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CONVERGENCE OF FINITE DIFFERENCE METHOD FOR THE GENERALIZED SOLUTIONS OF SOBOLEV EQUATIONS

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ABSTRACT. In this paper, finite difference method is applied to approximate the generalized solutions of Sobolev equations. Using the Steklov mollifier and Bramble-Hilbert Lemma, *a priori* error estimates in discrete L^2 as well as in discrete H^1 norms are derived first for the semidiscrete methods. For the fully discrete schemes, both backward Euler and Crank-Nicolson methods are discussed and related error analyses are also presented.

1. Introduction

Let Ω be a rectangular domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, and T be $0 < T < \infty$. We consider finite difference approximations for the generalized solutions of differential equations of the form

$$u_t + Au_t + Bu = f, \quad (x,t) \in \Omega \times (0,T], \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.1b}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{1.1c}$$

where $f = f(x,t)$, A and B are of the following forms

$$A(x)u = -\sum_{l,q=1}^{2} \frac{\partial}{\partial x_l}(a_{lq}(x)\frac{\partial u}{\partial x_q}),$$

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and

$$B(x,t)u = -\sum_{l,q=1}^{2} \frac{\partial}{\partial x_l}\left(b_{lq}(x,t)\frac{\partial u}{\partial x_q}\right) + \sum_{l=1}^{2} b_l(x,t)\frac{\partial u}{\partial x_l} + b(x,t)u.$$

We now make the following assumptions.

(1) The coefficients of $A(x)$ and $B(x,t)$, together with f, are smooth and bounded as far as the ensuring analysis demands.

(2) The coefficients $a_{lq} = a_{ql}$ satisfy

$$\sum_{l,q=1}^{2} a_{lq}\xi_l\xi_q \geq \sum_{l=1}^{2} \xi_l^2, \quad \forall\,(\xi_1,\xi_2) \in \mathbb{R}^2.$$

(3) There exists a unique generalized solution of the problem (1.1) with smoothness corresponding to that of the generalized solution.

The problem of this type arises in the study of consolidation of clay, heat conduction, homogeneous fluid flow in fissured material and shear in second order fluids. For existence, uniqueness and applications of (1.1), we refer to Ewing [5] and the extensive literatures contained therein.

Finite element methods for (1.1) have been studied by Ewing [5], Ford [6], Arnold *et al.* [1], Lin and Zhang [10] and Nakao[11]. For the analysis of finite difference schems, Ford and Ting [7]- [8] have obtained an order $O(k + h^2)$ of convergence for the backward Euler method and $O(k^2 + h^2)$ for the Crank-Nicolson method under the assumption that the exact solution $u, u_t \in C^4(\Omega)$ and $\Omega \subset \mathbb{R}$. For the problem in several space variables, Ewing [4] has obtained L^2 error estimates of order $O(k^2 + h^2)$ for the Crank-Nicolson method under the assumption that $u, u_t \in C^4(\Omega)$. In all these articles [4], [7]-[8], traditional Taylor's expansion is used for convergence analysis, which imposes sever smoothness conditions on the solution.

In this paper, using Steklov mollifier and a nonclassical discrete projection method we derive rates of convergences for the finite difference schemes and obtain orders of convergence compatible with the smoothness of the solution. After giving preliminaries in Section 2, we consider the semidiscrete scheme, its stability and error analysis in Section 3. In Section 4, we introduce a nonclassical discrete projection and obtain $O(h^2)$ convergence in the L^2-norm. In Section 5, we discuss fully discrete schemes which are optimal. We obtain an order $O(k+h^2)$ of convergence for the backward Euler method and an oredr $O(k^2+h^2)$ for the Crank-Nicolson scheme under the assumption that $u, u_t \in H^2(\Omega)$.

2. Preliminaries

Without loss of generality, it is assumed that the domain Ω is the unit square in $\mathbb{R}^2$. We select a mesh of width $h = \frac{1}{M}$, where M is a positive integer, and cover $\bar{\Omega} = \Omega \cup \partial\Omega$ with a square grid of mesh points $x_{ij} = (ih, jh), i, j = 0, 1, ..., M$. Let $\Omega_h = \{x_{ij} : x_{ij} \in \Omega\}$ and $\partial\Omega_h = \{x_{ij} : x_{ij} \in \partial\Omega\}$.

For a function w defined on Ω_h, the following notations will be used: for $x \in \partial\Omega_h$ and $l = 1, 2$,

$$w^{\pm l} = w(x \pm he_l), \quad w^{+l,-q} = w(x + he_l - he_q),$$

and

$$\nabla_l w(x) = \frac{w(x+he_l) - w(x)}{h}, \quad \bar{\nabla}_l w(x) = \frac{w(x) - w(x-he_l)}{h},$$

where e_l is the l-th unit vector in $\mathbb{R}^2$.

