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COHOMOLOGY OF GROUPS AND TRANSFER THEOREM

EUNMI CHOI

ABSTRACT. In this paper, we study the dependence of corestriction (or transfer) map on the choice of transversals. We also study transfer theorems with respect to some commutative subgroups.

1. Introduction

Representation theory of a finite group G strongly involves cohomology theory. There are three fundamental maps over cohomology groups, such as restriction, inflation and corestriction (or called, transfer). The first two maps were been studied in many places, the latter one, however, still remains ackward. There are two ways to define the corestriction; one is via abstract complexes and the other is cohomology groups, and each of these implies the others. In both ways, transversal sets are being used; however since the sets are not unique, calculations involving transversal are very troublesome. For this reason, character-theoretic corestriction which is a dual of ordinary corestriction has been introduced [10]. But this shall not be our concern, so that transversals are still key factor of the corestriction.

In this paper, we ask how corestriction maps depend on choices of transversal. Since corestriction maps on cohomology groups are very related to the group theoretical transfer map, which is one of the basic techniques of finite group theory, we shall also study transfer theorems with respect to some commutator subgroups.

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2. Preliminaries

Let H be a subgroup of G with $|G : H| = \mu < \infty$, and let $S = \{s_i\}_{i=1}^{\mu}$, $s_1 = 1$, be a right transversal of H in G. Then $G = \bigcup_{i=1}^{\mu} Hs_i = \bigcup_{i=1}^{\mu} s_i^{-1}H$. For any left G-module M, let M^G be a set of elements in M fixed by all $g \in G$. A homomorphism $S_{H,G} : M^H \to M^G$ defined by $S_{H,G}(m) = \prod_{i=1}^{\mu} s_i^{-1} \cdot m$ for $m \in M^H$, is called the trace map from H to G and is known to be independent of the choices of transversal.

LEMMA 1. [9] *Let A be a left G-module. Then there is a homomorphism ϕ : $Hom_H(A, M) \to Hom_G(A, M)$ defined by $(\phi(t))a = \prod_{i=1}^{\mu} s_i^{-1} \cdot t(s_i a)$.*

The map ϕ is a trace map, too. Let $\mathcal{C}(G)$ and $\Sigma = Z[G]$ denote standard complex and integer group ring for G, respectively, and consider

$$(1)\qquad \begin{array}{llll} \mathcal{C}(G) : & \cdots & \Sigma[\cdot_G] = X_0 \xleftarrow{\partial_1} X_1 \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_k} X_k \leftarrow \cdots \\ \quad \Lambda \downarrow & Z & \\ \mathcal{C}(H) : & \cdots & \Sigma[\cdot_H] = X_0' \xleftarrow{\partial_1'} X_1' \xleftarrow{\partial_2'} \cdots \xleftarrow{\partial_k'} X_k' \leftarrow \cdots \end{array}$$

where each X_k is a free left G-module with generator $[g_1, \cdots, g_k]$ for $g_i \in G$, and $[\cdot_G]$ is a generator of X_0. Let Λ be an H-homomorphism from $\mathcal{C}(G)$ to $\mathcal{C}(H)$, which makes (1) commutes, i.e., $\partial_k' \Lambda_k = \Lambda_{k-1}\partial_k$, for $k > 0$, and $\varepsilon'\Lambda_0 = \varepsilon$. (Here, ε and ε' are mappings to Z from $\Sigma[\cdot_G]$ and $\Sigma[\cdot_H]$, respectively.) Let $S_{H,G}$ be a trace map $\mathrm{Hom}_H(\mathcal{C}(G), M) \to \mathrm{Hom}_G(\mathcal{C}(G), M)$, and let

$$\Upsilon = S_{H,G}\Lambda^* : \ \mathrm{Hom}_H(\mathcal{C}(H), M) \to \mathrm{Hom}_G(\mathcal{C}(G), M),$$

where $\Lambda^* = (\Lambda, 1)$ is the induced map of Λ. For any standard k-cochain u of H in M, $\Upsilon_k : \mathrm{Hom}_H(X_k', M) \to \mathrm{Hom}_G(X_k, M)$ is given by

$$(2)\qquad \Upsilon_k(u) = (S_{H,G}\Lambda_k^*)(u) = S_{H,G}(\Lambda_k^*(u)) = S_{H,G}(u\Lambda_k).$$

