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A GLOBALITY OF A HOPF BIFURCATION IN A FREE BOUNDARY PROBLEM

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ABSTRACT. A globality of the Hopf bifurcation in a free boundary problem for a parabolic partial differential equation is investigated in this paper. We shall examine the global behavior of the Hopf critical eigenvalues and and apply the center-index theory to show the globality.

1. Introduction

In [3], it was shown the existence of Hopf bifurcation for the following parabolic free boundary problem with a bifurcation parameter τ

$$(1)\qquad \begin{cases} v_t = Dv_{xx} - c^2 v + H(x - s(t)) & \text{for } (x,t) \in \Omega^- \cup \Omega^+, \\ v_x(0,t) = 0 = v_x(1,t) & \text{for } t > 0, \\ v(x,0) = v_0(x) & \text{for } 0 \leq x \leq 1, \\ \tau \frac{ds}{dt} = C(v(s(t),t)) & \text{for } t > 0, \\ s(0) = s_0, \end{cases}$$

where $v(x,t)$ and $v_x(x,t)$ are required to be continuous in Ω. Here $\Omega = (0,1) \times (0,\infty)$, $\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t)\}$ and $\Omega^+ = \{(x,t) \in \Omega : s(t) < x < 1\}$. Furthermore, τ is the bifurcation parameter and $H(y)$ denotes the Heaviside unit step function. It was shown that at a critical value τ^* of τ, the stationary solution loses stability and a branch of stable periodic solutions appears for a finite diffusion constant D.

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Moreover, the steady state is stable for $\tau > \tau^*$ and unstable for $\tau < \tau^*$, and τ^* is a bifurcation point for a stable branch of periodic orbits which turns in the direction $\tau < \tau^*$. This Hopf bifurcation guarantees the existence of small amplitude, nontrivial periodic curve bifurcating from the Hopf point $(v^*(x), s^*, \tau^*)$. Results of numerical experiments in [3] with this problem indicate that this branch of periodic solutions persists as τ increases and the amplitude of the free boundary increases. We shall consider the question of a globality of the Hopf bifurcation for this problem. The term global means that there is a continuum $\mathcal{P}$ of periodic orbits such that $\mathcal{P}$ is unbounded or $\mathcal{P}$ contains arbitrary large virtual periods (which are multiples of the minimal period).

In this paper, we shall prove the globality of the Hopf bifurcation in this problem . In order to do so, we examine the global behavior of Hopf critical eigenvalues and make use of a center index introduced by Mallet-Paret & Yorke [2]. We now recall the notation and the systems of equations appearing in [3].

An abstract evolution system corresponding to (1) is given by

$$\text{(F)} \qquad \begin{cases} v_t + Av = H(x-s), & (x \in (0,1)\setminus\{s\}\,,\ t>0) \\ \tau s'(t) = C(v(s(t),t)), & (t>0) \\ v(x,0) = v_0(x)\,,\ s(0) = s_0\,. \end{cases}$$

Here A is the operator $Av = -v_{xx} + c^2 v$ together with Neumann boundary conditions $v_x(0) = v_x(1) = 0$. Note that we can assume that $D = 1$ for a finite diffusion constant D by a rescaling of t in (1).

We recall the regularization of the system (F) from [3]. Let $G : [0,1]^2 \to \mathbf{R}$ be Green's function of the operator A. Define $g : [0,1]^2 \longrightarrow \mathbf{R}$

$$g(x,s) := \int_s^1 G(x,y)\,dy = A^{-1}(H(\cdot - s))(x),$$

and $\gamma : [0,1] \longrightarrow \mathbf{R}$

$$\gamma(s) := g(s,s)\,.$$

If we define

$$u(t)(x) := v(x,t) - g(x,s(t))$$

the regularization problem was obtained by

$$(\mathrm{R})\qquad \begin{cases} \frac{du}{dt} + Au = \frac{1}{\tau}G(x,s)C(u(s)+\gamma(s)) \\ s'(t) = \frac{1}{\tau}C(u(s)+\gamma(s)) \\ u(0) = u_0, \ \ s(0) = s_0. \end{cases}$$

2. The behaviors of real eigenvalues

In order to examine the global behavior of the Hopf critical eigenvalues which are Hopf points, we need to investigate the properties of the real part of eigenvalues and to show the Hopf point is unique for the infinite time.