The Steklov mollifiers are defined in the following manner: $S = S_1^2 S_2^2$ with $S_l^2 = S_l^+ S_l^-, l = 1, 2$, where

$$S_l^+ \phi(x) = \int_0^1 \phi(x + she_l)\, ds, \quad S_l^- \phi(x) = \int_{-1}^0 \phi(x + she_l)\, ds.$$

The operators $S_l^\pm$ commute and the following relationships hold:

$$S_l^+ \frac{\partial\phi}{\partial x_l} = \nabla_l \phi, \quad S_l^- \frac{\partial\phi}{\partial x_l} = \bar{\nabla}_l \phi. \tag{2.1}$$

We now introduce the discrete L^2 space, denoted by $L_h^2(\Omega_h)$, with an inner product and the norm given by:

$$\langle w, v\rangle = h^2 \sum_{x\in\Omega_h} w(x)v(x) \text{ and } \|w\|_{0,h} = \langle w, w\rangle^{\frac{1}{2}}, \text{ for } v, w \in L_h^2(\Omega_h).$$

Further, let $H_h^1 = H_h^1(\Omega_h)$ denote the discrete analogue of H^1-Sobolev space with norm

$$\|w\|_{1,h}^2 = \|w\|_{0,h}^2 + \sum_{l=1}^{2} \|\nabla_l w\|_{0,h}^2$$

We also introduce a discrete H^2-Sobolev space with the following norm

$$\|w\|_{2,h}^2 = \|w\|_{1,h}^2 + \sum_{l,q=1}^{2} \|\nabla_l \bar{\nabla}_q w\|_{0,h}^2,$$

and denote it by $H_h^2 = H_h^2(\Omega_h)$. Whenever there is no confusion, we write $\|w\|$ and $\|w\|_j$, in place of $\|w\|_{0,h}$ and $\|w\|_{j,h}$. Throughout the paper, $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^m}$ will denote the norm in L^2 and the Sobolev space $H^m(\Omega)$, respectively. Further, let $|\cdot|_{W^{m,p}(\Omega)}$ denote the seminorm in $W^{m,p}(\Omega)$.

For functions v and w defined on Ω_h, the following identity is an easy consequence of summation

$$\langle \nabla_l v, w\rangle = -\langle v, \bar{\nabla}_l w\rangle, \quad l = 1, 2. \tag{2.2}$$

Along with the usual Bramble-Hilbert Lemma [2], the following bilinear version of it will be needed for our convergence analysis. For a proof, we refer the reader to Ciarlet[3].

LEMMA 2.1. *Let $P_{[r]}$ be the set of all polynomials of degree $\leq [r]$, where $[r]$ denotes the largest integer less than $r > 0$. If η is a bounded linear functional on $W^{\alpha,p}(\Omega) \times W^{\beta,q}(\Omega)$, with $\alpha, \beta \in (0, \infty)$ and $p, q \in [1, \infty]$ such that*

$$\eta(U, v) = 0, \quad \forall U \in P_{[\alpha]}(\Omega), \quad \forall v \in W^{\beta,q}(\Omega),$$
$$\eta(u, V) = 0, \quad \forall u \in W^{\alpha,p}(\Omega), \quad \forall V \in P_{[\beta]}(\Omega),$$

then there exists a positive constant C such that

$$|\eta(u, v)| \leq C|u|_{W^{\alpha,p}(\Omega)}|v|_{W^{\beta,q}(\Omega)}, \quad \forall u \in W^{\alpha,p}(\Omega), \forall V \in P_{[\beta]}(\Omega).$$

In the proofs below, the inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad a, b \in \mathbb{R}, \epsilon \geq 0, \tag{2.3}$$

will be used frequently and C will denote a generic positive constant whose dependence can be easily established from the proofs.

3. Semidiscrete schemes

Let A_h and B_h be defined for $(x,t) \in \Omega_h \times [0,T]$ as

$$A_h V = -\frac{1}{2} \sum_{l,q=1}^{2} \left[\nabla_l \left(a_{lq}(x) \bar{\nabla}_q V \right) + \bar{\nabla}_l \left(a_{lq}(x) \nabla_q V \right) \right],$$

and

$$\begin{aligned} B_h V = &-\frac{1}{2} \sum_{l,q=1}^{2} \left[\nabla_l \left(b_{lq}(x,t) \bar{\nabla}_q V \right) + \bar{\nabla}_l \left(b_{lq}(x,t) \nabla_q V \right) \right] \\ &+ \frac{1}{2} \sum_{l=1}^{2} b_l(x,t) \left[\nabla_l V + \bar{\nabla}_l V \right] + S\left(b(x,t)\right) V. \end{aligned}$$