Let $\bar{g}$ denote the unique s_i such that $g \in Hs_i$. Then $g\bar{g}^{-1} \in H$ for all $g \in G$, thus $\Upsilon_0(u)[\cdot_G] = \prod_{i=1}^{\mu} s_i^{-1} \cdot u[\cdot_H]$, and

(3)
$$\Upsilon_k(u)[g_1, .., g_k]$$
$$= \prod_{i=1}^{\mu} s_i^{-1} \cdot u[s_i g_1 \overline{(s_i g_1)}^{-1}, \cdots, \overline{(s_i g_1 \cdot \cdot g_{k-1})} g_k \overline{(s_i g_1 \cdot \cdot g_k)}^{-1}]$$

for $k > 0$. This is a corestriction map on cochain group relative to the transversal S, denoted by $\mathrm{Cor}_{G,H}$. Since a corestriction map sends cocycles to cocycles, and coboundaries to coboundaries, there is an induced map on cohomology groups, called a corestriction $\mathrm{Cor}^*_{G,H}$ over cohomology groups. Clearly, we have for $f \in Z^k(H, M)$,

$$\mathrm{Cor}^*_{G,H}(fB^k(H,M)) = (\mathrm{Cor}_{G,H} f)B^k(G,M)$$

for $f \in Z^k(H, M)$.

3. Corestriction map

In this section we shall study the following questions.

(a) Are the corestriction maps $\mathrm{Cor}_{G,H}$ over complexes or over cocycle groups independent of the choices of transversal?

(b) What conditions give independency of the choices of transversal?

For the map $\mathrm{Cor}^*_{G,H}$ over cohomology groups, this question is answered positively by Eckmann [3].

THEOREM 2. *Corestriction map on cochain group does depend on the choices of transversal.*

Proof. Since corestriction map over cochain group is a composition map of $\Lambda^* = (1, \Lambda)$ and the trace map $S_{H,G}$ as in (2), and since $S_{H,G}$ is known to be independent of the choice of transversal, we need to know how the H-homomorphism $\Lambda_k : X_k \to X'_k$ which satisfies $\partial'_k \Lambda_k = \Lambda_{k-1}\partial_k$ is defined explicitly, for all k.

Let $S = \{s_i\}_{i=1}^{\mu}$ be a transversal chosen for $\mathrm{Cor}_{G,H}$, and suppose $k = 0$. Since $s_i[\cdot_G]$ $(i = 1, \cdots, \mu)$ is a base for X_0. $\varepsilon'\Lambda_0 = \varepsilon$ implies that $(\varepsilon'\Lambda_0)(s_i[\cdot_G]) = \varepsilon(s_i[\cdot_G]) = s_i\varepsilon([\cdot_G]) = s_i 1 = 1 = \varepsilon'([\cdot_H])$, so that we may take $\Lambda_0(s_i[\cdot_G]) = [\cdot_H]$. Thus for any $h_i \in H$, the H-homomorphism Λ_0 is given by

$$\Lambda_0(h_i s_i[\cdot_G]) = h_i \cdot \Lambda_0(s_i[\cdot_G]) = h_i[\cdot_H]. \tag{4}$$

If $S' = \{s_i'\}_{i=1}^{\mu}$, $s_1' = 1$ is another transversal of H in G such that $s_i' \in Hs_i$, then $s_i' = h_i s_i$ for some $h_i \in H$ and $h_1 = 1$. Hence, $\Lambda'_0 : X_0 \to X_0'$ with respect to S' is given by $\Lambda'_0(h_i s_i[\cdot_G]) = \Lambda'_0(s_i'[\cdot_G]) = [\cdot_H]$, and this implies that Λ_0 depends on the choices of transversal.

