $p/(p-1) \leq r$, $(\alpha+1)p \leq r$, $R' = h + Q'$ and $QR'' = QQ'' = |k|^2 - (Q')^2$. When $t = 0$, we have

$$G''(0) \leq \left(|k|^2 - h^2\right) r(p-1)\left(x + |y|\right)^{p-2}.$$

This finishes the proof of Lemma 3.4. □

4. Proof of the inequality in Theorem 2.1

We may assume $\|u\|_p < \infty$. And we may further assume that

(iv) $u > 0$ and $|v| > 0$ on Ω.

Indeed, for each $\epsilon > 0$, the functions $u + \epsilon$ and (v, ϵ), where (v, ϵ) has value in the standard product Hilbert space $\mathbb{H} \times \mathbb{R}$, satisfy this extra assumption as well as the assumptions of the theorem. Now, the inequality

$$\|(v, \epsilon)\|_p \leq (r-1)\|u + \epsilon\|_r$$

yields, as $\epsilon \to 0$, the inequality in Theorem 2.1.

Let the functions U and V be as in the previous section. By the assumption (iv) we have $(u, v) \in S$ on Ω. The inequality in Theorem 2.1 is equivalent to

$$\int_{\partial D} V(u, v) d\mu \leq 0.$$

According to Lemma 3.1, it suffices to prove

$$\int_{\partial D} U(u, v) d\mu \leq 0.$$

Also, Lemma 3.2 and the assumption (i) imply $U\left(u(0), v(0)\right) \leq 0$. Hence the proof is complete if we show

$$\int_{\partial D} U(u, v) d\mu \leq U\left(u(0), v(0)\right)$$

which follows from the superharmonicity of $U(u, v)$.

Put $w = U(u, v)$. In order to show that w is superharmonic on Ω it suffices to check $\Delta w \leq 0$ on Ω. For $1 \leq i \leq n$ we use the chain rule to get

$$w_i = U_x(u, v) u_i + U_y(u, v) \cdot v_i \quad \text{and} \quad w_{ii} = U_x(u, v) u_{ii} + U_y(u, v) \cdot v_{ii} + A,$$

where

$$A_i = U_{xx}(u,v)u_i^2 + 2U_{xy}(u,v)\cdot u_i v_i + U_{yy}(u,v)v_i\cdot v_i.$$

Thus

$$\Delta w = U_x(u,v)\Delta u + U_y(u,v)\cdot\Delta v + \sum_{i=1}^n A_i.$$

From Lemma 3.3, the assumption (iii), the Cauchy-Schwarz inequality and the assumption that u is subharmonic we get

$$\begin{aligned} U_x(u,v)\Delta u + U_y(u,v)\cdot\Delta v &\le U_x(u,v)\Delta u + |U_y(u,v)|\,|\Delta v| \\ &\le \Big(U_x(u,v) + \alpha|U_y(u,v)|\Big)\Delta u \le 0. \end{aligned}$$

Fix $1\le i\le n$ and put $x=u$, $h=u_i$, $y=v$ and $k=v_i$. The assumption (iv) and Lemma 3.4 imply

$$\begin{aligned} &U_{xx}(u,v)u_i^2 + 2U_{xy}(u,v)\cdot u_i v_i + U_{yy}(u,v)v_i\cdot v_i \\ &\le \big(|v_i|^2 - u_i^2\big)r(p-1)\big(u+|v|\big)^{p-2}. \end{aligned}$$

Hence

$$\sum_{i=1}^n A_i \le \big(|\nabla v|^2 - |\nabla u|^2\big)r(p-1)\big(u+|v|\big)^{p-2} \le 0$$

by the assumption (ii). This proves that $\Delta w\le 0$ on Ω and finishes the proof of the inequality.

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