We recall Proposition 3.1 from [3]. Let $(v^*(x), s^*)$ denote the uniquely determined stationary solution of (1). The linearized eigenvalue problem for (1) is given by

$$(2)\qquad \begin{cases} (A+\lambda)v = -\delta_{s^*} \\ \rho\cdot\lambda = \gamma'(s^*) + G(s^*,s^*) + v(s^*) \end{cases}$$

where δ_{s^*} is the Dirac delta function and $\rho = \tau/4$.

We define a set $S_\nu := \{\lambda \in C \,|\, \mathrm{Re}\lambda > -\nu,\ \nu > 0\}$. In the first equation of (2), $A+\lambda$ is invertible in S_{c^2} and hence has a unique solution $v = -G_\lambda(\cdot, s^*)$, where G_λ is Green's function for the operator $A+\lambda$. It follows that the second equation of (2) can be written by

$$(3)\qquad \rho\cdot\lambda = \gamma'(s^*) + G(s^*,s^*) - G_\lambda(s^*,s^*).$$

We now establish some properties of the function $G_\lambda(s^*,s^*)$. Let $\alpha = \mathrm{Re}\lambda$ and $\beta = \mathrm{Im}\lambda$ where $Re\lambda$ is the real part of eigenvalue and $Im\lambda$ is the imaginary part of eigenvalue.

LEMMA 2.1. *The function $G_\alpha(s^*,s^*)$ is a strictly decreasing convex function of α, $\alpha > -c^2$, and*

$$\lim_{\alpha\to -c^2} G_\alpha(s^*,s^*) = \infty, \qquad \lim_{\alpha\to\infty} G_\alpha(s^*,s^*) = 0.$$

Furthermore, $\dfrac{dG_\lambda}{d\lambda}(s^*,s^*) \neq 0$ *for λ with $Im\lambda \neq 0$.*

Proof. Since the operator $(A+\lambda)^{-1}$ exists for $Re\lambda > -c^2$,

$$\lim_{\alpha \to -c^2} (A+\alpha)^{-1} = \infty \quad \textit{and} \quad \lim_{\alpha \to \infty} (A+\alpha)^{-1} = 0.$$

Therefore, we obtain

$$\lim_{\alpha \to -c^2} G_\alpha(s^*, s^*) = \infty \quad \textit{and} \quad \lim_{\alpha \to \infty} G_\alpha(s^*, s^*) = 0.$$

In order to show that $\alpha \mapsto G_\alpha(s^*, s^*)$ is a strictly decreasing function, we define $h(\lambda)(x) := G_\lambda(x, s^*) - G(x, s^*)$. Then (in the weak sense at first)

$$(A+\lambda)h(\lambda) = -\lambda G(\cdot, s^*).$$

It follows that $h(\lambda) \in D(A)$ and $h : \mathbf{R}^+ \to D(A)$ is differentiable with $h(\lambda) + (A+\lambda)h'(\lambda) = -G(\cdot, s^*)$, and hence

$$(A+\lambda)h'(\lambda) = -G_\lambda(\cdot, s^*).$$

Therefore

$$\begin{aligned} -\overline{h'(\lambda)(s^*)} &= \int_0^1 (A+\lambda)^2 h'(\lambda)\overline{h'(\lambda)(x)}\, dx \\ &= \int_0^1 \left(|Ah'(\lambda)|^2 + \lambda^2|h'(\lambda)|^2 + 2\lambda A|h'(\lambda)|^2\right) dx. \end{aligned}$$

It follows that

$$\int_0^1 \left(|Ah'(\lambda)|^2 + (\alpha^2 - \beta^2)|h'(\lambda)|^2 + 2\alpha A|h'(\lambda)|^2\right) dx = -Re\,(h'(\lambda)(s^*))$$

and

$$2\beta\Big(\int_0^1 (A+\alpha)|h'(\lambda)|^2\, dx\Big) = \operatorname{Im}(h'(\lambda)(s^*)). \tag{4}$$

For $\beta = 0$, the equation (4) become

$$\int_0^1 |(A+\alpha)h'(\alpha)|^2\, dx = -h'(\alpha)(s^*) > 0.$$

From the definition of h, we have $h'(\lambda)(s^*) = \dfrac{dG_\lambda}{d\lambda}(s^*, s^*)$ which implies that G_α is a strictly decreasing function of α. Moreover, we obtain $\mathrm{Im}(\dfrac{dG_\lambda}{d\lambda}) \neq 0$ holds iff $\beta \neq 0$ from (4).