Given the choice of Λ_0 as in (4), we have

$$(\mathrm{Cor}_{G,H}\ u)[\cdot_G] = (S_{H,G}(u\Lambda_0))\,[\cdot_G] = \prod_{i=1}^{\mu} s_i^{-1} \cdot (u\Lambda_0)(s_i[\cdot_G])$$

$$= \prod_{i=1}^{\mu} s_i^{-1} \cdot u\,(\Lambda_0(s_i[\cdot_G])) = \prod_{i=1}^{\mu} s_i^{-1} \cdot u([\cdot_H]) \ \in M,$$

for $u \in \mathrm{Hom}_H(X_0', M) = C^0(H, M)$, thus the choice of transversal does change Cor map on cochain group. $\square$

This is the answer of (a).

For the second question, we concentrate mainly on the lower dimensions 0, 1, and 2, which are essential for the application to the study of projective representations of groups [6],[8].

THEOREM 3. *Corestriction map on cochain group of dimension 0 is independent of the choices of transversal if one of the following holds.*

(1) *G acts trivially on M.*

(2) *For $u \in \mathrm{Hom}_H(X_0', M) = C^0(H, M)$, $u[\cdot_H]$ is contained in M^G.*

(3) *Corestriction map is defined on a cocycle group $Z^0(H, M)$ in $C^0(H, M)$.*

Proof. In cases of (1) and (2), we have

$$(\mathrm{Cor}_{G,H}\ u)[\cdot_G] = \prod_{i=1}^{\mu} s_i^{-1} \cdot u[\cdot_H] = (u[\cdot_H])^{\mu},$$

and this is independent of the choice of transversal. Note that the condition (2) can be replaced by a condition $u[\cdot_H] \in M^H$. For (3), we remark that $Z^0(H,M) \cong M^H \quad (u \mapsto u[\cdot_H])$, and $Z^0(G,M) \cong M^G$. It is known that a trace map $\rho : M^H \longrightarrow M^G$ defined by $\rho(m) = \prod_{i=1}^{\mu} s_i^{-1} \cdot m$ for $m \in M^H$ is independent of the choices of transversal S. So is the $\mathrm{Cor}_{G,H}$ map from $Z^0(H,M)$ to $Z^0(G,M)$. □

THEOREM 4. *Corestriction map on cochain group of dimension 1 is independent of the choices of transversal if one of the following holds.*

(1) *G acts trivially on M.*

(2) *Corestriction map is defined on a cocycle group $Z^1(H,M)$ in $C^1(H, M)$.*

Proof. If G acts on M trivially, then $H^1(G,M) = Z^1(G,M) \cong \mathrm{Hom}(G,M)$ so that (1) follows from (2).

We denote a commutative subgroup of G by G'. Consider a map

$$\psi : \ G/G' \to H/H', \quad \psi(gG') = \prod_{i=1}^{\mu} \left(s_i g \overline{(s_i g)}^{-1} \right) H'$$

for $g \in G$. Then it is a reduced group theoretical transfer, and is known to be independent of the choices of transversal (refer to [4]). The dual map ψ^* of ψ is a homomorphism $\mathrm{Hom}(H/H',M) \to \mathrm{Hom}(G/G',M)$ defined by,

$$(\psi^*(f))\,(gG') = f \cdot \psi(gG') = \prod_{i=1}^{\mu} f\left(s_i g \overline{(s_i g)}^{-1} \right) H',$$

for $f \in \mathrm{Hom}(H/H',M)$. Since there is an isomorphism from $\mathrm{Hom}(H/H', M)$ to $\mathrm{Hom}(H,M)$ defined by $f \to f'$ where $f(hH') = f'(h)$ for $h \in H$, the ψ^* is a map $\mathrm{Hom}(H,M) \to \mathrm{Hom}(G,M)$ such that $(\psi^*(u))(g) = \prod_{i=1}^{\mu} u(s_i g \overline{(s_i g)}^{-1})$, for $u \in \mathrm{Hom}(H,M)$. Furthermore since $\mathrm{Hom}(H, M) \cong Z^1(H,M)$, every 1-cocycle is a homomorphism, and $\psi^* = \mathrm{Cor}_{G,H} : Z^1(H,M) \to Z^1(G,M)$. Thue the fact that ϕ is independent of a choice of transversal yields the independence of ψ^* of a choice of transversal. □

4. Examples

Our aim here is to determine conditions on which corestriction map over cochain group of dimension 2 does not depend on the choices of transversal. Though we do not establish explicit conditions, there are many examples of groups.