Now, the semidiscrete approximation u_h of (1.1) is determined as a solution of

(3.1a) $$u_{h,t} + A_h u_{h,t} + B_h u_h = Sf, \quad (x,t) \in \Omega_h \times [0,T],$$

(3.1b) $$u_h(x,0) = u_0(x), \quad x \in \Omega_h,$$

(3.1c) $$u_h(x,t) = 0, \quad (x,t) \in \partial\Omega_h \times (0,T].$$

Let $H^1_{0,h} = \{v \in H^1_h : v = 0 \text{ on } \partial\Omega_h\}$. From the assumptions on $A(x)$ and $B(x,t)$, the following lemma can be easily verified using summation by parts.

LEMMA 3.1. *For $v, w \in H^1_{0,h}$, there exist constants C such that*

(1) *the discrete Poincaré inequality :* $\|v\|^2 \le C \sum_{l=1}^{2} \|\nabla_l v\|^2$,
(2) $\langle A_h v, v \rangle \ge C \|v\|_1^2$,
(3) $\langle B_h v, w \rangle \le C \|v\|_1 \|w\|_1$.

For subsequent error estimates, we derive stability results for the modified semidiscrete version of (3.1); namely,

(3.2) $$u_{h,t} + A_h u_{h,t} + B_h u_h = Sf + \sum_{l=1}^{2} \bar{\nabla}_l F,$$

where F is a function defined on $\Omega \times [0,T]$ which vanishes on $\partial\Omega_h$ and $F(0) = 0$.

The stability result for (3.2) is stated in the following theorem.

THEOREM 3.1. *Let u_h be a solution of* (3.2). *Then there exists a constant C such that*

$$\|u_h(t)\|_1 \leq C\{\|u_h(0)\|_1 + (\int_0^t \|Sf(s)\|^2 ds)^{1/2} + (\int_0^t \|F(s)\|^2 ds)^{1/2}\}.$$

Proof. Forming the inner product between (3.2) and u_h, we obtain

$$\frac{d}{dt}\|u_h\|_1^2 \leq C\{\|u_h\|_1^2 + \|Sf\|\|u_h\| + \|F\|\|u_h\|_1\}.$$

Integrating with respect to t, we find that

$$\begin{aligned}\|u_h(t)\|_1^2 \leq & C\{\|u_h(0)\|_1^2 + \int_0^t \|u_h(s)\|_1^2 ds \\ & + \int_0^t \|Sf(s)\|^2 ds + \int_0^t \|F(s)\|^2 ds\}.\end{aligned}$$

An application of Gronwall's Lemma now completes the proof. □

Using the above stability result, we shall derive the following error estimate.

THEOREM 3.2. *Let u and u_h be the solution of* (1.1) *and* (3.1), *respectively. Let $u, u_t \in H^\alpha(\Omega), 1 \leq \alpha \leq 3$, and for $t \in (0, T]$. Then there exists a constant C such that for the error $e(t) = u(t) - u_h(t)$ the following estimate*

$$\|e(t)\|_1 \leq C(u, T)h^{\alpha-1}$$

holds.

Proof. From (1.1) and (3.1), we obtain

$$\begin{aligned} e_t + A_h e_t + B_h e &= (u_t - Su_t) + (A_h u_t - SAu_t) + (B_h u - SBu) \\ &= I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

Following Jovanović *et al.*[9], $I_3(t)$ is rewritten as

$$I_3(t) = \sum_{l,q=1}^{2} \bar{\nabla}_l \xi_{lq}(t) + \sum_{l=1}^{2} \xi_l(t) + \xi(t),$$

where $\xi_{lq} = \xi_{lq}^{(1)} + \xi_{lq}^{(2)} + \xi_{lq}^{(3)} + \xi_{lq}^{(4)}$ with

$$\begin{aligned}
\xi_{lq}^{(1)} &= S_l^+ S_{3-l}^2 (b_{lq} \frac{\partial u}{\partial x_q}) - (S_l^+ S_{3-l}^2\, b_{lq})(S_l^+ S_{3-l}^2\, \frac{\partial u}{\partial x_q}),\\
\xi_{lq}^{(2)} &= [S_l^+ S_{3-l}^2\, b_{lq} - \frac{1}{2}(b_{lq} + b_{lq}^{+l})](S_l^+ S_{3-l}^2\, \frac{\partial u}{\partial x_q}),\\
\xi_{lq}^{(3)} &= \frac{1}{2}(b_{lq} + b_{lq}^{+l})[S_l^+ S_{3-l}^2\, \frac{\partial u}{\partial x_q} - \frac{1}{2}(\nabla_q u + \bar{\nabla}_q u^{+l})],\\
\xi_{lq}^{(4)} &= -\frac{1}{4}(b_{lq} - b_{lq}^{+l})(\nabla_q u - \bar{\nabla}_q u^{+l}),
\end{aligned}$$