Finally, to show the convexity of G_α, we differentiate the equation $h(\alpha) + (A+\alpha)h'(\alpha) = -G(\cdot, s^*)$ with respect to α, then we have

$$(A+\alpha)h''(\alpha) = -2h'(\alpha).$$

Multiplying $(A+\alpha)^2 h''(\alpha)$ and integrating both sides, then

$$\begin{aligned}\int_0^1 (A+\alpha)^3\ h''(\alpha)^2\,dx =& -2\int_0^1 (A+\alpha)^2 h'(\alpha)h''(\alpha)\,dx\\ =& -2\int_0^1 (A+\alpha)(-G_\alpha(x,s^*))h''(\alpha)\,dx\\ =& 2h''(\alpha)(s^*).\end{aligned}$$

Since $h''(\alpha) = \dfrac{d^2G_\alpha}{d\alpha^2}(s^*, s^*)$, the convexity of G_α is shown. □

LEMMA 2.2. *For some negative number $-\hat{\lambda}$, the function $\dfrac{dG_\lambda}{d\lambda}(s^*, s^*)$ which is evaluated at some complex eigenvalues have the following property*

$$\begin{aligned}\frac{dG_\lambda}{d\lambda}(s^*,s^*)\Big|_{(Re\lambda=-\hat{\lambda},\ \mathrm{Im}\lambda=0)} &> -\frac{dG_\lambda}{d\lambda}(s^*,s^*)\Big|_{(Re\lambda=0,\ \mathrm{Im}\lambda=0)}\\ &> Im\, G_\beta(s^*,s^*)|_{(Re\lambda=0,\ \mathrm{Im}\lambda=\beta)}.\end{aligned}$$

Proof. We use the cosine Fourier representation of G_λ which is given by

$$G_\lambda(s^*,s^*) = \frac{1}{c^2+\lambda} + 2\sum_{k=1}^{\infty}\frac{(\cos k\pi s^*)^2}{k^2\pi^2+c^2+\lambda}.$$

Differentiate this with respect to λ,

$$\frac{dG_\lambda}{d\lambda}(s^*,s^*)\Big|_{(Re\lambda=-\hat{\lambda},\ \mathrm{Im}\lambda=0)} = -\frac{1}{(c^2+\lambda)^2} - 2\sum_{k=1}^{\infty}\frac{(\cos k\pi s^*)^2}{(k^2\pi^2+c^2+\lambda)^2}.$$

Let $-\hat{\lambda}$ be a negative constant and $\beta \neq 0$, then

$$\begin{aligned}
\frac{dG_\lambda}{d\lambda}(s^*,s^*)\Big|_{(Re\lambda=-\hat{\lambda},\ \mathrm{Im}\lambda=0)} &= \frac{1}{(c^2-\hat{\lambda})^2} + 2\sum_{k=1}^{\infty}\frac{(\cos k\pi s^*)^2}{(k^2\pi^2+c^2-\hat{\lambda})^2} \\
&> \frac{1}{c^4} + 2\sum_{k=1}^{\infty}\frac{(\cos k\pi s^*)^2}{(k^2\pi^2+c^2)^2} \\
&= -\frac{dG_\lambda}{d\lambda}(s^*,s^*)\Big|_{(Re\lambda=0,\ \mathrm{Im}\lambda=0)} \\
&> \frac{1}{c^4+\beta^2} + 2\sum_{k=1}^{\infty}\frac{(\cos k\pi s^*)^2}{(k^2\pi^2+c^2)^2+\beta^2)^2} \\
&= \frac{1}{\beta}\,\mathrm{Im}G_\beta(s^*,s^*)\Big|_{(Re\lambda=0,\ \mathrm{Im}\lambda=\beta)}
\end{aligned}$$

where $G_\beta(s^*,s^*)$ is Green's function of the operator of $A+i\beta$. Thus, we have