(i) Let $G = \langle x \rangle \times \langle y \rangle$ be of order 4. For transversals of $H = \langle x \rangle$, we can take $S_1 = \{1, y\}$, $S_2 = \{1, xy\}, S_3 = \{x, y\}, S_4 = \{x, xy\}$. Write Cor_i with respect to S_i. Using relations that

$$\overline{x} = \overline{x}y\overline{xy}^{-1} = yy^{-1} = \overline{yx}y\overline{yxy}^{-1} = yy = 1$$
$$x\overline{x}^{-1} = yx\overline{yx}^{-1} = xyy^{-1} = x, \quad \overline{xy} = y$$

and a normalized $f \in Z^2(H, Q)$, we have
$(\mathrm{Cor}_1 f)(x, y) = f(x\overline{x}^{-1}, \overline{x}y\overline{xy}^{-1})f(yx\overline{yx}^{-1}, \overline{yx}y\overline{yxy}^{-1}) = (f(x, 1))^2$
$(\mathrm{Cor}_2 f)(x, y) = (f(x, x))^2 = (\mathrm{Cor}_3 f)(x, y)$,
$(\mathrm{Cor}_4 f)(x, y) = (f(x, 1))^2$.
Since $f(1, 1) = f(1, x) = f(x, 1) = 1$ and $f(x, x) = -1$, $\mathrm{Cor}_i(x, y)$ have same values for any S_i. Similar calculations show that corestriction map is independent of the choices of transversal.

(ii) Let $G = \langle x \rangle$ be of order 4, and $H = \langle x^2 \rangle$. We may take 4 transversals of H that $S_1 = \{1, x\}$, $S_2 = \{1, x^3\}$, $S_3 = \{x, x^2\}$ and $S_4 = \{x^2, x^3\}$. Calculations involving each transversals are:
with S_1, $x\overline{x}^{-1} = 1$, $x^2\overline{x^2}^{-1} = x^3\overline{x^3}^{-1} = x^2$;
with S_2, $x^3\overline{x^3}^{-1} = 1$, $x\overline{x}^{-1} = x^2\overline{x^2}^{-1} = x^2$;
with S_3, $x^3\overline{x^3}^{-1} = x^2$, $x\overline{x}^{-1} = x^2\overline{x^2}^{-1} = 1$;
and with S_4, $x\overline{x}^{-1} = x^2$, $x^2\overline{x^2}^{-1} = x^3\overline{x^3}^{-1} = 1$.
We thus have the followings that

$$(\mathrm{Cor}_1 f)(x, x^2) = f(x, 1)f(x, 1) = (\mathrm{Cor}_i f)(x, x^2), \ (i = 1, \cdots, 4)$$
$$(\mathrm{Cor}_1 f)(x, x^3) = f(1, 1)f(x^2, x^2) = (\mathrm{Cor}_i f)(x, x^3), \ (i = 1, \cdots, 4).$$

This shows that Cor_i does not depend on the choises of transversal.

We need to remark that corestriction map plays very important role in studying group representation theory, specially for determining relations between representations of a group and those of its subgroups [6],[8]. Because corestriction map depends on the choices of transversal, it is fundamental to choose proper transversal.

5. Transfer of groups

Let H be a subgroup of G. If $k = 1$ in (3) then the corestriction map on cohomology groups is the group theoretical transfer map. Let A be any abelian group, and θ be a homomorphism $H \to A$. A homomorphism $\theta^* : G \to A$ defined by

$$\theta^*(g) = \prod_{i=1}^{\mu} \theta(s_i g \overline{s_i g}^{-1}),\ g \in G$$

is the transfer map of θ. When θ is a homomorphism from H to H/H', the θ^* is called the transfer of G into H. In connection with transfer, conjugacy classes of H play important role in studying fusion of H or the focal subgroup of H in G.

