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UNIQUE FACTORIZATION IN TWO-DIMENSIONAL COMPLETE INTERSECTIONS

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ABSTRACT. Unique factorization of 2-dimensional complete intersection is investigated by using the determinant method introduced by D. Eisenbud.

1. Introduction

It is often quoted in undergraduate class that prime integers are no longer prime elements in some integral extensions of the ring of the integers. For example, neither 2 nor 3 remains as a prime in $\mathbb{Z}[\sqrt{-5}]$, even 2 is not irreducible in the ring of Gaussian integers $\mathbb{Z}[i]$. On the other hand $R[x]$ is factorial if R is so, and an element in R is a prime element of R if and only if it is a prime element of $R[x]$ (due to Gauss). For the formal power series ring $R[[x]]$, the corresponding question is not true.

PROBLEM 1.1. *For which U.F.D.'s R, is the formal power series ring $R[[x]]$ a U.F.D. ?*

The problem has been invesgated since early 1900s and substantial answers were made both for the positive case and the negative case in 1960s. The first success was made by E. Lasker in 1905 for an infinite field R. Quite generally Samuel and Buchsbaum proved that the problem is true if R is a locally regular U.F.D. ([13], [2]). Attributed to another result of Samuel (Theorem 1.2), 2-dimensional local factorial rings have been brought into focus and extensive results were given by Scheja, Danilov and Lipman.

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THEOREM 1.2. (Samuel [13]) *Let R be a locally Cohen-Macaulay factorial domain. If $R_p[[x]]$ is factorial for any height 2 prime ideal p in R, then $R[[x]]$ is factorial.*

Samuel found counterexamples to Problem 1.1, which were not complete ([13], [S_2]). So he thought that Problem 1.1 is true if R is a complete local U.F.D. It is true for local complete U.F.D.'s when depth$R$ is not equal to 2. Note that R is either a field or a DVR if R is a local U.F.D. with depth$R \leq 1$. Then the problem is known to be true by Lasker, Krull and Cohen.

THEOREM 1.3. (Scheja, [15]) *If R is a complete local factorial domain with depth $R \geq 3$. Then $R[[x]]$ is factorial.*

As a consequence of Theorem 1.3, if $R[[x]]$ is factorial for a complete local U.F.D. R, then so is $R[[x_1, \cdots, x_n]]$ for any $n \geq 1$. However, the 2-dimensional case was still remained open. Scheja found essentially all the 2-dimensional local complete U.F.D.'s for which Problem 1.1 is true.

The first counterexample to Problem 1.1 was introduced by Salmon. Let k be any field and u, x, y and z are variables. Then $R = k(u)[[x, y, z]]/(x^2+y^3+uz^6)$ is a UFD, but $R[[T]]$ is not [12]. But if we replace the field k by its algebraic closure, then R is no longer factorial. Lipman described the situation that 'R is not a genuine U.F.D., in that the divisor class group of R has many non-zero elements which happen to be concealed thinly'. He has given a remarkable result on the 2-dimensional complete local U.F.D.s'.

THEOREM 1.4. (Lipman [9]) *Let (R, m) be a 2-dimensional local ring such that R/m is algebraically closed of characteristic $\neq 2, 3, 5$. Assume that R is not regular. Then the completion $\hat{R}$ is factorial if and only if $\hat{R} \cong S/(x^2 + y^3 + z^5)$ for a 3-dimensional regular local ring S and a regular system of parameters x, y, z of S.*

If R is not regular and R/m is real closed, then he showed that R has rational singularity and is factorial if and only if m is generated by x, y, and z satisfying one of the following relations:

$$x^2 + y^3 + z^5 = 0$$

$$x^2 + y^3 + z^4 = 0$$

$$x^2 + y^2 + z^{2n} = 0$$

$$x^2 + y^2 + z^{2n+1} = 0, \quad n \geq 1.$$

In this note, we are going to disuss some of the 2-dimensional complete local U.F.D.'s that is not in the above list.

2. Determinant

Let (R, m) be a noetherian local ring. An ideal I of R is said to be *perfect* if $\mathrm{ht}I = \mathrm{pd}R/I$. By the Auslander-Buchsbaum formula [1], if I is perfect, then

$$\mathrm{depth}R/I + \mathrm{ht}I = \mathrm{depth}R.$$

If I is a proper ideal of R with $\mathrm{grade}I \geq 2$. Then the Hilbert-Burch Theorem says that I is perfect of height 2 if and only if I is the ideal of $n \times n$ minors of a $n \times (n+1)$ matrix with entries in R [3].