$$\begin{aligned}
-\frac{dG_\lambda}{d\lambda}(s^*,s^*)\Big|_{(\mathrm{Re}\,\lambda=-\hat{\lambda},\ \mathrm{Im}\,\lambda=0)} &> -\frac{dG_\lambda}{d\lambda}(s^*,s^*)\Big|_{(\mathrm{Re}\,\lambda=0,\,\mathrm{Im}\,\lambda=0)} \\
&> \mathrm{Im}G_\beta(s^*,s^*)|_{(\mathrm{Re}\,\lambda=0,\,\mathrm{Im}\,\lambda=\beta)}.
\end{aligned}$$

□

From (3), a real eigenvalue $\lambda = \alpha$ satisfies the equation

(5) $$\gamma'(s^*) + G(s^*,s^*) - \rho\alpha = G_\alpha(s^*,s^*).$$

Here $\gamma'(s^*) + G(s^*,s^*)$ is a positive constant. The real eigenvalues of (5) can be determined by the locating the intersection of the curve $G_\alpha(s^*,s^*)$ with the straight line $\gamma'(s^*) + G(s^*,s^*) - \rho\alpha$ (see Figure 1).

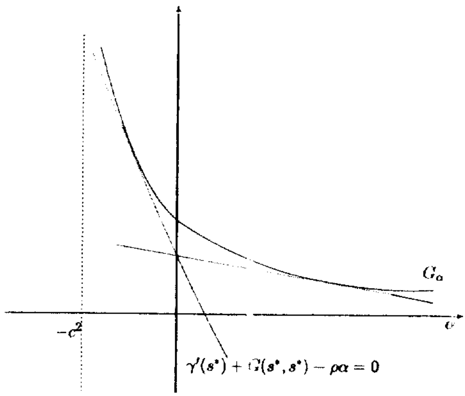

Figure 1: The graphs of G_α and $\gamma'(s^*) + G(s^*, s^*) - \rho\alpha$.

Let ρ_n be defined by

$$\begin{aligned}\rho_n := \min\{\rho \in \mathbf{R} :\ & \text{there exists at least one negative real eigenvalue} \\ :\ & \text{between the line } \gamma'(s^*) + G(s^*, s^*) - \rho\alpha \text{ and } G_\alpha \text{ for } \alpha > -c^2\}.\end{aligned}$$

We obtain the next lemma from a simple geometrical analysis.

LEMMA 2.3. *There exists a positive constant ρ_T in S_{c^2} with $\rho_T < \rho_n$:*

(i) *there are no real eigenvalues of* (5) *for $\rho_T < \rho < \rho_n$*
(ii) *there exists a unique real positive eigenvalue λ_T at $\rho = \rho_T$*
(iii) *there exist exactly two real eigenvalues $\lambda_1(\rho)$ and $\lambda_2(\rho)$ for $\rho < \rho_T$ where $-\rho_T$ is the slope of the line which is tangent to the curve $G_\alpha(s^*, s^*)$.*

REMARK 1. If we define

$$F(\lambda, \rho) := \lambda\rho - (\gamma'(s^*) + G(s^*, s^*)) + G_\alpha(s^*, s^*)$$

then

$$\frac{\partial F}{\partial \lambda}(\lambda^*, \rho) = 0 \quad \text{holds} \iff \lambda^* \quad \text{must be real}$$

by the Lemma 2.3. Therefore, for $\rho < \rho_n$,

$$\frac{\partial F}{\partial \lambda}(\lambda^*, \rho) = 0 \quad \text{holds} \iff (\lambda^*, \rho) = (\alpha_T, \rho_T)$$

where α_T is a real eigenvalue. When ρ is close to ρ_T in the right hand side, the real eigenvalues are expected to be changed to complex eigenvalues. The local behavior of eigenvalues near (α_T, ρ_T) is described as follows.