and

$$\xi = (Sb)u - S(bu).$$

Further, we decompose ξ_l as $\xi_l(t) = \xi_l^{(1)}(t) + \xi_l^{(2)}(t) + \xi_l^{(3)}(t)$ with

$$\begin{aligned}
\xi_l^{(1)} &= b_l(t)[\frac{1}{2}(\nabla_l u + \bar{\nabla}_l u) - S\frac{\partial u}{\partial x_l}],\\
\xi_l^{(2)} &= b_l(t)\, S\frac{\partial u}{\partial x_l} - S(b_l(t))S\frac{\partial u}{\partial x_l},\\
\xi_l^{(3)} &= S(b_l(t))S\frac{\partial u}{\partial x_l} - S(b_l(t)\,\frac{\partial u}{\partial x_l}),
\end{aligned}$$

Similarly, we also rewrite I_2 as

$$I_2(t) = \sum_{l,q=1}^{2} \bar{\nabla}_l \eta_{lq}(t),$$

where $\eta_{lq} = \eta_{lq}^{(1)} + \eta_{lq}^{(2)} + \eta_{lq}^{(4)} + \eta_{lq}^{(4)}$. Here η_{lq} are the same as ξ_{lq} except that b_{lq} are replaced by a_{lq}.

Altogether, we have

$$e_t + A_h e_t + B_h e = I_1(t) + \sum_{l=1}^{2} \xi_l + \xi + \sum_{l,q=1}^{2} \bar{\nabla}_l(\eta_{lq} + \xi_{lq}). \tag{3.3}$$

Setting $F(t) = \sum_{l,q=1}^{2}(\eta_{lq}(t) + \xi_{lq}(t))$ and the first term on the right hand side of the above equation as Sf, we apply Theorem 3.1 to obtain

$$\begin{aligned}\|e(t)\|_1 \le C[\|e(0)\|_1 &+ (\int_0^t \|I_1(s)\|^2 ds)^{1/2}\\ &+ (\int_0^t \|\xi(s)\|^2 ds)^{1/2} + \sum_{l=1}^{2}(\int_0^t \|\xi_l(s)\|^2 ds)^{1/2}\\ &+ \sum_{l,q=1}^{2}\{(\int_0^t \|\eta_{lq}(s)\|^2 ds)^{1/2} + (\int_0^t \|\xi_{lq}(s)\|^2\, ds)^{1/2}\}].\end{aligned}$$

Since $I_1(t)$ is a bounded linear functional on $H^\beta(D)$ with its kernel contained in $P_1(D)$, where $D = \{(s_1, s_2) \in \mathbb{R}^2 : -1 \le s_l \le 1, l = 1, 2\}$. The Bramble-Hilbert Lemma, therefore, yields

$$\|I_1(t)\| \le Ch^\beta |u_t|_{H^\beta(\Omega)}, \quad 1 \le \beta \le 2. \tag{3.4}$$

To estimate ξ, we first note that

$$\xi = (Sb)(u_t - Su_t) + (Sb)(Su_t) - S(bu_t).$$

Again a use of Lemma 2.1 yields, for $1 \le \beta \le 2$,

$$\int_0^t \|\xi(s)\|\, ds \le C\left(\|b\|_{L^\infty(W^{\beta-1,\infty})}\right) h^\beta \int_0^t \|u_t(s)\|_\beta ds. \tag{3.5}$$

As in Jovanović *et al.* [9], we obtain an estimate for ξ_{lq} of the form

$$\sum_{l,q=1}^{2}(\int_0^t \|\xi_{lq}(s)\|^2\, ds)^{1/2} \le Ch^{\alpha-1}(\int_0^t \|u_t(s)\|_\alpha^2 ds)^{1/2}, \tag{3.6}$$

where C depends on $\max_{l,q}\|b_{lq}\|_{L^\infty(W^{\alpha-1,\infty})}$ and $1 \le \alpha \le 3$.

Following the estimates of ξ_{lq}, the estimation of η_{lq} can be easily obtained with similar bounds. Further, the estimates of $\xi_l^{(i)}$ are similar to those of $\xi_{lq}^{(i)}$ for $i = 1, 2, 3$. This completes the rest of the proof. □

REMARK. Since $\|e\| \leq \|e\|_1$, we obtain from the previous Theorem

$$\|e(t)\| \leq C(u,T)h^{\alpha-1}, \qquad 1 \leq \alpha \leq 3.$$

In order to achieve an order $O(h^2)$ of convergence, it is to be noted that we need $u, u_t \in L^2(H^3(\Omega))$. In contrast to papers [4], in which Taylor's expansion is used to derive the convergence, the above result is a substantial improvement. However, using a discrete auxiliary projection, we shall, in the next section, prove a similar result when $u \in L^\infty(H^2)$ and $u_t \in L^2(H^2)$.