In this section, we shall study transfer theorems with resepct to a generalized term, called "F-conjugate". For this we may generally refer to [1].

Let F be a field of characteristic $p > 0$ and E be a normal closure of F. For a finite group G, let G_p be a p-part of G and $G_{p'}$ be a p'-part of G. If $p = 0$ then $G_p = 1$. Choose any positive integer n divisible by $\exp(G)$. Write $n = n_{p'}n_p$ where $n_{p'}$ [resp. n_p] is a p' [resp. p']-part of n, and let $\zeta_{n_{p'}}$ be a primitive $n_{p'}$-th root of unity. Two elements x and y in G are said to be F-conjugate if

$$y = z^{-1}x^{m(\sigma)}z \text{ for some } z \in G, \sigma \in \mathrm{Gal}(E/F) = \mathcal{G},$$

where $m(\sigma)$ is an integer satisfying both $\zeta_{n_{p'}} = \zeta_{n_{p'}}^{m(\sigma)}$ and $m(\sigma) \equiv 1 \pmod{n_p}$. The element $[x, y]_\sigma = x^{-1}y^{-1}x^{m(\sigma)}y = x^{-1}(x^{m(\sigma)})^y$ is an F-commutator element of G, and the group generated by all F-commutator elements is an F-commutator subgroup, denoted by $G'(F)$.

In terms of F-conjugate, two elements $x, y \in H$ are said to be F-fused in G if they are F-conjugate in G, i.e., $y = (x^{m(\sigma)})^g$ for some $g \in G, \sigma \in \mathcal{G}$. Furthermore an F-focal subgroup $\mathrm{Foc}_{F,G}(H)$ of H in G is the subgroup generated by the quotients of pairs of elements of H which are F-fused in G. That is,

$$\begin{aligned}\mathrm{Foc}_{F,G}(H) &= \langle h^{-1}k | h, k \in H, \text{ which are } F\text{-fused in } G\rangle \\ &= \langle h^{-1}(h^{m(\sigma)})^g | h, (h^{m(\sigma)})^g \in H, g \in G, \sigma \in \mathcal{G}\rangle.\end{aligned}$$

If F is algebraically closed, F-conjugate is nothing but conjugate, so that $\mathrm{Foc}_{F,G}(H) = \mathrm{Foc}_G(H)$ the focal subgroup of H in G.

THEOREM 5. *Let H be a subgroup of G. Let $H'(F)$ be an F-commutator subgroup of H. Then*
(1) *H' is normal in $\mathrm{Foc}_G(H)$, and $H'(F)$ is normal in $\mathrm{Foc}_{F,G}(H)$.*
(2) *$H'(F) \cdot \mathrm{Foc}_G(H) = \mathrm{Foc}_{F,G}(H)$, and $H'(F) \cap \mathrm{Foc}_G(H) = H'$.*
(3) *$\mathrm{Foc}_{F,G}(H)/H'(F) \cong \mathrm{Foc}_G(H)/H'$.*

Proof. For (1), we shall only prove the second statement. Clearly $H'(F) \subseteq \mathrm{Foc}_{F,G}(H)$. For any $h^{-1}(h^{m(\tau)})^g \in \mathrm{Foc}_{F,G}(H)$ and $[a,b]_\sigma \in H'(F)$ with $\sigma, \tau \in \mathcal{G}$ and $a, b, h, h^g \in H$, $g \in G$, $(h^{-1}(h^{m(\tau)})^g)^{-1}\ [a,b]_\sigma\ h^{-1}(h^{m(\tau)})^g = [a^{h^{-1}(h^{m(\tau)})^g}, b^{h^{-1}(h^{m(\tau)})^g}]_\sigma$, which is contained in $H'(F)$.
Choose any $h^{-1}(h^{m(\sigma)})^g \in \mathrm{Foc}_{F,G}(H)$ with $g \in G, \sigma \in \mathcal{G}$. Then

$$\begin{aligned} h^{-1}(h^{m(\sigma)})^g &= h^{-1}h^{m(\sigma)}(h^{m(\sigma)})^{-1}(h^{m(\sigma)})^g \\ &= [h,1]_\sigma\ (h^{m(\sigma)})^{-1}(h^{m(\sigma)})^g \in H'(F) \cdot \mathrm{Foc}_G(H). \end{aligned}$$