LEMMA 2.4. *The positive real eigenvalue α_T corresponding to $\rho = \rho_T$ is of multiplicity two. Near $\rho = \rho_T$, α_T splits into two eigenvalues since*

$$\begin{aligned} \lambda \simeq & \alpha_T \pm i\sqrt{\Delta_T(\rho - \rho_T)} \quad \text{for} \quad \rho > \rho_T \\ \lambda \simeq & \alpha_T \pm \sqrt{\Delta_T(\rho_T - \rho)} \quad \text{for} \quad \rho < \rho_T \end{aligned}$$

with $\Delta_T = \left. \dfrac{2\alpha_T}{\frac{d^2 G_\lambda}{d\lambda^2}}(s^*, s^*) \right|_{(Re\lambda = \alpha_T, \ \text{Im}\lambda = 0)}$

Proof. We use the Taylor series for $F(\lambda^*, \rho) = 0$ at (α_T, ρ_T), then we have

$$\begin{aligned} F(\lambda^*, \rho) =& \lambda\rho - (\gamma'(s^*) + G(s^*, s^*)) + G_\lambda(s^*, s^*) \\ \simeq & \alpha_T(\rho - \rho_T) + (\lambda - \alpha_T)^2 \cdot \left. \frac{d^2 G_\lambda}{2\, d\lambda^2}(s^*, s^*) \right|_{(Re\lambda = \alpha_T, \ \text{Im}\lambda = 0)}. \end{aligned}$$

The conclusion follows easily from the above equation. □

Since there was a Hopf bifurcation at $\rho = \rho^*$, the critical point ρ^* must lie in (ρ_T, ρ_n). In the following lemma, we determine a subinterval of (ρ_T, ρ_n) containing ρ^*.

LEMMA 2.5. *There exist positive constants ρ_s and $\hat{\lambda}$ such that there are no eigenvalues in $S_{\hat{\lambda}}$ for $\rho_n > \rho \geq \rho_s$.*

Proof. Let ρ_0 be the slope of $-G_\lambda(s^*, s^*)$ at $Re\,\lambda = 0$ i.e.,

$$\rho_0 = -\left. \frac{dG_\lambda}{d\lambda}(s^*, s^*) \right|_{Re\lambda = 0}$$

then $\rho_n > \rho_0$. Let ρ_s be a value satisfying $\rho_n > \rho_s > \rho_0$. For $\rho_n > \rho \geq \rho_s$, there are no real eigenvalues. Thus we need to show that there are no complex eigenvalues for $\rho_n > \rho \geq \rho_s$ where $-\hat{\lambda}$ is a negative constant determined by

$$\rho_s = -\frac{dG_\lambda}{d\lambda}(s^*, s^*)\bigg|_{Re\lambda=-\hat{\lambda},\ \ \mathrm{Im}\lambda=0}.$$

By the Lemma 2.3, we see that if $Re\lambda > -\hat{\lambda}$ and $\mathrm{Im}\lambda > 0$, then

$$\begin{aligned}\rho_s &= -\frac{dG_\lambda}{d\lambda}\bigg|_{(Re\lambda=-\hat{\lambda},\ \mathrm{Im}\lambda=0)} > -\frac{dG_\lambda}{d\lambda}\bigg|_{(Re\lambda=0,\ \mathrm{Im}\lambda=0)} \\ &= \frac{1}{\beta}\ \ \mathrm{Im}\, G_\beta\bigg|_{(Re\lambda=0,\ \mathrm{Im}\lambda=\beta)}\end{aligned}$$

which implies that there are no complex eigenvalues in $S_{\hat{\lambda}}$ since $\mathrm{Im}G_\beta$ $(s^*, s^*) = \rho \cdot \beta$ has a solution ρ^* when $\rho^* \geq \rho_s$ (see the equation (21) in [3]). □

REMARK 2. It is clear that $\rho_s > \rho_T$ and $\rho_s > \rho^*$.

3. A global Hopf bifurcation theorem

We can now describe the global behavior of the complex eigenvalues with respect to ρ after the imaginary axis crosses. From Lemma 2.3, there is a unique real positive eigenvalue at $\rho = \rho_T$. For $\rho > \rho_T$, a pair of complex conjugate eigenvalues appears. At this stage, there may exist other complex eigenvalues, however, we can avoid such an existence in the following sense.