4. Error estimates with reduced regularity

In this section, we shall derive the error estimate whose order of convergence is compatible with the spatial regularity on the generalized solution u.

Let us define $\tilde{u}$ as the solution of the following auxiliary discrete problem

$$A_h\tilde{u}_t + B_h\tilde{u} = S(f - u_t), \tag{4.1a}$$

$$\tilde{u}(0) = u_0(x). \tag{4.1b}$$

Since A_h is positive definite, there exists a unique solution $\tilde{u}$ of (4.1). Let $\rho = u - \tilde{u}$. We can therefore rewrite (4.1) as

$$A_h\rho_t + B_h\rho = (A_hu_t - SAu_t) + (B_hu - SBu) = I_2 + I_3. \tag{4.2}$$

LEMMA 4.1. *Let $u, u_t \in H^\alpha(\Omega)$, with $1 \leq \alpha \leq 2$ and $t \in [0,T]$. Then there exists a constant C such that*

$$\|\rho(t)\|_1,\ \|\rho_t(t)\|_1 \leq C(u,T)h^{\alpha-1}.$$

Proof. For the estimation of $\|\rho\|_1$, it follows from the discrete inner product of (4.2) with ρ that

$$\frac{d}{dt}\|\rho\|_1^2 \leq C\{\|\rho\|_1^2 + |\langle I_2 + I_3, \rho\rangle|\}.$$

On integrating with respect to time and then following (3.5)–(3.6), we obtain using Gronwall's Lemma

$$\|\rho(t)\|_1 \leq C(u, T)h^{\alpha-1}, \quad 1 \leq \alpha \leq 2.$$

Similarly, forming an inner product between (4.2) and ρ_t, we have

$$\|\rho_t(t)\|_1 \leq Ch^{\alpha-1}, \quad 1 \leq \alpha \leq 2. \qquad \square$$

For error estimate in L^2-norm, below we shall discuss the discrete Aubin-Nitsche duality argument, see, Pani *et al.* [12]-[13].

LEMMA 4.2. *Suppose that $u, u_t \in H^\alpha(\Omega)$, for $1 \leq \alpha \leq 2$ and for $t \in [0, T]$. Then there is a constant C such that*

$$\|\rho(t)\|, \ \|\rho_t(t)\| \leq Ch^\alpha.$$

Proof. Let Φ be a solution of the following second order problem

$$A_h\Phi = \rho_t, \quad x \in \Omega_h, \qquad \Phi = 0, \quad x \in \partial\Omega_h. \tag{4.3}$$

Because of the coercivity of A_h, Φ is the unique solution of (4.3) and it satisfies a discrete regularity

$$\|\Phi\|_2 \leq C\|\rho_t\|. \tag{4.4}$$

Forming an inner product between (4.3) and ρ_t, we obtain

$$\langle \rho_t, \rho_t \rangle = \langle A_h\rho_t, \Phi \rangle = \langle I_2 + I_3 - B_h\rho, \Phi \rangle.$$

The estimates for $\|\rho_t(t)\|$ given below can be proved easily using the Steklov mollifier and the Bramble-Hilbert Lemma 2.1. For a complete proof, we refer Pani *et al.* [12]–[13].

$$\|\rho_t(t)\|^2 \leq C(h^\alpha + \|\rho(t)\|)\|\Phi\|_2,$$

and hence, using discrete regularity, we obtain

$$\|\rho_t(t)\| \leq C(h^\alpha + \|\rho(t)\|). \tag{4.5}$$

Note that

$$\|\rho(t)\|^2 \le C\{\|\rho(0)\|^2 + \int_0^t \|\rho_t(s)\|^2 ds\}$$
$$\le C\{\|\rho(0)\|^2 + h^{2\alpha} + \int_0^t \|\rho(s)\|^2 ds\}.$$

It now follows from Gronwall's Lemma that

$$\|\rho(t)\| \le C\{\|\rho(0)\| + h^\alpha\} \le Ch^\alpha. \tag{4.6}$$

Finally, we obtain the L^2-error estimate for $\|\rho_t\|$ from (4.5)–(4.6). □

Let $\theta(t) = u_h(t) - \tilde{u}(t)$, then the error $e(t) = u(t) - u_h(t) = \rho(t) - \theta(t)$.