LEMMA 2.1. *Let (R, m) be a regular local ring and I an ideal of R with $\mathrm{ht}I \geq 2$. If I is perfect, then it is unmixed. The converse holds if* $\dim R \leq 3$.

Proof. Suppose that I is perfect. Then

$$\begin{aligned} \mathrm{depth}R/I &= \mathrm{depth}R - \mathrm{pd}R/I \\ &= \dim R - \mathrm{ht}I \\ &= \dim R/I. \end{aligned}$$

Thus I is unmixed.

Assume that $\dim R = 3$ and I is unmixed If $\mathrm{ht}I = 3$. Then $\mathrm{depth}R/I = \dim R/I = 0$ and the assertion is trivial. So suppose that $\mathrm{ht}I = 2$. Since I is unmixed m is not an associated prime of I and $\mathrm{depth}R/I = 1$. Then $\mathrm{pd}R/I = \mathrm{depth}R - \mathrm{depth}R/I = 2$. Thus I is perfect. □

An element f of a noetherian local ring (R, m) is called a *determinant* if f can be expressed as the determinant of an $k \times k$ matrix ($k \geq 2$) with entries in the maximal ideal m. Eisenbud has found that the factoriality of surfaces of embdding dimension 3 is decided by computing the determinant.

THEOREM 2.2. (Eisenbud [7]) *Let (R, m) be a regular local ring and $S = R/(f)$ for some f in R. If S is a UFD then f is not a determinant in R. Conversely, if f is not a determinant and* $\dim R \leq 3$, *then S is a UFD.*

If f is a determinant of a $k \times k$ matrix A, $k \geq 2$. then let B be the $(k-1) \times k$ matrix obtained from A by deleting the first row. The ideal I of $(k-1) \times (k-1)$ minors of B is unmixed ideal of height 2 by the Hilbert-Burch Theorem and Lemma 2.1. Thus $I/(f)$ is an unmixed ideal of height 1 in S that is not principal. So S is not factorial.

Now suppose that R is of dimension 3 and f in not determinant in R. For any unmixed ideal $I/(f)$ of S, I is the ideal of $k \times k$ minors of a $k \times (k+1)$ matrix A with entries in m (by the Hilbert-Burch Theorem and Lemma 1). If f is not a determinant, then some $(k-1) \times (k-1)$ minor must be a unit in S and I is generated by two elements including f. Thus $I/(f)$ is principal and S is factorial.

If R is a regular local ring R with $\dim R \geq 4$ and f is not a determinant in R. Then it is not necessarily true that $R/(f)$ is a UFD.

THEOREM 2.3. *Let K be a real closed field and R be the 4-dimensional regular local ring $K[x, y, z, w]_{(x,y,z,w)}$. Then $f = x^2 + y^2 + z^2 + w^2$ is not a determinant in R.*

Proof. Suppose f is a determinant in R. Then it must be a determinant of a 2×2 matrix A since $f \notin m^3 R$ where $m = (x, y, z, w)K[x, y, z, w]$. Consider the entries of A as rational functions in x, y, z and w and clear the denominators in each entry of A. Thus $f(1+g) = \det B$ for some $g \in m$ and a matrix B with entries in $K[x, y, z, w]$. Put

$$B = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}.$$

Notice that the elementary row and column operations do not change the determinant up to multiplication by units. So we may assume that

$$f_1 = x + ay + bz + cw + \text{higher degree terms} \quad a, b, c \in K.$$

Using the elememtary operations again, remove the linear term of x in the entries f_2 and f_3. Comparing the coefficients of each side of the equation $f(1+g) = \det B$ we obtain

$$f_4 = x - ay - bz - cw + \text{higher degree terms.}$$

Therefore,

$$f(1+g) = \det B =$$

$$\begin{vmatrix} x + ay + \cdots + \text{higher terms} & a'y + \cdots + \text{higher terms} \\ a''y + \cdots + \text{higher terms} & x - ay - \cdots + \text{higher terms} \end{vmatrix}.$$