We now trace the behavior of these other complex eigenvalues as ρ increases from ρ_T. Since there are no real eigenvalues for $\rho_n > \rho > \rho_T$, they remain as complex eigenvalues and can be uniquely expressed as functions of ρ. By Lemma 2.4, they must cross the imaginary axis from right to left at some point $\rho = \hat{\rho}$ before ρ reaches ρ_s. However, because of the uniqueness of pure imaginary eigenvalues, $\hat{\rho}$ must be equal to ρ^* and the corresponding eigenvalues must be $\lambda(\rho^*)$. This establishes the global behavior of Hopf critical eigenvalues with respect to ρ. The following theorem summarizes what we have proved:

THEOREM 3.1. *Suppose that* $0 < \dfrac{1-2a}{2} < 1/c^2$. *Then we have*

(i) *At* $\rho = \rho^*$, *all other eigenvalues lie strictly in the left half-plane in* $\mathbf{C}$.

(ii) *Following the Hopf bifurcation, the pure imaginary eigenvalues behave as follows:* $\lambda(\rho)$ *and* $\overline{\lambda(\rho)}$ *combine to make a real eigenvalue* α_T *of multiplicity two at* $\rho = \rho_T (< \rho^*)$, *which then it splits into the two real eigenvalues, say* $\lambda_1(\rho)$ *and* $\lambda_2(\rho)$ *for* $\rho < \rho_T$. *Moreover, for* $\rho \leq \rho^*$, *there are no eigenvalues except for those constructed above with some constant* $\hat{\lambda}$.

We note that the last claim of (ii) implies (i). In fact, suppose there are eigenvalues (that must be complex) other than those constructed above; they must move to the left half-plane when ρ approaches ρ_s. By Remark 2, they cannot join to the above Hopf critical eigenvalues. Therefore, they cross the imaginary axis at different points from the critical eigenvalue $\pm i \ \mathrm{Im}\,\lambda(\rho^*)$. This contradicts the uniqueness of the pure imaginary eigenvalues $\pm i \ \mathrm{Im}\,\lambda(\rho^*)$.

Now, in order to show the global Hopf bifurcation, we shall use the center index ⊕. A global Hopf bifurcation asserts the existence of a bifurcating continuum $\mathcal{P}$ of periodic orbits with the property that $\mathcal{P}$ is unbounded, or $\mathcal{P}$ contains another center $(\hat{u}, \hat{s}, \hat{\rho})$ which is not a Hopf point. Here, a center means that some eigenvalues of the linear part of (R) are purely imaginary and not zero. Let $E(\rho)$ denote the sum of the multiplicities of the eigenvalues of the linearization of (R) having strictly positive real parts. Let $E(\hat{\rho}+)$ and $E(\hat{\rho}-)$ denote right- and left-hand limits of E at $\hat{\rho}$. Define the crossing number χ, the net number of pairs of eigenvalues crossing the imaginary axis at $\hat{\rho}$ by

$$\chi = \frac{1}{2}\Big(E(\hat{\rho}+) - E(\hat{\rho}-)\Big).$$

We define the center index of a center $(\hat{u}, \hat{s}, \hat{\rho})$ to be the product

$$⊕\,(\hat{u}, \hat{s}, \hat{\rho}) = \chi \cdot (-1)^{E(\hat{\rho})}.$$

Essentially, a nonzero H-index,

$$H := \Sigma \ ⊕ \quad \neq 0$$

implies the global Hopf bifurcation in [1]. Therefore, we must show that H-index is not zero.

Because of the global behavior of Hopf critical eigenvalues in Theorem 3.1, the Hopf point $(0, s^*, \rho^*)$ is the only center of (R). Thus, $E(\rho^*) = 0$ and $\chi = 1$ imply that a center index at $(0, s^*, \rho^*)$ is equal to 1. Hence, the H-index, $H = \Sigma$ ⊕ $= 1 \neq 0$. Therefore, we now have a global Hopf bifurcation.

References

[1] B. Fiedler, *Global Hopf Bifurcation of Two-Parameter Flows*, Arch. Rational Mech. Anal. (1985), 59-81.

[2] J. Mallet-Paret, J. Yorke, *Snakes: Oriented Families of Periodic Orbits, Their Sources, Sinks, and Continuation*, J. Differential Equations **43** (1982), 419-450.

[3] Y. M. Lee, R. Thompson and R. Schaaf, *A Hopf bifurcation in a parabolic free boundary problem*, J. of Comput. Appl. Math. **52** (1994), 305-325.

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