Certainly, $H'(F) \cdot \mathrm{Foc}_G(H) \subseteq \mathrm{Foc}_{F,G}(H)$, and it follows $H'(F) \cdot \mathrm{Foc}_G(H) = \mathrm{Foc}_{F,G}(H)$. Finally, let $y \in H'(F) \cap \mathrm{Foc}_G(H)$. Then $y \in H'(F)$ implies $y = [a,b]_\sigma = a^{-1}(a^{m(\sigma)})^b$ for some $a, b \in H$, $\sigma \in \mathcal{G}$. Further since $y \in \mathrm{Foc}_G(H)$, y is a quotient of pairs of elements of H which are fused, so that we may generally assume that $m(\sigma) = 1$. Hence $y = [a,b] \in H'$ and $H'(F) \cap \mathrm{Foc}_G(H) = H'$. □

By an F-group, we mean all irreducible characters of G in E have values in F. It is shown [1] that for an abelain group G, G is an abelian F-group if and only if F contains $\zeta_{\exp(G)_{p'}}$; that is, $\exp(G) | m(\sigma) - 1$ for all $\sigma \in \mathcal{G}$. Thus for a normal subgroup N of G, G/N is an abelian F-group if and only if $G'(F) \subseteq N$.

COROLLARY 6. *If H is an either abelian F-group or abelian p-group of G then $\mathrm{Foc}_G(H) = \mathrm{Foc}_{F,G}(H)$.*

Proof. It is shown that H is an abelian F-group if and only if $a^{m(\sigma)} = a$ for any $a \in H$. Further since every abelian p-group is an abelian F-group [1], the proof follows immediately. □

We denote by G_F the F-kernel of G, which is the intersection of the kernels of all elements in $\mathrm{Hom}(G, F^*)$. It is proved [1] that for a normal subgroup N of G, the factor group G/N is an abelian F,p'-group if and only if N contains the G_F. We refer $G'(p)$ and $G'(p')$ to p-commutator and p'-commutator subgroups of G, respectively. For relations between these groups, it has been studied and proved in [1] that

$$G'(p)\cdot G_F = G,\ \ G'(p)\cap G_F = G'(F)$$
$$G'(F)\cdot G'(p') = G_F,\ \ G'(F)\cap G'(p') = G'.$$

Now we shall add one more relation with $\mathrm{Foc}_{F,G}(H)$.

COROLLARY 7. *$Foc_{F,G}(H)$ is a subgroup of $H'(p)$.*

Proof. Let K be a subgroup of H containing $\mathrm{Foc}_{F,G}(H)$ with $K/\mathrm{Foc}_{F,G}(H) = (H/\mathrm{Foc}_{F,G}(H))_{p'}$. Then H/K is a p-group and $K/\mathrm{Foc}_{F,G}(H)$ is a p'-group. Since $H'(p)$ is the smallest subgroup of H to be its factor group $H/H'(p)$ an abelian p-group, we have $H'(p)\subseteq K$. Furthermore these groups are equal because $[K : H'(p)]$ is divisible by both p and p', and this implies $\mathrm{Foc}_{F,G}(H)\subseteq H'(p)$ (refer to diagram 1). □

Thus, combining all results in [1], Theorem 5 and Corollary 7, we have diagram 2.

$$\begin{tikzcd}
H \arrow[r] \arrow[d, dash] & H/Foc_{F,G}(H) \arrow[d, dash] \\
K \arrow[r] \arrow[d, dash] & (H/Foc_{F,G}(H))_{p'} \arrow[d, dash] \\
Foc_{F,G}(H) & 1
\end{tikzcd}$$