Now compute the quadratic terms of each side of the equation $f(1+g) = \det B$. Then

$$a^2 + a'a'' = -1, \tag{1}$$

$$b^2 + b'b'' = -1, \tag{2}$$

$$2ab + a'b'' + a''b' = 0. \tag{3}$$

Since K is real closed, the equation (1) and (2) impliy that

$$a'a''b'b'' \neq 0.$$

Substitute for a'' and b'' in (3) and multiply a' and b'. Then

$$\begin{aligned} 0 &= a'^2(b^2+1) + b'^2(a^2+1) - 2aba'b' \\ &= (ab' - a'b)^2 + a'^2 + b'^2. \end{aligned}$$

This forces that $a' = b' = 0$ which is absurd. □

Let K be a real closed field. The Proof of Theorem 2.3 shows that none of

$$\begin{aligned} f &= x^2 + y^2 + z^2 + w^2 \\ f' &= x^2 + y^2 + z^2 \\ f'' &= x^2 + y^2 + z^2 - w^2 \end{aligned}$$

are a determinant in $R = K[x, y, z, w]_{(x,y,z,w)}$. Thus $R/(f')$ as well as $R'/(f')$ is a UFD for a 3-dimensional regular local ring $R' = K[x, y, z]_{(x,y,z)}$ (Theorem 2).

Note that $R/(f'')$ is also a UFD [F, 11.6]. But $S = R/(f)$ is not a UFD. The divisor class group $\mathrm{Cl}(S)$ of S is infinite cyclic. Let F

be the algebraic closure of K. Then f is a determinant in $R_F = F[x, y, z, w]_{(x,y,z,w)}$. That is,

$$x^2 + y^2 + z^2 + w^2 = \begin{vmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{vmatrix}.$$

Let $S_F = R_F/(f)$. Then $\mathrm{Cl}(S_F) \cong \mathbb{Z}$ and it is generated by $\mathrm{cl}(p)$ of height 1 prime ideal $p = (x + iy, z + iw)S_F$ of S_F. Let $q = (x^2 + y^2, z^2 + w^2, xz + yw, xw - yz)S$, then q is a height 1 prime of S and $\mathrm{cl}(qS_F) = 2\mathrm{cl}(p)$. Thus the natural map $\mathrm{Cl}(S) \longrightarrow \mathrm{Cl}(S_F)$ is not the zeor map. Therefore $\mathrm{Cl}(S)$ is infinite cyclic [F, 11.6].

3. Two-dimensional complete intersections

Let (S, n) be a 3-dimensional regular local ring with x, y and z a regular system of parameters of S. If the residue class field S/n is neither algbraically closed nor real closed, Then the complete intersection $R = S/(f)$ is factorial with $f = x^2 + y^3 + uz^6$ or $f = x^2 + y^3 + uz^3$ for suitable u in S.

THEOREM 3.1. *Let K be a field and $u \in K$. Then $K[[x, y, z]]/(x^2 + y^3 + uz^6)$ is a U.F.D. if and only if there are no solutions in K for the equation $T_1^3 - T_2^2 - u = 0$ in two variables T_1 and T_2.*

Proof. Suppose that there are no solutions in K for the equation $T_1^3 - T_2^2 - u = 0$. It's enough to prove that $f = x^2 + y^3 + uz^6$ is not a determinant in $S = K[[x, y, z]]$. Assume f is a determinant in S. Then it must be a determinant of a 2×2 matrix A since $f \notin n^3$. Put

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For each element $a \in S$, write $a = \sum a_{x^i y^j z^k} x^i y^j z^k$ for $a_{x^i y^j z^k} \in K$. Notice that the elementary row and column operations do not change the determinant up to multiplication by units. So we may assume that

$$a = x + a_y y + a_z z + \text{higher order terms}.$$

Using the Gauss elimination, remove the terms of positive powers in x in the entries b and c. That is, we reduce to the case that $b_{x^i y^j z^k} = c_{x^i y^j z^k} = 0$ for $i \geq 1$.

Either b_y or c_y is nonzero, otherwise $f_{y^3} = 0$. Say $c_y \neq 0$ and we may write

$$c = y + c_z z + \text{higher order terms}.$$

By the Gauss elimination again, $a_{x^i y^j z^k} = d_{x^i y^j z^k} = 0$ for $j \geq 1$.