THEOREM 4.1. *Let u and u_h be the solutions of* (1.1) *and* (3.1), *respectively. Further, let $u \in L^\infty(H^\alpha(\Omega))$ and $u_t \in L^2(H^\alpha(\Omega))$, $1 \le \alpha \le 2$. Then there exists a constant C such that*

$$\|e(t)\| \le C(u, T)h^\alpha.$$

Proof. Since the estimate for $\rho(t)$ is given in Lemmas 4.1 and 4.2, it is enough to estimate $\theta(t)$. From (1.1), (3.1) and (4.1), it follows that

$$\theta_t + A_h\theta_t + B_h\theta = Su_t - \tilde{u}_t = -I_1(t) + \rho_t. \tag{4.7}$$

It follows from (3.4) and Lemmas 4.1–4.2 that

$$\|\theta(t)\|_1 \le C\{\|\theta(0)\|_1 + h^\alpha \|u_t\|_{L^2(H^\alpha)}\}, \quad 1 \le \alpha \le 2.$$

Because of the choice of $u_h(0)$, we have $\theta(0) = 0$. For L^2-error estimate, form the inner product between (4.7) and θ and obtain

$$\langle\theta_t, \theta\rangle + \langle A_h\theta_t, \theta\rangle = \langle -I_1 + \rho_t, \theta\rangle - \langle B_h\theta, \theta\rangle.$$

Since A_h is coercive, we obtain

$$\frac{d}{dt}\|\theta(t)\|^2 \le C\{\|I_1\|\|\theta\| + \|\rho_t\|\|\theta\| + \|\theta\|_1^2\}.$$

It follows from the integration with respect to t and Gronwall's Lemma that

$$\|\theta(t)\|^2 \le C(T)\int_0^t (\|I_1(s)\|^2 + \|\rho_t(s)\|^2)ds.$$

Hence, we obtain the required result from Lemma 4.2. □

5. Fully discrete schemes

In this section, we shall consider the stability and error analysis for the fully difference schemes which are based on the Euler and Crank-Nicolson methods. Let $k = \frac{T}{N}$ denote the size of the time discretization for a given positive integer N and $t_n = nk$, for $n = 0, 1, 2, ..., N$. For any function ϕ, denote $\phi^n = \phi(t_n)$ and

$$\bar{\partial}_t \phi^n = \frac{\phi^n - \phi^{n-1}}{k}.$$

The backward Euler method. The backward Euler scheme is now defined by

$$(5.1a) \qquad \bar{\partial}_t U^n + A_h\left(\bar{\partial}_t U^n\right) + B_h U^n = Sf^n, \quad x \in \Omega_h,$$

$$(5.1b) \qquad U^n = 0, \quad x \in \partial\Omega_h,$$

$$(5.1c) \qquad U^0 = u_0(x), \quad x \in \Omega_h.$$

Below, we shall prove a stability result in discrete H^1- norm not for (5.1 a) but for a modified equation

$$(5.2) \qquad \bar{\partial}_t U^n + A_h\left(\bar{\partial}_t U^n\right) + B_h U^n = Sf^n + \sum_{l=1}^{2} \bar{\nabla}_l F^n.$$

THEOREM 5.1. *Let U^n be a solution of* (5.1). *Then there are positive constants C and k_0 such that for $0 < k \le k_0$*

$$\|U^J\|_1 \le C\{\|U^0\|_1 + (k\sum_{n=1}^{J} \|Sf^n\|^2)^{\frac{1}{2}}\} + (k\sum_{m=1}^{n} \|F^n\|^2)^{\frac{1}{2}}\},$$
$$J = 1, 2, \cdots, N.$$

Proof. Form a discrete L^2-inner product between (5.2) and U^n and then use Lemma 3.1 with summation by parts for the last term to have

$$\bar{\partial}_t\|U^n\|^2 + \bar{\partial}_t\|\nabla U^n\|^2 \le C\{\|Sf^n\|^2 + \|F^n\|^2 + \|U^n\|_1^2\}.$$

Summing from $n = 1$ to J, we obtain

$$(1 - Ck)\|U^J\|_1^2 \le C\{\|U^0\|_1^2 + k\sum_{n=1}^{J}(\|Sf^n\|^2 + \|F^n\|^2) + k\sum_{n=1}^{J-1}\|U^1\|_1^2\}.$$

Choose k_0 in such a way that $(1 - Ck) > 0$ for $0 < k \le k_0$. An application discrete Gronwall's Lemma now completes the proof. $\square$

LEMMA 5.1. *Let $u, u_t, u_{tt} \in L^2(\Omega)$ for $t \in [0, T]$. Then there exists a constant C such that*

$$\|\tilde{u}_{tt}(t)\|_1 \leq C, \quad t \in [0, T].$$

Proof. Forming an inner product between (4.1) and $\tilde{u}$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\tilde{u}\|_1 &\leq C\{\|S(f - u_t)\|\|\tilde{u}\| + \|\tilde{u}\|_1\} \\ &\leq C\{\|S(f - u_t)\|^2 + \|\tilde{u}\|_1^2\}.\end{aligned}$$

Integrating with respect to t and applying Gronwall's inequality, we have

$$\|\tilde{u}\|_1^2 \leq C\{\|\tilde{u}(0)\|_1^2 + \int_0^t \|S(f - u_t)\|^2 ds\}.$$

For the estimate of $\tilde{u}_t$, we obtain

$$\|\tilde{u}_t\|_1^2 \leq C\{\|S(f - u_t)\|^2 + \|\tilde{u}\|_1^2\} \tag{5.3}$$

by taking innner product with $\tilde{u}_t$ and using (2.3).