dagram 1

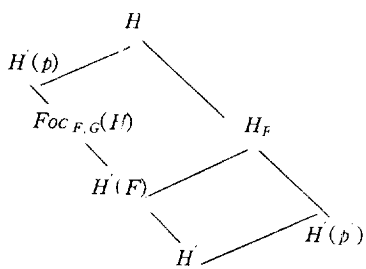

diagram 2

Furthermore, we have that

$$H_F\cdot \mathrm{Foc}_{F,G}(H) = H'(F)\cdot H'(p')\cdot \mathrm{Foc}_{F,G}(H) = H'(p')\cdot \mathrm{Foc}_{F,G}(H)$$

and

$$\begin{aligned} H_F\cdot \mathrm{Foc}_{F,G}(H) &= H'(F)\cdot H'(p')\cdot H'(F)\cdot \mathrm{Foc}_G(H)\\ &= H_F\ \ \mathrm{Foc}_G(H).\end{aligned}$$

Thus it follows that

$$H_F \cdot \mathrm{Foc}_{F,G}(H) = H'(p') \cdot \mathrm{Foc}_{F,G}(H) = H_F \cdot \mathrm{Foc}_G(H).$$

The study of fusion of p-elements is very close to the question of whether a given group G posesses a nontrivial p-factor group, that is, a proper normal subgroup of index a power of the prime p. Grün and Alperin gave criterions for the existence of nontrivial p-factor groups.

LEMMA 8. ([4, p.245]) *G has a nontrivial p-factor group if and only if G has a nontrivial abelian p-factor group. Let P be a Sylow p-subgroup of G, and K be normal in G such that G/K is an abelian p-group. Then $P \cap G' \subseteq K$ and G/K is isomorphic to the homomorphic image of $P/(P \cap G')$. Further there is a normal subgroup H of G such that $G/H \cong P/(P \cap G')$.*

This theorem shows the significance of subgroup $P \cap G'$, and tell us that G possesses a unique maximal abelian p-factor group isomorphic to $P/P \cap G'$. It has been proved $P \cap G'$ is a focal subgroup of P in G, and the focal subgroup provides us more useful expressions for the kernel of the transfer into Sylow subgroup.

We shall study analogues of lemma involving abelian F-group.

THEOREM 9. *A group G has a nontrivial F-factor group if and only if G has a nontrivial abelian F-factor group.*

Proof. One direction is very obvious. Suppose that G has a normal subgroup N and $G/N = L$ is an F-group. Consider an F-commutator subgroup $L'(F)$ of L. Then $L' \subset L'(F)$ and $L/L'(F)$ is an abelian F-group. The corresponding theorem shows the existence of subgroups H and K of G containing G' such that $H/N = L'(F)$ and $K/N = L'$. Since L is an F-group, so are all subgroups $L'(F)$ and L' (see [1]). Thus, $G/H \cong L/L'(F)$ is an abelian F-group, and $G/K \cong L/L'$ is an abelian group. □

COROLLARY 10. *The condition 'F-factor group' in Theorem 9 can be replaced by F, p-factor group, or F, p'-factor group.*

Proof. If $G/N = L$ is a p'-group, then so are $L/L'(F)$ and L/L'. If L is a p-group, then $L = L_F = L'(p')$ and $L'(p) = L'(F) = L'$. Thus, $L/L'(F) = L/L'$ is an abelian F, p-group. □

THEOREM 11. *Let P be a Sylow p-subgroup of G, and let K be a normal subgroup of G such that G/K is an abelian F-group. Then $P \cap G' \subset P \cap G'(F)$ is a subgroup of K, and G/K is isomorphic to the homomorphic image of $P/(P \cap G'(F))$.*

Proof. Since G/K is an abelian F-group, $G'(F)$ is contained in K, and $P \cap G' \subset P \cap G'(F) \subset P \cap K$. And it follows that $G/K \cong P/(P \cap K)$. Consider a canonical projection $\pi : P \to P/(P \cap K)$. For $x \in P \cap G'(F)$, $\pi(x) = x(P \cap K) = 1$ thus $P \cap G'(F) \subset \mathrm{Ker}\pi$. Hence there is a surjection $P/(P \cap G'(F)) \to P/(P \cap K) \cong G/K$. Therefore this is in fact $P \cap G'(F) = P \cap G'$. □

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Department of Mathematics
HanNam University
Ojungdong, Taejon