Comparing the quadratic terms of the each side of the equation $f = ad - bc$, one can find that there are no linear terms in the entries a, b and d except $a_x = d_x = 1$. Compute the cubic and higher order terms:

$$f_{y^2 z} = c_z - b_{yz} = 0,\ f_{yz^2} = -b_{yz}c_z - b_{z^2} = 0 \text{ and } f_{z^3} = -b_{z^2}c_z = 0.$$

This forces that $c_z = b_{yz} = b_{z^2} = 0$. From the computation of the coefficients of xz^2 and z^4, we obtain $a_{z^2} = d_{z^2} = 0$. Thus $b_{z^3} = -f_{yz^3} = 0$. Thus

$$x^2 + y^3 + uz^6 = \begin{vmatrix} x + a_{z^3}z^3 + \cdots & -(y^2 + b_{yz^2}yz^2 + b_{z^4}z^4 + \cdots) \\ y + c_{z^2}z^2 + \cdots & x - a_{z^3}z^3 + \cdots \end{vmatrix}.$$

This forces the equaton:

$$c_{z^2} + b_{yz^2} = 0,\ c_{z^2}b_{yz^2} + b_{z^4} = 0, \text{ and } c_{z^2}b_{z^4} - a_{z^3}^2 - u = 0.$$

Therefore $c_{z^2}^3 - a_{z^3}^2 - u = 0$. This is impossible.

Now suppose that there are α and β in K satisfying $\beta^3 - \alpha^2 - u = 0$. Then

$$x^2 + y^3 + uz^6 = \begin{vmatrix} x + \alpha z^3 & -(y^2 - \beta yz^2 + \beta^2 z^4) \\ y + \beta z^2 & x - \alpha z^3 \end{vmatrix}.$$

Thus $f = x^2 + y^3 + uz^6$ is a determinant and $K[[x, y, z]]/(f)$ is not a U.F.D. □

As a corollary of Theorem 3.1 $R = k(u)[[x, y, z]]/(x^2 + y^3 + uz^6)$ is factorial. However, if the algebraic closure F of $k(u)$ is substituted for $k(u)$, then $R' = F[[x, y, z]]/(x^2 + y^3 + uz^6)$ is not a UFD. It is easy to express $x^2 + y^3 + uz^6$ as a determinant in $F[[x, y, z]]$. That is,

$$x^2 + y^3 + uz^6 = \begin{vmatrix} x + (-u)^{1/2}z^3 & -y^2 \\ y & x - (-u)^{1/2}z^3 \end{vmatrix}.$$

Note that $p' = (y, x - (-u)^{1/2}z^3)R'$ is a prime ideal of height 1 in R' that is not principal. It is said to be 'concealed thinly' in the divisor class group of R [10].

Finally, we examine the relation $x^2 + y^3 + uz^5 = 0$.

THEOREM 3.2. *Let K be a field and $u \in K$. Then $K[[x,y,z]]/(x^2 + y^3 + uz^3)$ is a U.F.D. if and only if there are no solutions in K for the equation $T^3 + u = 0$.*

Proof. Suppose that $T^3 + u$ is irreducible over K. Following the same procedure as in the Proof of Theorem 3.1, we are forced to solve the equation $c_z^3 + u = 0$. It is a contradiction and $x^2 + y^3 + uz^3$ is not a determinant in $K[[x,y,z]]$.

Assume that $T^3 + u$ is reducible over K, so $u^{1/3} \in K$. Then

$$x^2 + y^3 + uz^3 = \begin{vmatrix} x & -(y^2 - u^{1/3}yz + u^{2/3}z^2) \\ y + u^{1/3}z & x \end{vmatrix}.$$

That is, $x^2+y^3+uz^3$ is a determinant in $K[[x,y,z]]$ and $K[[x,y,z]]/(x^2 +y^3 + uz^3)$ is not a U.F.D. □

Notice that $\mathbb{Q}[[x,y,z]]/(x^2+y^3+2z^3)$ is factorial, but $\mathbb{Q}[[x,y,z]]/(x^2 +y^3 + 2z^6)$ is not. In fact, for any 3-dimensional regular local ring S and a regular system of parameters x, y, z of S, $x^2 + y^3 + 2z^6$ is a determinant in S. That is,

$$x^2 + y^3 + 2z^6 = \begin{vmatrix} x + 5z^3 & -(y^2 - 3yz^2 + 9z^4) \\ y + 3z^2 & x - 5z^3 \end{vmatrix}.$$

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