Differentiate (4.1) with respect to t and take an inner product with $\tilde{u}_{tt}$, then as in (5.3) we obtain

$$\|\tilde{u}_{tt}\|_1^2 \leq C\{\|S(f_t - u_{tt})\|^2 + \|\tilde{u}\|_1^2 + \|\tilde{u}_t\|_1^2\}.$$

This completes the proof. □

It is here that we exploit the full potential of the Steklov mollification and the discrete projection. Let $\Theta^n = U^n - \tilde{u}^n$ and $E^n = u^n - U^n = \rho^n - \Theta^n$.

THEOREM 5.2. *Let u^n and U^n be the solution of (1.1) and (5.1), respectively. Further, let $u, u_t \in L^\infty(H^\alpha(\Omega))$ and $u_{tt} \in L^\infty(L^2(\Omega))$ for $1 \leq \alpha \leq 2$. Then there are positive constants C and k_0 such that the error $E^J = u(t_J) - U^J$*

$$\|E^J\| \leq C(u, T)(h^\alpha + k), \quad J = 1, 2, \cdots, N$$

holds for $0 < k \leq k_0$.

Proof. Since the estimate for ρ^J can be found out from Lemma 4.1, it is sufficient to obtain an estimate for Θ^J. From (4.1) and (5.1), it follows that

(5.4)
$$\begin{aligned}
&\bar{\partial}_t\Theta^n + A_h\left(\bar{\partial}_t\Theta^n\right) + B_h\Theta^n \\
&\quad = (\bar{\partial}_t u^n - \bar{\partial}_t\tilde{u}^n) + (u_t^n - \bar{\partial}_t u^n) + (Su_t^n - u_t^n) + A_h(\tilde{u}_t^n - \bar{\partial}_t\tilde{u}^n) \\
&\quad = \bar{\partial}_t\rho(t_n) + (\tilde{u}_t^n - \bar{\partial}_t\tilde{u}^n) - I_1^n + A_h(\tilde{u}_t^n - \bar{\partial}_t\tilde{u}^n).
\end{aligned}$$

Apply Theorem 5.1 to (5.4) to obtain

$$\begin{aligned}
\|\Theta^J\|_1^2 \leq C\{\|\Theta^0\|_1 + k\sum_{n=1}^{J}(\|\bar{\partial}_t\rho(t_n)\|^2 + \|\tilde{u}_t^n - \bar{\partial}_t\tilde{u}^n\|^2 \\
+ \|I_1^n\|^2 + \|\tilde{u}_t^n - \bar{\partial}_t\tilde{u}^J\|_1^2)\}.
\end{aligned}$$

Note that

$$k\sum_{n=1}^{J}\|\bar{\partial}_t\rho(t_n)\| \leq \sum_{n=1}^{J}\int_{t_{n-1}}^{t_n}\|\rho_t\|\,ds \leq Ch^\alpha\|u_t\|_{L^1(H^\alpha)}, \quad 1\leq\alpha\leq 2.$$

Further, using Lemma 5.1, we have

$$k\sum_{n=1}^{J}\|I_1^n\| \leq Ch^\alpha\|u_t\|_{L^\infty(H^\alpha)},$$

and

$$k\sum_{n=1}^{J}\|\tilde{u}_t^n - \bar{\partial}_t\tilde{u}^n\|_1 \leq Ck\sum_{n=1}^{J}\|k\tilde{u}_{tt}\|_1 \leq Ck\|\tilde{u}_{tt}\|_{L^\infty(H^1)}.$$

This completes the rest of the proof. □

The Crank-Nicolson scheme

For a second order accurate in time, we consider the Crank-Nicolson scheme for (1.1). Let $U^{n-\frac{1}{2}} = (U^n + U^{n-1})/2$ and $f^{n-\frac{1}{2}} = f(t_{n-\frac{1}{2}})$. Define the fully discrete scheme as

$$\text{(5.5a)} \quad \bar{\partial}_t U^n + A_h \bar{\partial}_t U^n + B_h(t_{n-\frac{1}{2}}) U^{n-\frac{1}{2}} = S f^{n-\frac{1}{2}}, \quad x \in \Omega_h,$$

$$\text{(5.5b)} \quad U^0 = u_0(x), \quad x \in \Omega_h,$$

$$\text{(5.5c)} \quad U^n = 0, \quad x \in \partial\Omega_h.$$

Below, we shall prove a stability result in discrete H^1- norm not for (5.5a) but for a modified equation

(5.6)

$$\begin{aligned} &\bar{\partial}_t U^n + A_h \left(\bar{\partial}_t U^n\right) + B_h U^{n-\frac{1}{2}} \\ &\quad = S f^{n-\frac{1}{2}} + \sum_{l=1}^{2} \bar{\nabla}_l F^{n-\frac{1}{2}}, \quad (x, t_n) \in \Omega_h \times (0, T], \end{aligned}$$

THEOREM 5.3. *Let U^n be a solution of* (5.5). *Then there are positive constants C and k_0 such that for $0 < k \le k_0$*

$$\|U^J\|_1 \le C\{\|U^0\|_1 + (k \sum_{n=1}^{J} \|S f^{n-\frac{1}{2}}\|^2)^{\frac{1}{2}} + (k \sum_{n=1}^{J} \|F^{n-\frac{1}{2}}\|^2)^{\frac{1}{2}}\}.$$

Proof. Forming the inner product between (5.6) and $U^{n-\frac{1}{2}}$, it follows that

$$\bar{\partial}_t \|U^n\|^2 + \bar{\partial}_t \|U^n\|_1^2 \le C\{\|S f^{n-\frac{1}{2}}\|^2 + \|F^{n-\frac{1}{2}}\|^2 + \|U^{n-\frac{1}{2}}\|_1^2\}.$$

Summing from $n = 1$ to J, we obtain

$$\begin{aligned} (1 - Ck)\|U^J\|_1^2 \le & C\{\|U^0\|_1^2 + k \sum_{n=1}^{J} (\|S f^{n-\frac{1}{2}}\|^2 + \|F^{n-\frac{1}{2}}\|^2) \\ & + k \sum_{m=0}^{J-1} \|U^n\|_1^2\}. \end{aligned}$$

Choosing k_0 appropriately so that $(1 - Ck) > 0$ for $0 < k \le k_0$, we obtain the desired result using discrete Gronwall's Lemma. □

Below, we shall present an error analysis using discrete projection. Let u^n and U^n be the solutions of (1.1) and (5.5), respectively. Let $E^n = u^n - U^n$.

THEOREM 5.4. *Let $u, u_t \in L^\infty(H^\alpha(\Omega))$ and $u_{tt} \in L^\infty(L^2(\Omega))$ for $1 \le \alpha \le 2$. Then there are positive constants C and k_0 such that the error $E^J = u(t_J) - U^J$*

$$\|E^J\| \le C\{h^\alpha + k^2\}. \qquad J = 1, 2, \cdots, N.$$

Proof. Since $E^J = \rho^J - \Theta^J$ and we know the estimate for ρ^J, we have only to estimate for $\Theta^J = U^J - \tilde{u}^J$ due to the triangle inequality. From (4.1) and (5.5), it follows that

(5.6)
$$\begin{aligned}
&\bar{\partial}_t\Theta^n + A_h\bar{\partial}_t\Theta^m + B_h(t_{m-\frac{1}{2}})\Theta^{m-\frac{1}{2}} = (\bar{\partial}_t u^n - \bar{\partial}_t\tilde{u}^n)\\
&+ (u_t^{m-\frac{1}{2}} - \bar{\partial}_t u^n) + (Su_t^{m-\frac{1}{2}} - u_t^{m-\frac{1}{2}}) + A_h(\tilde{u}_t^{m-\frac{1}{2}} - \bar{\partial}_t\tilde{u}^m)\\
&= \bar{\partial}_t\rho(t_m) + (u_t^{m-\frac{1}{2}} - \bar{\partial}_t u^n) - I_1^n + A_h(\tilde{u}_t^{m-\frac{1}{2}} - \bar{\partial}_t\tilde{u}^m).
\end{aligned}$$

It follows that as in Theorem 5.3

$$\begin{aligned}
\|\Theta^J\|_1^2 \le& C\{\|\Theta^0\|_1^2 + k\sum_{m=1}^{J}[\|\bar{\partial}_t\rho(t_m)\|^2\\
&+ \|u_t^{m-\frac{1}{2}} - \bar{\partial}_t u^n\|^2 + \|I_1^n\|^2 + \|\tilde{u}_t^{m-\frac{1}{2}} - \bar{\partial}_t\tilde{u}^m\|_1^2\}.
\end{aligned}$$

It completes the rest of proof. □

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